

MESM 2006

**Modeling the Dispersion of Demand  
when Simulating  
the Newsboy Problem**

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# the newsboy paradigm

## introduction

- (Scarf's notations)  $y$  : order quantity,  $\Phi(\xi)$  : demand cdf,  $c$  : unit cost,  $r$  : unit selling price
- unsold units are discarded
- the satisfied demand is :  $\xi^+ = \min(y, \xi)$
- naive solution :  $\hat{G} \doteq G(\mu, \mu)$ ... but actual gain (ex post) :

$$G(y, \xi) = r \xi^+ - c y$$

## cost of a given choice

- define  $\theta_y \doteq \int_y^\infty d\Phi(\xi)$  together with:

$$\xi_y^a \doteq E(\xi \mid y < \xi) \quad ; \quad \xi_y^b \doteq E(\xi \mid \xi < y)$$

- define  $G(y, \Phi) \doteq E(G(y, \xi))$ , obtain:

$$\begin{aligned} \widehat{G} - G(y, \Phi) = \\ \theta_y (\xi_y^a - y) (r - c) + (1 - \theta_y) (y - \xi_y^b) c \end{aligned}$$

- and conclude:

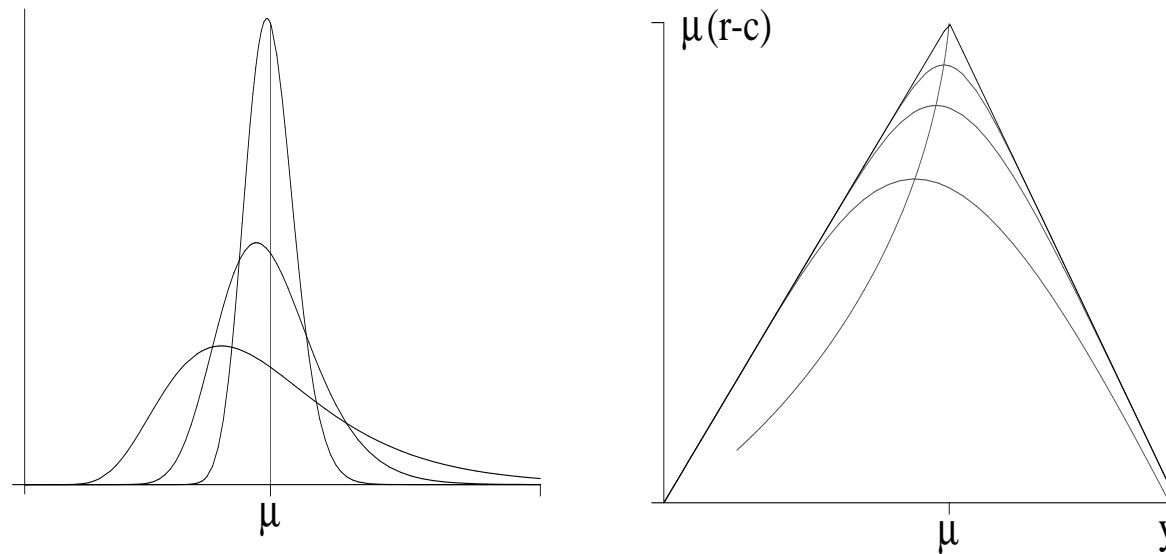
$$\forall \Phi \forall y : G(y, \Phi) \leq \widehat{G}$$

## cost of uncertainties

- knowing  $\Phi$ , we have  $y^* = \arg \max_y G(y, \Phi)$
- exact analytical solution :  $\theta_{y^*} = \theta_* = 1 - \Phi(y^*) = c/r$
- the corresponding (least) miss to gain can be rewritten as:

$$\hat{G} - G(y^*, \Phi) = \theta_* (1 - \theta_*) (\xi_*^a - \xi_*^b) r$$

## example : lognormal $\Phi$ , given $\mu$



- positive values, but assume multiplicative independence
- curve  $\sigma \mapsto (y^*, G^*)$ , assuming  $c/r < 1/2$

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# robust solutions

## knowledge versus facilities

- probability distributions can be used to impersonate our actual knowledge about the real world... or about the limits of our knowledge
- but, too often, side assumptions are introduced that does not come from the actual framework, but only from computing easiness

## basic questions

- does  $\Phi$  model a lack of knowledge due e.g. to their cost or model the intrinsic wild behavior of the markets ?
- is  $\Phi$  guessed from many parallel independent worlds or induced from historical data (questionable ergodicity) ?
- can  $\xi$  be ever measured, even afterwards, when the demand overflows the inventory ?
- robust solution against a family  $\mathcal{F}$  of distributions :

$$G^* \doteq G(y^*, \mathcal{F}) \doteq \max_y \min_{\Phi \in \mathcal{F}} G(y, \Phi)$$

## Scarf's theorem

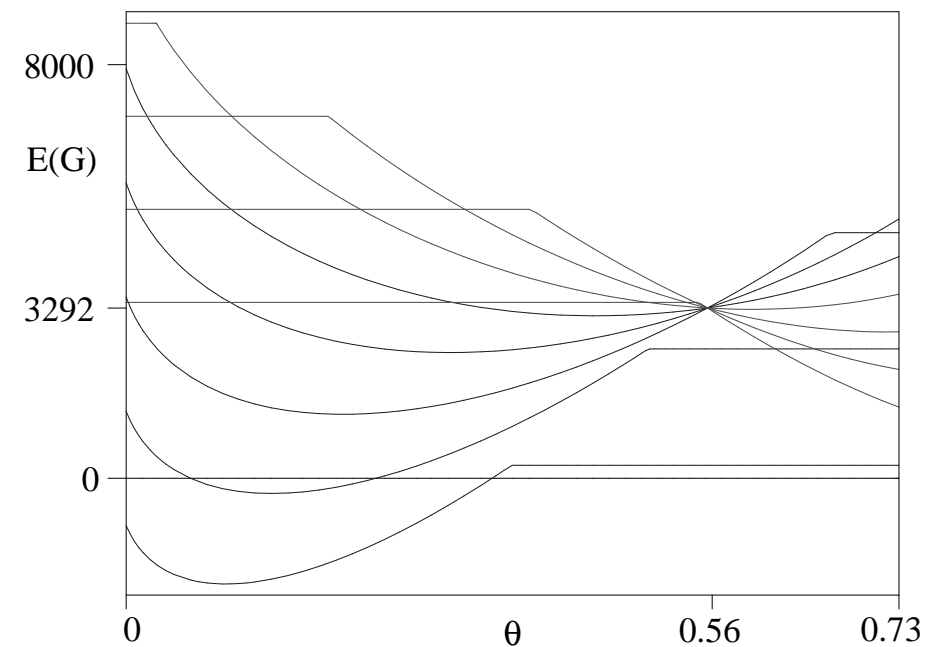
- when playing against  $\mathcal{F}(\mu, \sigma)$ , the worst case is a Dirac two points distribution where  $\xi$  is either  $\xi^a$  or  $\xi^b$
- when  $(c/r)^{-1} < 1 + \sigma^2/\mu^2$ , then better buy nothing
- otherwise, the robust decision is:

$$\begin{cases} y^* & = \mu + \sigma (r/2 - c) / \sqrt{c(r - c)} \\ G^* & = \mu(r - c) - \sigma \sqrt{c(r - c)} \end{cases}$$

## graphical proof

playing against the  $\mathcal{F}(\mu, \sigma, Dirac)$  family is:

- chose an  $y$ , i.e. a curve
- wait for answer  $G(y, \xi)$
- $E(G)$  depends on  $\theta$



all curves are going through the same point  $\theta = c/r$

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# the intermeans parameter

## cost of mean

- we have  $\hat{G} - G(\mu, \Phi) = \theta_\mu (1 - \theta_\mu) (\xi_\mu^a - \xi_\mu^b) \times r$
- this kind of factorization applies only to  $y^*$  from  $\theta_* = c/r$  and to  $\mu$  from the obvious  $\forall y : \theta_y \xi_y^a + (1 - \theta_y) \xi_y^b = \mu$
- thus  $\hat{G} - G^* \leq \delta r$ , independent of  $c/r$ , where

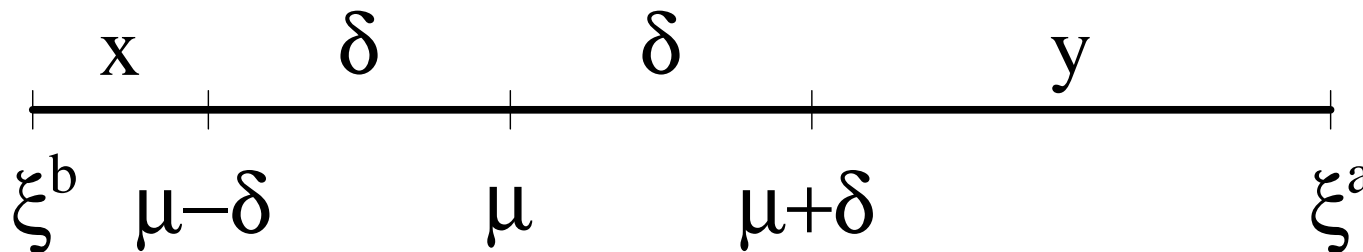
$$\delta \doteq \theta_\mu (1 - \theta_\mu) (\xi_\mu^a - \xi_\mu^b)$$

## a measure of dispersion

- from now on, all  $\theta$ ,  $\xi^a$ ,  $\xi^b$  are relative to  $\mu$

$$\delta \doteq \theta (1 - \theta) (\xi^a - \xi^b)$$

- this  $\delta$  has some similarities with the interquartile range
- properties :  $\xi^b < \mu - \delta < \mu + \delta < \xi^a$  and  $x y = \delta^2$



## comparison $\delta$ versus $\sigma$

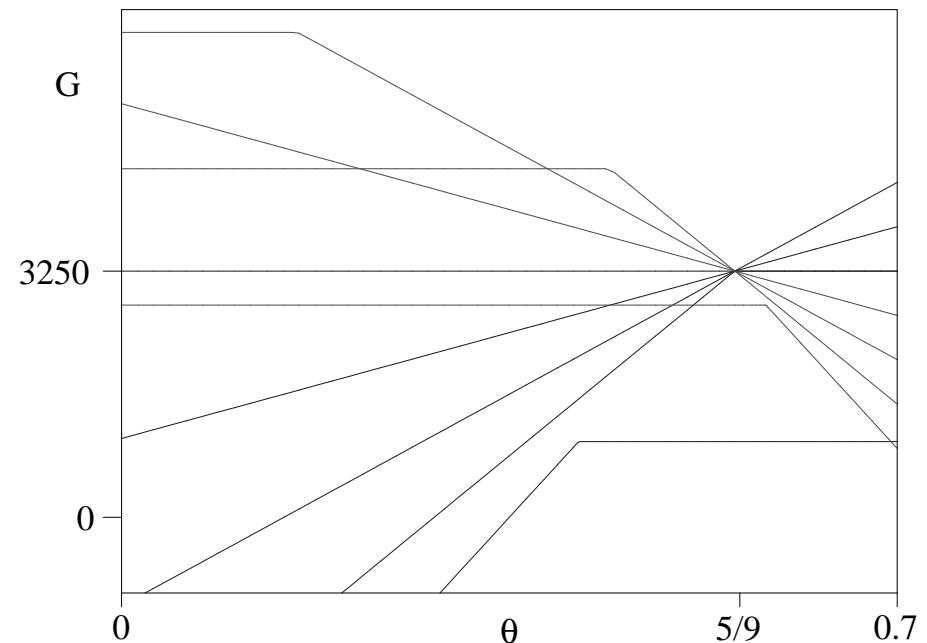
$\Phi$	$\delta/\sigma(\text{exact})$	$\delta/\sigma(\text{approx})$	0.25 gap
uniform	$\sqrt{3}/4$	$\approx 0.433$	
triangular	$1/\sqrt{6} \dots 8\sqrt{2}/27$	$.408 \dots .419$	
normal	$1/\sqrt{2\pi}$	$\approx 0.399$	
exp	$1/e$	$\approx 0.368$	
Dirac	$\sqrt{\theta(1-\theta)}$	$0 \dots 0.5$	$\theta = 7\%, 93\%$
lognormal	$\leq 1/\sqrt{2\pi}$	$0 \dots 0.399$	$\sigma/\mu \approx 2$

in all realistic situations :  $0.25 \sigma \leq \delta \leq 0.50 \sigma$

## a surprising result

- when playing against  $\mathcal{F}(\mu, \delta)$ , the worst case is not necessarily a two points distribution
- when playing against  $\mathcal{F}(\mu, \delta, Dirac)$ , all curves are going through the same point  $\theta = c/r$
- and now

$$G_{Dirac}^* = \mu$$



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# sampling properties

## sampling properties of variance

- obtain sample  $\omega$  by  $n$  independent drawings from  $\Phi$
- put  $s^2 \doteq \text{var}(\omega)$  and define  $S^2 \doteq s^2 \times n / (n - 1)$ . Then:

$$E(S^2) = \sigma^2 \quad ; \quad \text{var}(S^2) = \frac{1}{n} \left( M_4 - \frac{n-3}{n-1} \sigma^4 \right)$$

$$n \times \frac{\text{var}}{E^2}(S^2) = \left( \frac{M_4}{\sigma^4} - 1 \right) + \frac{2}{n-1}$$

## sampling properties of $\delta$

- $d \doteq \delta(\omega)$  is well defined, even if  $x_n = m$

$$d(\omega) = d(\omega \setminus \{x_n\}) \times (n-1)/n$$

- define  $D = d \times \text{bias\_factor}$  so that  $E(D) = \delta(\Phi)$  :

$\Phi$	$D/d$	$n \times \frac{\text{var}}{E^2}(D)$	$n \times \frac{\text{var}}{E^2}(S^2)$
Dirac's	$\frac{n}{n-1}$	$\frac{1}{\theta(1-\theta)} - 4 + \frac{2}{n-1}$	idem
unif.	$\frac{n}{n-2/3}$	$\frac{1}{3} + \frac{2/3}{n-26/45} + \dots$	$\frac{4}{5} + \frac{2}{n-1}$

- bias factor depends on  $\Phi$

## experimental behavior

- exact bias factors have not been found for other  $\Phi$
- experiments :  $n = 4, 7, 10, 13$  and each time  $N = 1600$
- using  $m(d) \approx E(d)$  and  $S^2(d) \approx \text{var}(d)$  leads to:

$\Phi$	$n \times \frac{\text{var}}{E^2}(d)$	$n \times \frac{\text{var}}{E^2}(S)$	$n \times \frac{\text{var}}{E^2}(S^2)$
unif.	$\approx 0.4$	$\approx 0.3$	$\frac{4}{5} + \frac{2}{n-1}$
gauss	$\approx 0.6$	$\approx 0.5$	$2 + \frac{2}{n-1}$
exp	$\approx 1.5$	$\approx 1.7$	$8 + \frac{2}{n-1}$
log	$\approx 1.5$	$\approx 1.7$	$\approx 10$

## conclusion

- the best decision for the newsboy problem depends heavily on the choice of the dispersion measure
- the well known "Scarf's rule" follows when assuming an exact knowledge of  $\mu, \sigma$
- but another strategy follows when assuming an exact knowledge of  $\mu, \delta$
- this happens while  $0.3 \leq \delta/\sigma \leq 0.5$  for all relevant  $\Phi$  and while estimator  $d$  doesn't behave worse than estimator  $S$

- an exact identification of reduced parameters concerning the demand models seems to be questionable
- extraction of knowledge from history cannot be model-free ( $\xi^a$  is beyond any experience)
- a description using larger families of pdf such as  $\mathcal{F}(\mu \pm \Delta\mu, \sigma \pm \Delta\sigma)$  or  $\mathcal{F}(\mu \pm \Delta\mu, \delta \pm \Delta\delta)$  seems a better way for a robust description

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