

# A new point on Euler line ?

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## 1 # 18419

Let  $H$  be the orthocenter, the intersection point of altitudes  $AH_1, BH_2, CH_3$ , and  $G$  be the centroid, the intersection point of medians  $AM_1, BM_2, CM_3$ . Draw the parallel to  $AC$  through  $H$ , which intersects  $BA$  and  $BC$  at  $C_1$  and  $A_2$ , respectively. Analogously, draw the parallel through  $H$  to  $BA$  (and to  $BC$ ), to find the points  $A_1$  and  $B_2$  (and  $B_1$  and  $C_2$ ).

Extend the lines  $C_1B_2, A_1C_2, B_1A_2$  to form a new triangle  $XYZ$  ( $X$  is opposite to  $A$ ,  $Y$  to  $B$ ,  $Z$  to  $C$ ). Denote midpoints of  $C_1B_2, A_1C_2, B_1A_2$  by  $T_1, T_2, T_3$ . It is known that  $H_1, H_2, H_3, M_1, M_2, M_3, T_1, T_2, T_3$  are cocyclic (Euler circle, center  $Oe$ ). Let this circle intersect again  $C_1B_2, A_1C_2, B_1A_2$  at points  $S_1, S_2, S_3$ .

I find that lines  $XS_1, YS_2, ZS_3$  intersect at a new point  $Y$  (Figure 1) which is collinear with  $H, G$  (on the Euler line). This point is a triangle center, but not yet in the Encyclopedia of Kimberling?

## 2 # 18421, #18423

This result can be extended.

Consider  $P = p : q : r$  (barycentrics, not on the sidelines) instead of  $H$ .

- (1) For  $U, V$  two vertices of  $ABC$ , denote by  $U_V$  the intersection of the side opposite to  $U$  with the parallel to  $UV$  drawn by  $P$ . We have  $A_b = A_1 = 0 : p + q : r$ ,  $A_c = A_2 = 0 : q : p + r$ , etc. By definition,  $T_a = \text{midpoint}(A, P) = \text{midpoint}(B_a, C_a)$  and the triangle of these points is (written in columns) :

$$T = \begin{array}{ccc} 2p + r + q & p & p \\ q & p + 2q + r & q \\ r & r & q + 2r + p \end{array}$$

This triangle is perspective in  $P$  with cevian( $P$ ). It is well known that the  $P_i$  (cevians of  $P$ ), the  $M_i$  (cevians of  $G$ ) and the  $T_i$  are on the same conic  $\Gamma$ , aka  $\text{conicev}(G, P)$ , that generalizes the nine-point circle,  $\text{conicev}(G, H)$  :

$$\Gamma : \sum (rqx^2 - p(q+r)yz) = 0$$



tion. It can be found that triangle  $S$  is :

$$\begin{array}{ccc} p(q^2 + r^2 + qp + rp) & p^2(q - r + p) & p^2(p + r - q) \\ q^2(q - r + p) & q(r^2 + p^2 + pq + rq) & q^2(r + q - p) \\ r^2(p + r - q) & r^2(r + q - p) & r(p^2 + q^2 + pr + qr) \end{array}$$

And we can check that triangle  $S$  and triangle  $X$  are perspective, with perspector

$$Y = \begin{array}{l} p^2(q^2 + r^2 - qr + rp + pq) \\ q^2(r^2 + p^2 + qr - rp + pq) \\ r^2(p^2 + q^2 + qr + rp - pq) \end{array}$$

4. (3) In order to detect the special cases, we compute the determinants of all these triangles and find :

$$\det T = 2(p + q + r)^3, \det X = pqr(r + p + q)^3 \\ \det S = 2p^2q^2r^2(r + p + q)^3$$

Factors  $pqr$  indicates that many cancellations have been done that require  $pqr \neq 0$ . Direct examination of the choice  $P = 0 : q : r$  leads to :

$$T = \begin{bmatrix} q+r & 0 & 0 \\ q & 2q+r & q \\ r & r & 2r+q \end{bmatrix}, X = \begin{bmatrix} 0 & 0 & 0 \\ q & -q^2 & q^2 \\ r & r^2 & -r^2 \end{bmatrix} \\ S = \begin{bmatrix} 0 & 0 & 0 \\ q^2 & q & q \\ -r^2 & r & r \end{bmatrix}, Y = \begin{bmatrix} 0 \\ q \\ r \end{bmatrix}$$

Triangles  $X$  and  $S$  don't even have *three* vertices. On the other hand, factor  $p + q + r$  indicates that, if  $P$  is at infinity, then everything collapses to  $P$  and we have  $T = X = S = Y = P$ .

5. (5bis) When  $P$  is not at infinity and not on the sidelines,  $Y = P$  happens only at point  $G$  (proof : elimination, that must be done with care, due to the many special cases). This special role of point  $P$  comes from the fact that points  $P$  and  $Y$  are ever collinear with  $G$ . Moreover, we can compute their barycentric ratio and obtain :

$$\overrightarrow{GY} \div \overrightarrow{GP} = \frac{K016(P)}{K016(P) - 3pqr}$$

where  $K016(P) = p^2q + p^2r + q^2r + q^2p + r^2p + r^2q$  is the standardized equation of cubic  $K016$ . This proves that  $Y = G$  if and only if  $P$  is either  $G$  itself or a finite point of  $K016$ . Moreover  $Y$  is at infinity when  $P$  is on the cubic  $K016(P) - 3pqr = 0$ .

6. Condition for  $Y$  lies on conic  $\Gamma$ . This condition factors into :

$$2q^2p^2r^2(p + q + r)^2(pq + pr + qr) = 0$$

When  $p = 0$ , conic  $\Gamma$  degenerates into  $x(x - y - z) = 0$  i.e. two parallel lines. When  $P$  is at infinity, there is no degeneracy, and the power two indicates that cevian conic  $CV(G, P)$  is a parabola. The last case is  $P$  on the Steiner circumconic  $CC(G)$ .

7. Using the Kimberling data basis for a brute force attack of the problem, we find 223 named points whose  $Y$  is named either. Except from  $G$ , points at infinity and points on the Steiner conic  $CC(G)$ , we only find the incenter  $I=X(1)$  that gives  $Y(I)=X(995)$ . Obviously, the true story is : brute force attack leads to all former results.
8. (4) When  $P$  is on the Steiner circumconic  $CC(G)$ , then  $Y$  is on  $CV(P, G)$  and also on the Steiner inconic  $CV(G, G)$ .
9. (6)  $P=I$  ... to be studied... (Figure 2).

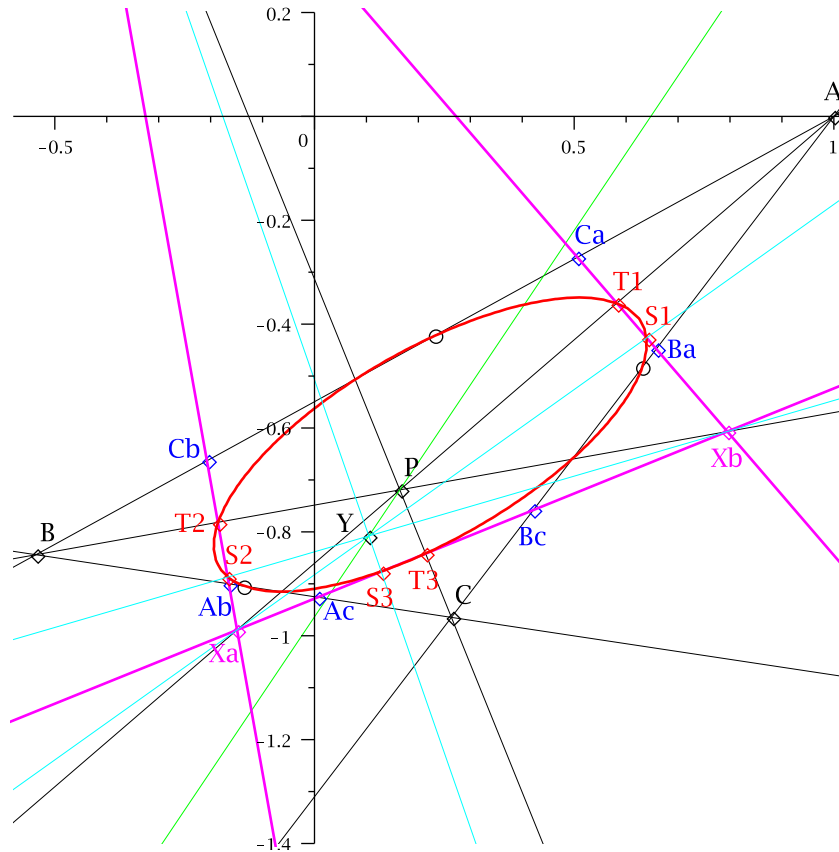


Figure 2: When  $P=I$

10. (7) Some simple (length  $L < 40$ ), but not named  $Y$  are:

<i>Kim</i>	<i>L</i>	<i>barycentrics</i>	$CC(G)$
190	12	$(-b + 2a - c)^2$	$y$
668	17	$(ab + ca - 2bc)^2$	$y$
3226	29	$(b - c)^2 (2a^2 + ab + ca - bc)^2$	$y$
2481	29	$(b - c)^2 (a^2 - ab - ca - 2bc)^2$	$y$
670	29	$(a^2b^2 + a^2c^2 - 2b^2c^2)^2$	$y$
664	31	$(2a^2 - ab - b^2 - ca + 2bc - c^2)^2$	$y$
1494	32	$(b - c)^2 (b + c)^2 (-b^2 - c^2 + a^2)^2$	$y$
75	35	$bc (bc^2 + b^2c - abc + ab^2 + ac^2)$	$n$
32	37	$a^8 (a^4b^4 + c^4a^4 - b^4c^4 + c^8 + b^8)$	$n$
31	37	$a^6 (a^3b^3 + c^3a^3 - b^3c^3 + c^6 + b^6)$	$n$
6	37	$a^4 (a^2b^2 + a^2c^2 - b^2c^2 + c^4 + b^4)$	$n$

### 3 remaining questions are

1. geometrical examination of  $P = H$  (are there special properties due to the special nature of the conic, namely a circle).
2. special case  $P = I$  ??
3. explanation of the results obtained when  $P$  is on circum-Steiner  $= CC(G)$
4. explanation of the role played by K016 (leading to  $Y=G$ ) and K016-3pqr (leading to  $Y$  at infinity, i.e. parallel lines)
5. checking if any of the "simplest unnamed pairs" have any significance
6. counterparts when general cevian conics are used.