Starting from Hyacinthos #18419 (work in progress, 2)

Yakub Aliyev, Pierre L. Douillet

2010-04-07

Remark : Sections 1-3 are from version 1, new stuff begins at Section 4.

1 Start point : Hyacinthos # 18419

The start point of the story is the following Hyacinthos message #18419 emitted by the first author :

> Let H be the orthocenter, the intersection point of altitudes AH_1, BH_2, CH_3, and G be the centroid, the intersection point of medians AM_1, BM_2, CM_3. Draw the parallel to AC through H, which intersects BA and BC at C_1 and A_2, respectively. Analogously, draw the parallel through H to BA (and to BC), to find the points A_1 and B_2 (and B_1 and C_2).
>
> Extend the lines C_1B_2, A_1C_2, B_1A_2 to form a new triangle XYZ (X is opposite to A, Y to B, Z to C). Denote midpoints of C_1B_2, A_1C_2, B_1A_2 by T_1, T_2, T_3. It is known that H_1, H_2, H_3, M_1, M_2, M_3, T_1, T_2, T_3 are cocyclic (Euler circle, center Oe). Let this circle intersect again C_1B_2, A_1C_2, B_1A_2 at points S_1, S_2, S_3.
>
> I find that lines XS_1, YS_2, ZS_3 intersect at a new point Y (Figure 1) which is collinear with H, G (on the Euler line). This point is a triangle center, but not yet in the Encyclopedia of Kimberling?

2 Generalization to any point in the plane # 18421, #18423

2.1 Notations

A slight modification of the notations has been introduced, in order to simplify the formal treatment of the problem. Triangles are uniformly noted $triU$ where U is some identifying letter and its vertices are noted Ua, Ub, Uc. By exception, the reference triangle is noted $triA$, but its vertices remain noted A, B, C as usual. Isolated points are noted by a single letter (P, D, Y, Z, G). The six feet of the parallels are noted Ab, Ba, Bc, Cb, Ca, Ac with the following rules : line $BaCa$ is the secant related to vertex A, while line $AbBa$ is parallel to sideline AB.

2.2 Barycentric formulas

Consider $P = p : q : r$ (barycentrics, not on the sidelines) instead of H.

1. Point Ab (formerly A_1) is the intersection of parallel to AB through P with sideline BC, while point Ac (formerly A_2) is the intersection of parallel to AC through P with sideline BC. We have :

$$Ab \simeq \left(\begin{array}{c} 0 \\ p+q \\ r \end{array}\right), \; Ac \simeq \left(\begin{array}{c} 0 \\ q \\ p+r \end{array}\right)$$

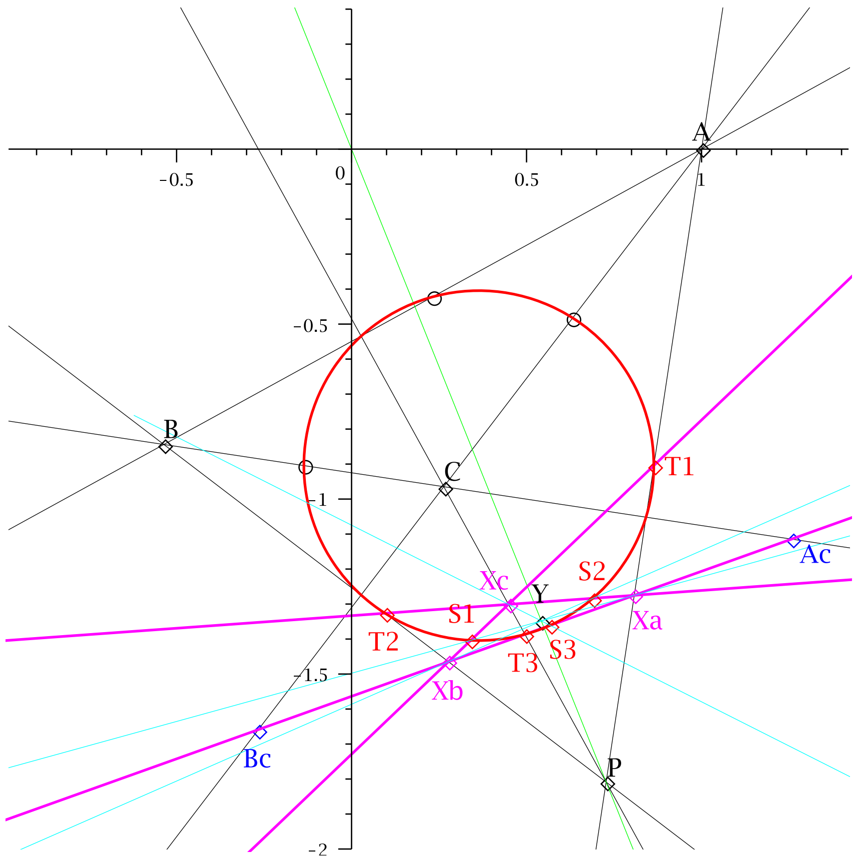


Figure 1: When P=H

2. Since $ABaPCa$ is a parallelogram, we have $Ta = midpoint\,(A, P) = midpoint\,(Ba, Ca)$ and the triangle of these points is (written in columns) :

$$triT \simeq \left(\begin{array}{ccc} 2p+r+q & p & p \\ q & p+2q+r & q \\ r & r & q+2r+p \end{array} \right)$$

This triangle is perspective in P with the cevian triangle $triP = (Pa, Pb, Pc)$. Denote by $triM = (Ma, Mb, Mc)$ the cevian triangle of the centroid $G =$X(2). It is well known that all the nine vertices of $triP$, $triM$, $triT$ are on the same conic $\Gamma = CV\,(G, P)$, the cevian conic of G and P, that generalizes the nine-point circle who is $CV\,(G, H)$:

$$\Gamma \ : \ \sum \left(rqx^2 - p(q+r)yz \right) = 0$$

The center of these conic is $D = (3G + P)\,/4$.

3. Triangles $triM$ and $triT$ are perspective at D. More precisely, D, the center of Γ, is the midpoint of any chord $TaMa$.

4. Barycentrics of line $CaBa$ are $Ca \wedge Ba$, etc, leading to the trigon (written in rows) :

$$\left(\begin{array}{ccc} -rq & r(p+r) & (p+q)q \\ (q+r)r & -rp & p(p+q) \\ q(q+r) & (p+r)p & -qp \end{array} \right)$$

Taking the adjoint of this matrix gives the vertices $XaXbXc$ of the triangle $triX$ (written in columns) :

$$triX \simeq \left(\begin{array}{ccc} -p^2 & (p+r)p & p(p+q) \\ q(q+r) & -q^2 & (p+q)q \\ (q+r)r & r(p+r) & -r^2 \end{array}\right)$$

5. Conic Γ and line $BaCa$ intersect at Ta. Call Sa their second intersection. It can be found that triangle $triS = (Sa,\, Sb,\, Sc)$ is :

$$triS \simeq \left(\begin{array}{ccc} p\left(q^2+r^2+qp+rp\right) & p^2(q-r+p) & p^2(p+r-q) \\ q^2(q-r+p) & q\left(r^2+p^2+pq+rq\right) & q^2(r+q-p) \\ r^2(p+r-q) & r^2(r+q-p) & r\left(p^2+q^2+pr+qr\right) \end{array}\right)$$

6. We can check that triangles $triM$ and $triS$ are perspective at P.

7. We can check that triangles $triS$ and $triX$ are perspective, with perspector

$$Y \simeq \left(\begin{array}{c} p^2\left(q^2+r^2-qr+rp+pq\right) \\ q^2\left(r^2+p^2+qr-rp+pq\right) \\ r^2\left(p^2+q^2+qr+rp-pq\right) \end{array}\right)$$

8. In order to detect the special cases, we compute the determinants of all these triangles and find :

$$\det T = 2\,(p+q+r)^3\,,\ \det X = pqr\,(r+p+q)^3$$
$$\det S = 2p^2q^2r^2\,(r+p+q)^3$$

Factors pqr indicates that many cancellations have been done that require $pqr \neq 0$. Direct examination of the choice $P = 0:q:r$ leads to :

$$T = \left[\begin{array}{ccc} q+r & 0 & 0 \\ q & 2\,q+r & q \\ r & r & 2\,r+q \end{array}\right],\ X = \left[\begin{array}{ccc} 0 & 0 & 0 \\ q & -q^2 & q^2 \\ r & r^2 & -r^2 \end{array}\right]$$
$$S = \left[\begin{array}{ccc} 0 & 0 & 0 \\ q^2 & q & q \\ -r^2 & r & r \end{array}\right],\ Y = \left[\begin{array}{c} 0 \\ q \\ r \end{array}\right]$$

Triangles $triX$ and $triS$ don't even have *three* vertices. But limit value $Y\,(P) = P$ remains acceptable (except from the reference vertices!!!) . On the other hand, factor $p+q+r$ indicates that, if P is at infinity, then everything collapses to P and we have $T = X = S = Y = P$.

9. Points P and Y are ever collinear with G. When nobody is at infinity, their barycentric ratio is given by :

$$\overrightarrow{GY} \div \overrightarrow{GP} = \frac{K016\,(P)}{K016\,(P) - 3pqr}$$

where $K016\,(P) = p^2q + p^2r + q^2r + q^2p + r^2p + r^2q$ is the standardized equation of cubic $K016$. This proves that

   (a) $Y = G$ if and only if P is either G itself or a finite point of $K016$.
   (b) Y is at infinity when P is on the cubic $K016(P) - 3pqr = 0$.
   (c) $Y = P$ occurs only for points at infinity, points on the sideline (except from the vertices) and G itself.

10. Condition for Y lies on conic $\Gamma = CV\,(P, G)$. This condition factors into :

$$2q^2p^2r^2\,(p+q+r)^2\,(pq+pr+qr) = 0$$

When $p = 0$, conic Γ degenerates into $x\,(x-y-z) = 0$ i.e. two parallel lines. When P is at infinity, there is no degeneracy, and the power two indicates that cevian conic $CV\,(G, P)$ is a parabola. The last case is P on the Steiner circumconic $CC\,(G)$. Then Y is on the Steiner inconic $CV\,(G, G)$.

11. Using the Kimberling data basis for a brute force attack of the problem, we find 223 named points whose Y is named either. Except from G, points at infinity and points on the Steiner conic $CC\,(G)$, we only find the incenter I=X(1) that gives $Y\,(I)$=X(995), as seen in Figure 2. Obviously, the true story is : a brute force attack has lead to all former results.

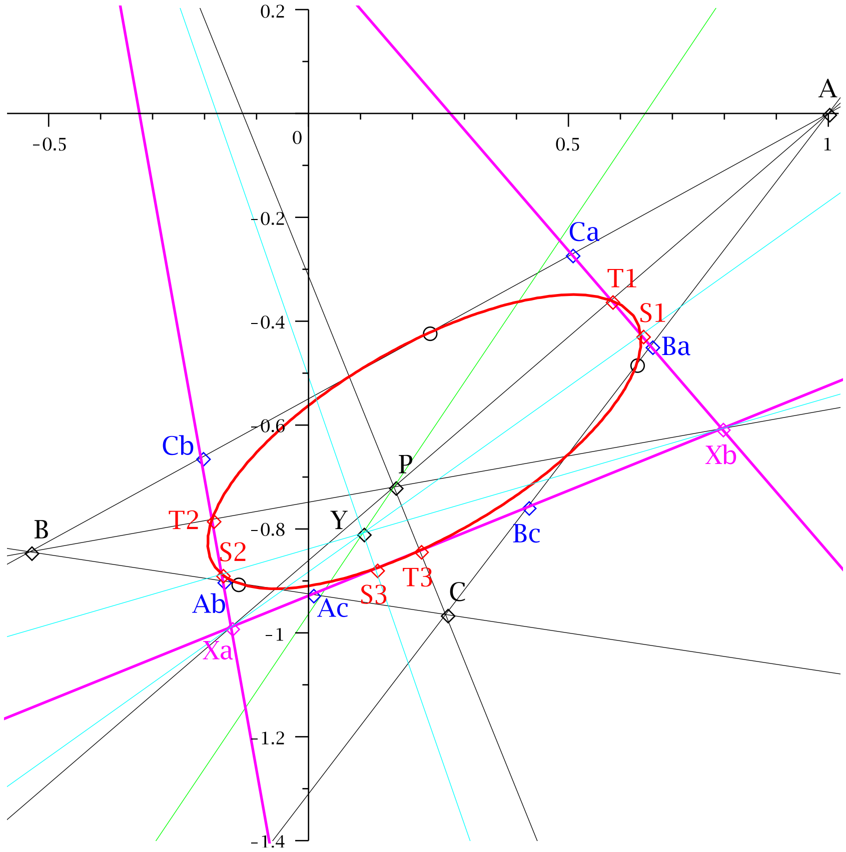


Figure 2: When P=I

12. Some simple (length $L < 40$), but not named Y are:

Kim	L	$barycentrics$	CC(G)
190	12	$(-b+2\,a-c)^2$	y
668	17	$(ab+ca-2\,bc)^2$	y
3226	29	$(b-c)^2\left(2\,a^2+ab+ca-bc\right)^2$	y
2481	29	$(b-c)^2\left(a^2-ab-ca-2\,bc\right)^2$	y
670	29	$\left(a^2b^2+a^2c^2-2\,b^2c^2\right)^2$	y
664	31	$\left(2\,a^2-ab-b^2-ca+2\,bc-c^2\right)^2$	y
1494	32	$(b-c)^2\,(b+c)^2\left(-b^2-c^2+a^2\right)^2$	y
75	35	$bc\left(bc^2+b^2c-abc+ab^2+ac^2\right)$	n
32	37	$a^8\left(a^4b^4+c^4a^4-b^4c^4+c^8+b^8\right)$	n
31	37	$a^6\left(a^3b^3+c^3a^3-b^3c^3+c^6+b^6\right)$	n
6	37	$a^4\left(a^2b^2+a^2c^2-b^2c^2+c^4+b^4\right)$	n

3 Synthetic confirmation of our results

J.L Ayme has reexamined the previous results, using a synthetic point of view. This 20 pages document is
http://pagesperso-orange.fr/jl.ayme/Docs/A%20new%20point%20on%20Euler%20line.pdf

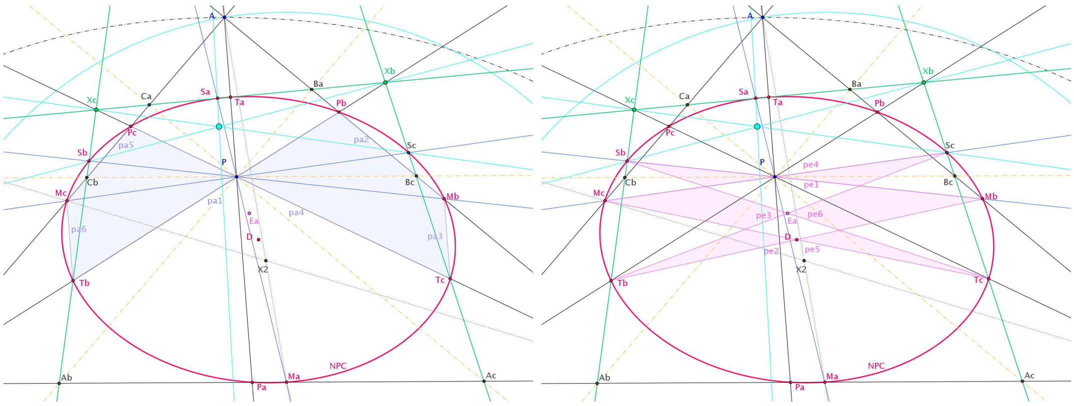


Figure 3: Some hexagons

Only the special case where $P = H$ =X(4) is discussed, using the fact that $CV(G, H)$ is a circle, and therefore angles like $MaSaTa$ are right angles. This is a strong limitation, discarding any possibility of generalization. Moreover, this document do not disclose any other specific property that would apply to this special choice, leaving unsolved our former "remaining question n°1" : does the special case $P = H$ as any specific property ?

Let us give now some synthetic insights over our general results. A tower of hexagons is required, in order to masquerade all these determinants that won the battle. It is not sure that the result is more convincing, but another sight is ever useful. Figure 3 shows two of these hexagons.

1. $triA$ is the reference triangle A, B, C ; P is a point not on the sidelines

2. $triP$ is the cevian triangle Pa, Pb, Pc of P wrt $triA$.

3. $triM$ is the cevian triangle Ma, Mb, Mc wrt $triA$ of G =X(2), the centroid

4. $triT$ is the triangle Ta, Tb, Tc of the middle points $Ta = middle(P, A)$, etc.

5. Consider hexagon Tb-1-Pb-2-Mb-3-Tc-4-Pc-5-Mc-6-Tb. We have $14 = TbPb \cap TcPc = P$ by definition, $25 = PbMb \cap PbMb = A$ by definition and $36 = MbTc \cap McTb = \mathcal{L}_\infty \cap PA$ by middle points triangles. Since 14, 25, 36 are on the same line, all six points are on a same conic. We call it NPC.

6. Define D as the middle of $TaMa$. We have $D = (Ta + Ma)/2 = ((A + P)/2 + (B + C)/2) = (3G + P)/4$. Thus D is also the middle of $TbMb$ and $TcMc$. When two chords have the same middle, this point is the center of the conic.

7. Consider the cevian conic $conicev(P, G)$. It goes through Pa, Pb, Pc, Ma, Mb, Mc (existence : see Galattly). $MaPa$ and $MbMc$ are parallel chords. A diameter is therefore $(m(Ma, Pa), m(Mb, Mc))$. This line goes through D, and circularly. Therefore D is also the center of this conic. Having four independent points in common and the center, both conics are the same : NPC circumscribes $triT$, $triM$, $triP$. This is the expected behavior of the nine point conic.

8. Define Ab, Ba by (Ab, Ba) is the parallel to AB through P, with $Ab \in AC$, $Ba \in AC$, etc. Point Ta is the middle of $CaBa$ since $CaABaP$ is a parallelogram. Sa is the second intersection of $CaBa$ with the conic, etc. It can be seen that triangle $triS = SaSbSc$ is perspective at P with triangle $triM$.

9. Consider the inscribed hexagon Mb-1-Sb-2-Tb-3-Mc-4-Sc-5-Tc-6-Pb. We have $14 = MbSb \cap McSc = P$ by definition, $25 = SbTb \cap ScTc \doteq Xa$ and $36 = TbMc \cap TcMb = \mathcal{L}_\infty \cap PAa$ as before. Therefore, Xa belongs to PAa and triangles $triA$, $triX$ are perspective at P. And also $triT$, $triX$ are perspective at P.

10. Since $triT$ is a cevian triangle wrt $triX$ and inscribed in a conic, the triangle of the remaining intersections$triS$ is also cevian wrt $triX$, proving the existence of perspector Y. Consider the inscribed hexagon Sb-1-Mb-2-Tb-3-Sc-4-Mc-5-Tc-6-Sb. We have $14 = SbMb \cap ScMc = P$, $25 = MbTb \cap McTc = D$ and $36 = TbSc \cap TcSb \doteq Ea$. Thus Ea belongs to line PG.

11. Consider the inscribed hexagon Tb-1-Xb-2-Sb-3-Tc-4-Xc-5-Sc-6-Tb. We have $14 = TbXb \cap TcXc = P$, $25 = XbSb \cap XcSc = Y$ and $36 = TbSc \cap TcSb \doteq Ea$. Therefore Y belongs to line PG.

4 A bonus result

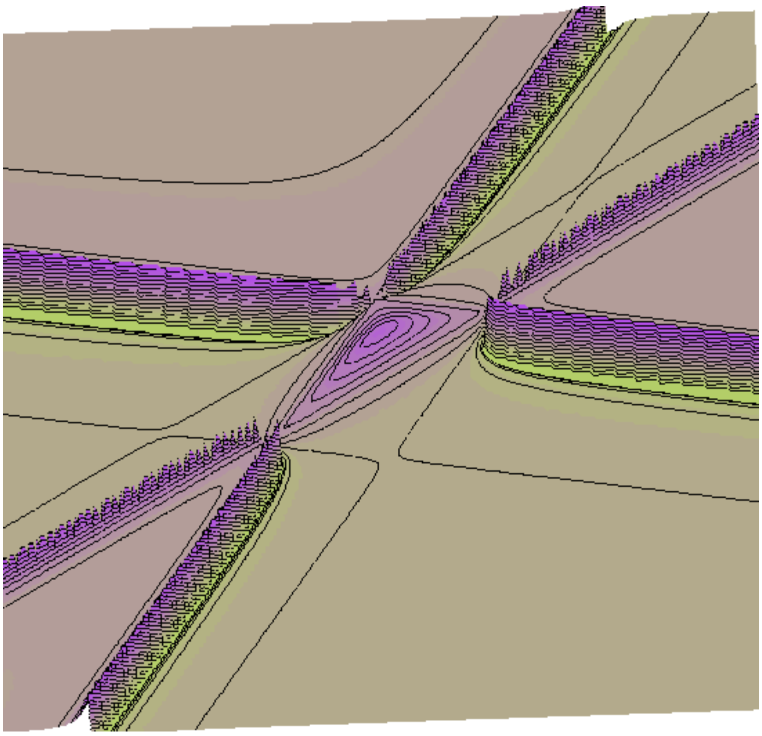
Triangle $triS$ is perspective with $triA$, and the perspector is :

$$Z \simeq \begin{pmatrix} \dfrac{p^2}{q+r-p} \\ \dfrac{q^2}{r+p-q} \\ \dfrac{r^2}{p+q-r} \end{pmatrix}$$

i.e. the image of the anticomplement of P in the isoconjugacy that fixes P.

5 Back to our key result

Let us go back to our key result as summarized by Figure 4.



Sharp walls are the sidelines of the reference triangle, center hill is roughly speaking bordered by the Steiner ellipse and roads in the grey plane are sidelines of the antimedial triangle.

Figure 4: Contour lines

For all points (except from vertices and points at infinity), we have :

$$\overrightarrow{GY} \div \overrightarrow{GP} = \frac{K016\,(P)}{K016\,(P) - 3pqr}$$

where $K016\,(P) = p^2q + p^2r + q^2r + q^2p + r^2p + r^2q$ is the Tucker cubic. The contour lines $\mathcal{C}\,(\lambda)$ of $\overrightarrow{GY} \div \overrightarrow{GP}$ are therefore the cubics :

$$\mathcal{C}\,(\lambda) \doteq \left(x^2y + x^2z + y^2z + y^2x + z^2x + z^2y\right)(1-\lambda) + 3\lambda\, xyz = 0$$

Here are some special values.

$\mathcal{C}\,(0)$ is the Tucker cubic itself ;

$\mathcal{C}\,(1)$ is the cubic $xyz = 0$, made of the three sidelines, where points are unchanged by the Y-transform ;

$\mathcal{C}\,(2/5)$ is $(x+y)\,(y+z)\,(z+x) = 0$, i.e. the sidelines of the antimedial triangle (parallels to the sidelines through the vertices of the reference triangle). In this case, the involved conic is an hyperbola degenerated in two lines, a line like $MbMc$ among them.

$\mathcal{C}\,(1/2)$ is $(x+y+z)\,(xy+yz+zx) = 0$, i.e. the union of the infinity line and the Steiner circumscribed ellipse. In the later case Y is on the Steiner inscribed ellipse, from $\overrightarrow{GY} = \overrightarrow{GP}/2$. Moreover, $Y\,(P)$ lies on the cevian conic $CV\,(G,P)$. This solves our "remaining question n° 3"... and gives a meaning to the ratio $\overrightarrow{GY} \div \overrightarrow{GP}$ when P is at infinity : the center of a conic that evolves into a parabola goes to infinity half fast as the contact point.

All the contour lines share the same asymptotic directions. More precisely, we have :

$$9\,\left(2\,\lambda-1\right)^2 C\,(\lambda) = \prod\left(\left(5\,\lambda-2\right)p - \left(\lambda-1\right)\left(q+r\right)\right) - \left(p+q+r\right)^3 \left(5\,\lambda-2\right)\left(\lambda-1\right)^2$$

Except from the case $\lambda = 1/2$, asymptotes are defining a triangle whose centroid is G. In the special case $\lambda = 0$, the Tucker cubic itself, asymptotes concur in G.

6 remaining questions are

1. Does it happens something special when $P = H$ =X(4), i.e. when Γ is a circle ?
2. Does it happens something special when $P = I$ =X(1) ?
3. Obtain more insights on the relationship between the geometrical construction and the family $C\,(\lambda)$ of contour lines.