Translation of the Kimberling’s Glossary into barycentrics
"Le Glossaire de Pierre"
(v96-dvi5)

pldx

January 3, 2024
Acknowledgement. This document began its life as a private copy of the Glossary accompanying the Encyclopedia of Clark Kimberling (1998-2021). This Glossary—as its name suggests—is organized alphabetically. As a never satisfied newcomer, I would have preferred a progression from the easiest to the hardest topics and I reordered this document in my own way. It is unclear whether this new ordering will be useful to someone else! In any case a detailed index is provided.

Second point, the Kimberling’s Glossary is written using trilinear coordinates. From an advanced point of view, these coordinates are neither better nor worse than the barycentric coordinates. Nevertheless, having some practice of the barycentrics, and none of the trilinears, I undertook to translate everything, from one system to another. In any case, this was a formative exercise, and this also puts the focus on the covariance/contra-variance properties that were subsequently systematized.

Drawings are the third point. Everybody knows—or should know—that geometry is not possible at all without drawings. Having no intention to pay royalties for using rulers and compasses, I turned to an open source software (kseg) in order to produce my own drawings. Thereafter, I have used Geogebra, together with pstricks. What a battle, but no progress without practice!

Subsequently, other elements have been incorporated from other sources, including materials about cubics, from the Gibert web site, and about Cremona transforms from the Déserti archive. Finally, original elements were also added. As it will appear at first sight, "pldx" is addicted to a tradition that requires a precisely specified universal space for each object to live in.

A second massive addiction of the author is computer algebra. Having at your fingertips a tool that gives the right answer to each and every expansion or factorization, and never lost the small paper sheet where the computation of the week was summarized is really great. Moreover, being constrained to explain everything to a computer helps to specify all the required details. For example, the "equivalence up to a proportionality factor" doesn't apply in the same way to a matrix whether this matrix describes a collineation, a triangle, a trigone or a set of incidence relations.

In this document, "beautiful geometrical proofs" are avoided as much as possible, since they are the most error prone. A safe proof of "the triangle of contacts of the inscribed circle and the triangle of the mid-arcs on the circumcircle admit the insimilicenter of the two circles as perspector" is:

\[
\text{ency(persp(matcev(vX(7)),matucev(vX(1))))} \mapsto 56
\]

where the crucial point is ency, i.e. a safe implementation of the Kimberling’s search key method to explore the database.

To summarize, the present document is rather a "derived work", where the elements presented are not intended to be genuine, apart perhaps from the way to assemble the ingredients and cook them together.
In a better world, this page wouldn’t be required, but...

1. All of this barnum requires –inter alia– my own version of `prettyref.sty` and `algorithmic.sty`

2. These files are stored in `ipse/.lyx/_modified_latex`

3. This directory is linked into `/usr/share/texmf/tex/latex/_modified_latex`

4. But nowadays, `texhash ∈ SuSE-15.5` behaves poorly when a directory is missing.

5. For each directory where `texhash` complains, create this directory, and empty `toto.sty` file together with a file named `ls-R` and containing the line

   ```latex
   \% ls-R -- filename database for kpathsea; do not change this line.
   ```
Index

-Table of Contents, 19
=well-known, 2
  circles, 172
  conics, 138
  cubics, 299
  lines, 36
  matrices, 89
  surds, 171
  triangles, 47

A
adjoint, 126
affix, 213
  complex, 213
  Morley, 212, 215
alephdivision, 232
Al-Kashi, 68, 82
Alt-Spieker circle, 176
angle between lines, 88
angle of two circles, 193
angular coordinates, 246
anticevian triangle, 46
anticomplement, 43, 231
anticomplementary conjugate, 345
anticomplementary triangle, 43, 47
antigonal conjugacy, 246
antimedial triangle, 43
antiorthic axis, 36
anti-pedal triangle, 354
Apollonian A,B,C circles, 161, 201, 256
Apollonius circle, 176
Apollonius configuration, 203
Arbelos, 182
area, 82
  complex affixes, 213
  Heron, 82
  matrix, 82
areal center, 359
auxiliary line (of an inconic), 134

B
backward matrix, 216
backward substitutions, 216
backward-2 matrix, 221
backward-2 substitutions, 221
barycentric division, 33
barycentric multiplication, 33
barycenters, 30, 33
barycenters (standard), 73
barymul
  as a collineation, 230
  as a formal operator, 344
  construction, 230
base points (pencil of circles), 195, 243
Bevan circle, 175
Bevan line, 36
Bezout theorem, 211
bicentric points, 39
Brisse Transform, 137
Brocard
  1st and 2nd LFIT’s, 369
  angle, 90
Brocard 3-6 circle, 178, 257, 338
Brocard ellipse, 148
Brocard line, 36
Brocard points, 90, 257
Brocard second circle, 179
Brocard triangles, 47

C
Cayley-Bacharach, 295
center
  rational, 40
  strong, 39
  weak, 40
center of a circle, 170
center of a conic, 128
central line, 39
central point, 39
central triangle, 41
ceva conjugacy, 56
cevadiv, 55, 231, 232, 343
cevamul, 55, 231, 343
cevapoint, 56
cevian collineation, 231
cevian conic, 139
cevian conjugacies, 232
cevian lines, 46
cevian nest, 53
cevian triangle, 46
circle, 169
  list of well-known circles, 172
circles
  angle of two, 193
circum-anticevian, 117
circumcevian, 117
circumcircle, 138, 172
  standard eqn, 84
circumconic, 132
  asymptote, 152
circum-hyperbola, 152
circumRH, 152
gudulic point, 140
parametrization, 132
circum-hyperbola, 152
circumpolar, 123, 219
circumRH, 152
cK(F,U), 314
Clawson-Schmidt homography, 49, 163, 407
coccevian triangle, 46
collinear, 30
collineation, 227
collineation algorithm, 227
Collings transform, 346
Comatrix, 126
combos, 45
complement, 43, 231
complementary conjugate, 345
completed line, 190
complex affix, 213
concurrent, 31
concyclic, 250
conic, 123
auxiliary circle, 158
by five points, 127
center, 128
cross-ratio, 137
degenerate, 154
dual of a conic, 128
FF, the focal tangential pencil, 143
LLLL, the Miquel pencil, 163
metric elements, 154
orthoptical cycle, 158
parametrization, 127
perspector, 128
pole and polar, 127
PPPP, the four points pencil, 142
proper, 126
PTPT, the bitangent pencil, 158
tangential conic, 129
ten determinants formula, 127
conic-pivotal isocubic, 314
Conway symbols, 42
cosinus of projection, 95
cot versus Conway, 91
covariant derivative, 290
Cremona, 237, 240
exceptional curves, 240
homography, 238
indeterminacy points, 240
cross conjugacy, 54
crossdiff, 344
crossdifference, 35
crossdiv, 54, 231, 343
crossmul, 54, 231, 343
crosspoint, 54
cross-ratio, 44
conic, 137
Laguerre formula, 214
of parameters, 44
Riemann sphere, 44
crosssum, 313
crostriangle, 50
cubic, 295
Cayley-Bacharach, 295
cK(F,U), 314
list of well-known cubics, 299
nK(P,U,k), 313
nK0(P,U), 313
Orion, 309
pK(P,U), 298
pK: 22 points property, 300
PKA, 303
tangential, 296
tangential quadruple, 303
triangular, 298
Van Rees, 404
cycle, 190, 250
cyclocevian conjugate, 184
cyclopedal conjugate, 109
D
dalethdivision, 234
Danneels (second) perspector, 58
Danneels perspector DP1, 57
Darboux cubic, 306
DC point, 119
de Longchamps axis, 36
degenerate conic, 154
deltoid, 408
tripolar circular cubics, 332
derivative, covariant, 290
Desargues
theorem, 51
triangle, 51
trigone, 51
Descartes
folium, 124
directrix, 149
distance line-parallel, 90
distance point-line, 90
distance point-point, 83
div formula, 344
doteq, 30
dual of a conic, 128
dual triangle, 119
duality, 34
E
eigencenter, 235
eigentransform, 312
Electrical notation, 97
embedded vector, 82
equal, 30
equicenter, 359
euclidian, 31, 81
Euler line, 36
Index

excentral triangle, 47
excentricity, 154–156
exceptional curves (Cremona), 240
Exceter point, 117
eccessiliticenter, 198
 extouch triangle, 47
extraversions, 40

F
Fermat axis, 36
Fermat points, 184
Feuerbach point, 174, 201
Feuerbach RH, 138, 153
focus, 141
 of an inconic, 144
 of an inscribed parabola, 150
 of LFT parabolas, 384
folium, 124
forward matrix, 216
forward substitutions, 216
forward-2 matrix, 221
forward-2 substitutions, 221
fourth harmonic, 44, 45
Fuhrmann 4-8 circle, 181
Fuhrmann triangle, 47, 339

G
gnogebra, 32
Gergonne axis, 36
Gibert-Simson transform, 315
gimeldivison, 233
Gram matrix, 203
Gt mapping, 165
gudulic point, 140

H
Hamilton, 266
harmonic division, 45
helethdivision, 234
Hexyl triangle, 47
Hirst inverse, 346
hirstpoint, 346
homography
 construct the middle, 237
 Cremona transform, 238
 multiplier, 239
 of parameters (thm), 238
 on the complex line, 237
homothety, 95
homothety centers (circles), 198
horizon, 190
hyperbola, 151
 circum-hyperbola, 152
 circumRH, 152
 in-hyperbola, 153
 inRH, 154
hyperbolic
 Klein conveyor, 273
 Klein metric, 274

Poicaré conveyor, 271
 Poicaré metric, 272
hyperbolic hyperboloid, 103

I
incenter
 exclusion curve, 180
 in-excenters, 244
incentral triangle, 47
incircle, 138, 139, 173
Incircle transform, 174
inconic, 134
 auxiliary line, 134
 center, 134
 in-hyperbola, 153
in-parabola, 150
inRH, 154
 parametrization, 134
perspector, 134
indeterminacy points (Cremona), 240
infinity line, 36
in-hyperbola, 153
inRH, 154
insimilicenter (of circles), 198
intouch triangle, 47
inversion in a circle
 barycentric-formula, 197
 complex-formula, 253
 definition, 197
involutory collineation, 228
isoconjugacy
 see sqrtdiv, 243
isodynamic points, 93, 179, 203
isogonal
 Morley formula, 245
isogonal conjugacy, 108
 Morley, 244, 298
isoptic of a cubic, 300
isoptic cubic, 404
isoptic pencil, 195
isosceles triangles, 182
 isotomic conjugacy, 48
 isotomic pencil, 195
isotropic lines, 85

J
Jerabek RH, 138, 153
joint-orthoptic circle, 158

K
K001 Neuberg, 304
K002 Thomson, 305
K003 McKay, 304
K004 Darboux, 306
K007 Lucas, 306
K010 Simson, 315
K060 cubic, 312
K155 EAC2, 310
K162 cubic, 315
K170 cubic, 300
K219, 154
K-ellipse, 148
Kiepert parabola, 138, 219
Kiepert RH, 138, 182
kitW, 170
Klein
hyperbolic conveyor, 273
hyperbolic metric, 274
quadrangle, 40
symmetry, 274
transforms, 40
K-matrix (Al-Kashi), 82

L
Laguerre formula, 214
Lemoine
transform, 221, 242
transforms, 40
Lemoine axis, 36, 138, 257
Lemoine first circle, 177
Lemoine inconic, 138
Lemoine second circle, 177
LFIT, 359, 360
adjunct circle, 377
adjunct points, 377
areal center, 359, 362
circle of similarity, 373
critical triangle, 363, 381
equicenter, 359
fixed point, 377
fixed points, 374
graphs
Catalan, 369
hexagonal, 365
temporal, 368
hexagonal conic, 365
incident motion, 372
observer, 391
parabolas, 383
parametrization, 364
polar conic, 367
polar point, 367
slowness center, 359, 362
temporal embedding, 370
temporal graphs, 368
Tucker’s associate, 370
velocities, 360
Vertex-Miquel circle, 373
limit points (pencil of circles), 195, 243
line, 30
at infinity, 31, 213
completed, 190
list of well-known lines, 36
Linear Families of Inscribed Triangles, 359
locusconic, 130
Longchamps circle, 175
Lubin parametrization, 215
Lubin-2 parametrization, 221
Lucas cubic, 306

M
MacBeath circumconic, 139
MacBeath-inconic, 139
major center, 40
matrix
K, 82
M, 85, 214
N, 410
OrtO, 84, 214
Pyth, 83, 214
Q, 190
Q^-1, 190
Q_p, 259
Q_s, 262
Q_z, 251
W, 81, 213
McKay cubic K003, 304
medial triangle, 43, 47
Menelaus theorem, 49
mimosa transform, 233
Miquel circle (quadrilateral), 400
Miquel point, 373
Miquel theorem, 49
mixtilinear circles, 261
Monge
alignment, 199
Morley affix, 212, 215
Morley space, 212
mul formula, 344
multiplier of an homography, 239

N
Nagel line, 36
Napoleon points, 184
Neuberg circles, 91
Neuberg K001 cubic, 304
Newton axis, 164, 400
Newton line, 49
nine-points circle, 139, 174
NK transform, 313
nK(P,U,k), 313
nK0(P,U), 313
non pivotal isocubic, 298

O
Orion cubic, 309
Orion transform, 308
orthic axis, 36, 112
orthic triangle, 47
orthoassociate, 115
orthocentroidal 2-4 circle, 179
orthocorrespondent, 114
orthodir, 85, 214
orthogonal
cycle, 193, 252
projector, 93, 228
Index

reflection, 93, 228
orthologic (triangles), 354
orthology (collineation), 354
orthopedal circle, 415
orthopoint, 84
orthopole, 410
orthoptic cycle, 156, 158
OrtO-matrix, 84

P
parabola, 129, 149
parallelogic (triangles), 353, 391
parallelogy (collineation), 353
Pascal’s theorem, 296
pedal
circle, 109, 144, 414
triangle anti-pedal, 354
triangle pedal, 107
Pelle à Tarte, 182
pencil, 194
Complex plane, 249
Orthogonality formula, 196
Triangle plane, 187
pencil of conics
FF, the focal tangential pencil, 143
LLLL, the Miquel pencil, 163
PPPP, the four points pencil, 142
PTPT, the bitangent pencil, 158
perspectivity, 50
perspectivity kit, 51
perspector, 51
perspector of a conic, 128
perspectrix, 51
pivot of a cubic, 299
pivotal conic, 314
pivotal umbilical pK, 304
PK≠ transform, 300
pK(P,U), 298
pK: 22 points property, 300
PKA cubics, 303
Plucker
formulas, 125
representation, 97
Poincaré
hyperbolic conveyor, 271
hyperbolic metric, 272
symmetry, 273
point, 30
polar circle, 157, 175
polar wrt a conic, 127
polar wrt a curve, 123
polardiv, 345
polarmul, 133
polarmul of lines, 345
polarmul of points, 345
poles of points triangle, 234
pole wrt a conic, 127
poles of lines triangle, 234
Poncelet
porism, 136, 183
porism
Poncelet, 136
Poulbot, first point, 226
Poulbot, second point, 226
power, 83
barycentric formula, 83, 169
Veronese formula, 193
projector
orthogonal, 93
psi-Kimberling, 232
Pythagoras theorem, 83
Pyth-matrix, 83

Q
quadrilateral, 398
diagonal triangle, 401
embedded triangles, 398
Miquel circle, 400
Miquel point, 400
Newton axis, 400
Newton pencil, 399
orthopedal circle, 415
polar circles, 400
reciprocal lines, 399
Steiner axis, 400
Steiner pencil, 400

R
radical center, 194
radical trace, 171
radius of a circle, 170, 192
rational center, 40
reciprocal conjugacy, 243
reciprocal of a line, 48
rectangular hyperbola, 152
reflection, 43
representative, 190
of a point-circle, 190
residual triangles, 336
of a cevian, 336
RH characterization, 152
Rigby point, 416
rotation, 229
3D-formula, 101
RS, 416

S
S=area(ABC), 42
S_a, 42
Saragossa points, 119
search key
Kimberling, 74
Morley, 79, 217
patched, 74
shadow, 268
shortest cubic, 305
sideline triangle, 51
sideline trig one, 51
simdoteq, 30
simeq, 30
similarity, 229
  circles, centers of, 198
direct, 229
Morley plane, 229
skew, 114, 229
three similarities theorem, 378
Simson cubic, 315
Simson line, 408
Simson-Moses point, 417
sine-triple-angle circle, 178
slowness center, 359, 362
SM, 417
Soddy (but not so) circles, 205
Soddy circles, 204
Soddy conic, 138, 142
Soddy line, 36
Soland’s porism, 279
Sondat, 349
Spiroier circle, 175
sqrtdiv
  construction, 57, 242
  formal definition, 242
  heuristic definition, 34
sqrtmul, 34, 57
SR, 416
star triangle, 47, 340
Steinbart transform, 117
Steiner
  angles, 91
  axis (quadrilateral), 400
circumellipse, 58, 138
deltoid, 408
in-ellipse, 138, 146
line (triangle), 111
Steiner triangle, 111
Stereographic projection, 137, 267
strong center, 39
symbolic substitution, 42
symmetric (metric) functions, 42
symmetric functions, 215
symmetry
  K-symmetry, 274
  orthogonal, 93
  P-symmetry, 273
T
tangent, 88
tangent of two lines, 88, 214
tangent to a curve, 123
tangential, 206
tangential conic, 129
tangential triangle, 47
Tarry point, 91
Taylor circle, 181
TCC-perspector, 117
tetrahedron
  altitudes, 104
circumcenter, 102
metric, 101
Monge point, 102
orthocentric, 104
volume, 102
Tg mapping, 165
Thomson K002 cubic, 305
transcendental center, 40
translation, 228
transpose, 35
triangle, 30
  list of well-known triangles, 47
  may be degenerate, 30
triangle center, 39
trigone, 31
trilinear pole (deprecated), 35
trilinears, 33
tripolar, 35
tripolar centroid, 50
tripolar curve, 319
tripolar line, 46
tripole, 35
Tucker’s associate, 370
type-crossing, 33
type-keeping, 33
U
umbilics, 73, 85, 250
  bold, 188
unary cofactor triangle, 235
V
Van Rees cubic, 404
vandermonde, 215
Veronese map
  barycentric, 189
  Morley space, 250
  Pedoe version, 258
  Spherical version, 261
vertex associate, 120
vertex triangle, 50
vertex trigone, 50
virtual circle, 196
visible point, 212
W
weak center, 40
wedge operator, 34
W-matrix, 81
Y
Yff parabola, 138
Z
Z(U) cubic, 299
Z+cubic, 314
zosma, 234
# List of Figures

1.1 Point P and line $A'B'C'$ are the tripolars of each other.  
36

3.1 Obtain Q from A,P,C and auxiliary D  
45

3.2 Cevian, anticevian and cocevian triangles  
49

3.3 Ceva’s Theorem  
50

3.4 Triangles ABC and UVW that admit P as perspector.  
52

3.5 P is cevamul(U,X)  
53

3.6 crossmul, crossdiv  
55

3.7 P is crossmul of U and X  
55

3.8 evamul, evadiv  
56

3.9 Lemoen’s construction of P=cevamul(U,X)  
57

7.1 Level curves of the Brocard angle of the pedal triangle of point M  
94

9.1 Cyclopedal conjugates are isogonal conjugates  
109

10.1 The orthopoint transform  
112

11.1 Saragossa points of point P  
120

12.1 Folium of Descartes  
124

12.2 Dual curve of the folium  
126

12.3 Starting from perspector $P$  
131

12.4 Conjugacy between a line and a circumconic  
134

12.5 How to generate an inscribed conic from its perspector.  
135

12.6 Stereographic projection  
138

12.7 Focus of an inconic  
144

12.8 The Steiner in ellipse  
147

12.9 The K-ellipse  
148

12.10 Confocal conics and orthoptic circles.  
158

12.11 The tangential pencil  
159

12.12 Two constructions of the focal cubic  
161

12.13 Two transformations of the focal cubic.  
162

12.14 Constructions of paired focuses  
163

12.15 The Miguel pencil and its focal cubic  
166

13.1 Sin triple angle circle  
178

13.2 The exclusion curve  
181

13.3 Cake server (Pelle a Tarte)  
183

13.4 Arbelos configuration  
183

14.1 No point-shadow fall outside of the IC(X76) inconic  
195

14.2 Three circles, six inversions  
198

14.3 Euler pencil and incircle  
200

14.4 Lemoine and Brocard pencils  
202

14.5 Apollonius circles of the not so Soddy configuration.  
206

14.6 Alt-Spieker configuration  
208
List of Figures

15.1 Poulbot's points ................................................. 225
16.1 Construction of barymul ........................................ 231
17.1 Construct the middle of a subtangent ............................ 237
17.2 Antigonal conjugacy ............................................ 247
18.1 The Euler pencil ................................................ 255
18.2 Lemoine and Brocard revisited ................................ 258
18.3 Both projections of A, B in the plane ........................... 264
18.4 Stereographic projection and apexes ............................ 269
19.1 The Poincaré to Klein transform ................................ 275
19.2 The Poincaré to Upper-half transform .......................... 279
20.1 Pascal’s theorem: A",B",C" are aligned ........................... 296
20.2 Using Cayley-Bacharach to prove the cubic associativity ......... 297
20.3 pK(2,4) ......................................................... 301
20.4 The Darboux and Lucas cubics .................................. 307
20.5 The Orion points ............................................... 309
20.6 The Simson cubic (as depicted in Gibert-CTP) ................. 315
20.7 The Simson diagram .............................................. 316
21.1 Does the focal conic intersect the orthogonal circle? ............ 321
21.2 Tripolar curve .................................................. 327
21.3 Tripolar curve, after the meltdown of two focuses ............... 329
21.4 Tripolar cubic .................................................. 330
21.5 Tripolar cubic .................................................. 331
21.6 Angels ......................................................... 332
21.7 Cartesian ovals by inversion .................................... 333
22.1 Special Triangles ................................................. 342
22.2 The star triangle ............................................... 342
23.1 The complementary and anticomplementary conjugates .......... 345
23.2 Collings configuration .......................................... 347
23.3 Collings locus is a ten points rectangular hyperbola ............ 348
24.1 Antipedal triangle ............................................... 355
24.2 Orthology and perspective ...................................... 357
25.1 The Neuberg construction ....................................... 361
25.2 Variable triangle abc is inscribed into fixed triangle ABC ....... 362
25.3 Hexagonal construction of the inscribed triangle ................ 365
25.4 Miquel circles .................................................. 374
25.5 The "three" similarities configuration ........................... 375
25.6 The RC hodograph .............................................. 377
25.7 Circle of similarity ............................................. 378
25.8 Constructing the critical triangle ............................... 382
25.9 Two parabola .................................................... 387
26.1 Starting from the four M_j ...................................... 405
26.2 van Rees cubic defined by two isogonal pairs .................... 407
26.3 Point E is the orthopole of line A'B'C' ........................ 410
26.4 Revisiting the Sister Mary Cordia Karl's correspondence .......... 413
26.5 Construct the 3 Simson lines through a given point ............. 413
# List of Tables

1.1 Some well-known lines ........................................ 36
2.1 Major centers .................................................. 41
3.1 Some well-known triangles ........................................ 47
3.2 Three cases of cevian nets ........................................ 54
7.1 All these matrices .............................................. 89
12.1 Some Inconics and Circumconics ................................. 138
12.2 Perspector and focuses of some in-conics ......................... 145
13.1 Some usual square roots (kitW) .................................. 171
13.2 Some usual notations (circle kit 2) ............................. 171
13.3 Some circles .................................................. 172
13.4 Blocks related to Kiepert adjunctions .......................... 184
13.5 Similicenters on Kiepert RH .................................... 185
15.1 Action of the Klein group ....................................... 224
17.1 Reduction of a Cremona transform .............................. 241
19.1 Indices and the associated variables ............................ 279
20.1 Some well-known cubics ......................................... 299
20.2 Some cubic shadows on EAC2 .................................... 311
28.1 Structure of the database ...................................... 429
30.1 The various species of points in the Kimberling’s database ......... 434
List of Algorithms

5.1 The reduce procedure .................................................. 70
5.2 The reducol procedure .................................................. 70
5.3 The redurow procedure .................................................. 70
5.4 The wedge procedure ................................................... 70
6.1 circle3 ................................................................. 73
6.2 The normalize procedure ............................................. 74
6.3 The reliresk procedure ............................................... 75
6.4 Procedure buildsk ..................................................... 76
6.5 buildencsort ........................................................... 77
6.6 The writesk procedure ................................................. 78
6.7 The build_sk_plex procedure ........................................ 78
6.8 The ency procedure ................................................... 79
6.9 The dichot procedure .................................................. 79
12.1 The locusconi procedure ............................................. 130
19.1 setvars tells our naming conventions to the computer .......... 280
19.2 The atens procedure (A means Action) .......................... 280
19.3 The dtens procedure (D means Derivation) ...................... 280
19.4 The cten procedure (C means Catenate) ......................... 281
19.5 The rtens procedure (R means Reduce) ........................... 281
19.6 The xtens procedure (X means Xcross) ........................... 282
19.7 The addtens2 procedure ............................................. 282
19.8 The chdotens procedure .............................................. 287
21.1 The geotcurv procedure ............................................. 323
21.2 The buildmeth procedure .......................................... 324
28.1 The structure.php file ................................................ 428
## List of Constructions

<table>
<thead>
<tr>
<th>Section</th>
<th>Description</th>
<th>Page</th>
</tr>
</thead>
<tbody>
<tr>
<td>3.2.12</td>
<td>Construct the fourth harmonic of three aligned points</td>
<td>45</td>
</tr>
<tr>
<td>3.4.7</td>
<td>Construct cev, cocev, anticev</td>
<td>46</td>
</tr>
<tr>
<td>3.11.7</td>
<td>Construct cevamul(U,X)</td>
<td>56</td>
</tr>
<tr>
<td>3.12.1</td>
<td>Construct sqrtdiv(F,U)</td>
<td>57</td>
</tr>
<tr>
<td>9.1.9</td>
<td>Construct the ABC-pedal triangle of a given shape</td>
<td>108</td>
</tr>
<tr>
<td>12.3.13</td>
<td>Construct the polar line of a point</td>
<td>128</td>
</tr>
<tr>
<td>12.7.2</td>
<td>Construct a circumconic from its perspector</td>
<td>132</td>
</tr>
<tr>
<td>12.8.4</td>
<td>Construct an inscribed conic from its perspector</td>
<td>134</td>
</tr>
<tr>
<td>12.8.7</td>
<td>Generate an inscribed conic using an arbitrary line</td>
<td>135</td>
</tr>
<tr>
<td>12.24.6</td>
<td>Construct the orthoptic of an inconic</td>
<td>157</td>
</tr>
<tr>
<td>16.4.4</td>
<td>Construct the barymul of two points</td>
<td>230</td>
</tr>
<tr>
<td>17.1.3</td>
<td>Construct the middle of a subtangent</td>
<td>237</td>
</tr>
<tr>
<td>17.4.5</td>
<td>Construct the isoconjugate knowing a pair of conjugates</td>
<td>242</td>
</tr>
<tr>
<td>17.4.7</td>
<td>Construct the fixed points of an isoconjugacy</td>
<td>242</td>
</tr>
<tr>
<td>17.4.8</td>
<td>Construct the fourth harmonic</td>
<td>243</td>
</tr>
<tr>
<td>25.1.6</td>
<td>Construct the Neuberg centers E,S</td>
<td>360</td>
</tr>
<tr>
<td>25.2.3</td>
<td>Compute the temporal parameter in a LFIT</td>
<td>360</td>
</tr>
<tr>
<td>25.3.3</td>
<td>Construct the hexagonal graphs of a LFIT</td>
<td>364</td>
</tr>
<tr>
<td>25.7.1</td>
<td>Construction of the inscribed triangles</td>
<td>374</td>
</tr>
<tr>
<td>25.10.12</td>
<td>Construct the critical triangle of a given LFIT</td>
<td>382</td>
</tr>
<tr>
<td>25.12.8</td>
<td>Construct the embedded pedal triangle</td>
<td>385</td>
</tr>
<tr>
<td>25.12.10</td>
<td>Construct the three embedded cevian triangles</td>
<td>386</td>
</tr>
</tbody>
</table>
# Table of Contents

Acknowledgement 2

Index 4

List of Figures 11

List of Tables 13

List of Algorithms 15

List of Constructions 17

Table of Contents 19

1 Introduction 29

1.1 Special remark for French natives .......................... 30
1.2 Basic objects: points and lines .......................... 30
1.3 Figures using geogebra .................................. 31
  1.3.1 Alternatives ...................................... 31
  1.3.2 Various versions of geogebra ......................... 32
  1.3.3 Using Geogebra .................................... 32
    1.3.3.1 Construction protocol ......................... 32
    1.3.3.2 3D geometry .................................. 32
    1.3.3.3 Macros ....................................... 33
1.4 Type-keeping and type-crossing functions .................. 33
1.5 Duality between point and lines ........................ 34
1.6 Isoconjugacy has moved ................................ 37

2 Central objects 39

2.1 Triangle centers ....................................... 39
  2.1.1 Favored concepts .................................. 39
  2.1.2 Deprecated concepts ................ .................. 40
2.2 Central triangle ....................................... 41
2.3 Symbolic substitution .................................. 42

3 Cevian stuff 43

3.1 Centroid stuff ......................................... 43
3.2 Cross-ratio and fourth harmonic .......................... 44
3.3 About combos ........................................... 45
3.4 Cevian, anticevian, cocevian triangles .................... 46
  3.4.1 Well-known triangles ................................. 47
  3.4.2 Isotomic and reciprocal conjugacies ................. 47
3.5 Transversal lines, Menelaus and Miquel theorems .......... 48
3.6 Tripolar centroid ...................................... 50
3.7 Cross-triangle ........................................ 50
3.8 Perspectivity ........................................... 50
3.9 Cevian nests ........................................... 53
3.10 The cross case (aka case I, cev of cev) .................. 54
# Table of Contents

3.11 The ceva case (aka case II, cev and acev) .................................. 55
3.12 The square case (aka case III, acev of acev) ............................... 57
3.13 Danneels perspectors ................................................................. 57

4 The French Touch 61
4.1 The "not so flying plane" ................................................................. 61
4.2 Algorithmic in the rantanplan .......................................................... 62
4.3 Thales antiquadratic form ................................................................. 64
4.4 Pythagoras quadratic form ............................................................... 65
4.5 Tangent of an angle between two lines .............................................. 66

5 Teaching Geometry to a computer 67
5.1 The random observer ................................................................. 67
5.2 Working out an example ................................................................. 67
5.3 An involved observer ................................................................. 68
5.4 From an involved observer to another one ........................................ 69
5.5 Reducing up to a factor ................................................................. 70
5.6 packages ......................................................................................... 71

6 Maple procedures about searchkeys 73
6.1 Procedure mkalgo ................................................................. 73
6.2 Standardized barycentrics ............................................................... 73
6.3 Numerical values ................................................................................ 74
6.3.1 The new reliresk ........................................................................ 75
6.4 The new buildsk ............................................................................. 75
6.5 Complex points ................................................................................. 77
6.5.1 Procedures ency and dichot ........................................................... 79
6.6 morley ......................................................................................... 79

7 Euclidian structure using barycentrics 81
7.1 Lengths and areas ................................................................. 81
7.2 Embedded euclidian vector space ..................................................... 82
7.3 About circumcircle and infinity line ................................................... 83
7.4 Orthogonality .................................................................................. 84
7.4.1 Matrices and formulas ............................................................... 84
7.4.2 Discussion ............................................................................. 86
7.5 Angles between straight lines ......................................................... 88
7.6 Distance from a point to a line ......................................................... 90
7.7 Brocard points and the sequel ........................................................... 90
7.7.1 Some results ........................................................................ 90
7.7.2 Results related to the Kiepert RH ................................................ 92
7.7.3 Spoiler: study of the Neuberg pencil ........................................ 92
7.7.4 Spoiler: Brocard angle of a pedal triangle .................................... 93
7.8 Orthogonal projector onto a line ....................................................... 93
7.9 The hortocenter romance ............................................................... 95

8 Brief extension to 3D spaces 97
8.1 Basic results ..................................................................................... 97
8.2 Euclidian cartesian metric ............................................................... 100
8.3 Rotations in the 3D Euclidean space ................................................ 101
8.4 Euclidian metric in the tetrahedron space ......................................... 101
8.5 HH: hyperbolic hyperboloids ........................................................... 103
8.6 HH: some examples ........................................................................ 105

9 Pedal stuff 107
9.1 Pedal triangle ........................................................................ 107
9.2 Isogonal conjugacy and Steiner triangle ......................................... 108
9.3 Cyclopedal conjugate ................................................................... 109

January 3, 2024 21:08 published under the GNU Free Documentation License
# 10 Orthogonal stuff

10.1 Steiner triangle ................................................................. 111
10.2 Steiner line ........................................................................ 111
10.3 Parallelogy ......................................................................... 113
10.4 Orthology ........................................................................... 113
10.5 Orthopole ........................................................................... 113
10.6 Spoiler: Moebius-Steiner-Cremona transform ......................... 113
10.7 Orthocorrespondents ............................................................. 114
10.8 Isoscelizer .......................................................................... 115

# 11 Circumcevian stuff

11.1 Circum-cevians, circum-anticevians ....................................... 117
11.2 Steinbart transform ............................................................... 117
11.3 Circum-eigentransform .......................................................... 118
11.4 Dual triangles, DC and CD Points ......................................... 119
11.5 Saragossa points ................................................................ 119
11.6 Vertex associates ................................................................ 120

# 12 About conics

12.1 Tangent to a curve ................................................................ 123
12.2 Folium of Descartes ............................................................... 124
12.3 General facts about conics ..................................................... 126
12.4 Tangential conics ................................................................ 129
12.5 Locusconi ............................................................................ 130
12.6 Founding configuration ......................................................... 130
12.7 Circumconics ..................................................................... 132
12.8 Inconics .............................................................................. 134
12.9 Poncelet porism ................................................................. 136
12.10 Conic cross-ratios ............................................................... 137
12.11 Some in- and circum-conics ............................................... 138
12.12 Cevian conics .................................................................. 139
12.13 Direction of axes ................................................................ 140
12.14 Focuses of a conic .............................................................. 141
12.14.1 Soddy Conic ................................................................. 142
12.15 PPPP, the four points pencil .............................................. 142
12.16 FF, the focal tangential pencil ........................................... 143
12.17 Focuses of an inconic ........................................................ 144
12.18 Focuses of the Steiner inconic ............................................ 146
12.18.1 Using barycentrics ........................................................ 146
12.18.2 Using Morley affixes ..................................................... 147
12.19 The Brocard ellipse, aka the K-ellipse .................................. 148
12.20 Parabola .......................................................................... 149
12.20.1 Inscribed parabola ........................................................ 150
12.20.2 Circumscribed parabola ............................................... 150
12.21 Hyperbola ........................................................................ 151
12.21.1 Circum-hyperbolas ...................................................... 152
12.21.2 Circum-rectangular-hyperbolas ................................... 152
12.21.3 Inscribed hyperbolas .................................................... 153
12.21.4 Inscribed-rectangular-hyperbolas ................................. 154
12.22 Metric elements ................................................................ 154
12.23 Diagonal conics ................................................................ 156
12.23.1 Pencils of diagonal conics ............................................ 156
12.24 Orthoptic cycle ................................................................. 156
12.25 PTPT, the bitangent pencil .............................................. 158
12.25.1 The focal cubic ........................................................... 159
12.25.2 More constructions of the focal cubic ......................... 160
12.26 LLLL, the Miquel pencil .................................................. 163
12.27 Tg and Gt mappings .......................................................... 165
### Table of Contents

#### 13 More about circles
- **13.1** General results ........................................ 169
- **13.2** Inversion in a circle .................................. 171
- **13.3** Antipodal Pairs on Circles ............................ 171
- **13.4** Circumcircle .......................................... 172
- **13.5** Incircle ................................................ 173
- **13.6** Nine-points circle .................................... 174
- **13.7** Polar circle ........................................... 175
- **13.8** Longehamps circle .................................... 175
- **13.9** Bevan circle ......................................... 175
- **13.10** Spieker circle ...................................... 175
- **13.11** Alt-Spieker circle ................................ 176
- **13.12** Apollonius circle .................................. 176
- **13.13** First Lemoine circle ................................ 177
- **13.14** Second Lemoine circle .............................. 177
- **13.15** Sine-triple-angle circle ............................. 178
- **13.16** Brocard 3-6 circle ................................ 178
- **13.17** Second Brocard circle .............................. 179
- **13.18** Orthocentroidal 2-4 circle .......................... 179
- **13.19** Fuhrmann 4-8 circle ................................ 181
- **13.20** Taylor circle ......................................... 181
- **13.21** Kiepert RH and isosceles adjunctions ............... 182
- **13.22** Cyclocevian conjugate ............................... 184
- **13.23** Mixtilinear circles ................................ 186

#### 14 Pencils of Cycles in the Triangle Plane
- **14.1** Introductory remarks ................................ 187
  - **14.1.1** How many points at infinity should be used ? .... 187
  - **14.1.2** Umbilics ........................................... 188
  - **14.1.3** Notations .......................................... 189
- **14.2** Cycles and representatives .......................... 189
- **14.3** Fundamental quadric and orthogonality .............. 191
- **14.4** Pencils of cycles .................................... 194
- **14.5** Classification of pencils ............................ 194
- **14.6** Apexes ................................................ 196
- **14.7** Inversion ............................................. 197
  - **14.7.1** One cycle .......................................... 197
  - **14.7.2** Two cycles ........................................ 198
  - **14.7.3** Three circles .................................... 198
- **14.8** Euler pencil and incircle ............................ 199
- **14.9** The Brocard-Lemoine pencils ........................ 201
- **14.10** The Apollonius configuration ....................... 203
  - **14.10.1** Tangent cycles in the representative space .... 203
  - **14.10.2** An example: the Soddy circles .................. 204
  - **14.10.3** An other example: the not so Soddy circles .... 205
  - **14.10.4** The three excircles .............................. 207
  - **14.10.5** The special case ................................ 208
- **14.11** Elementary properties of the Triangle Lie Sphere .... 209

#### 15 Morley spaces, or how to use complex numbers
- **15.1** How to deal with complex conjugacy ? ................ 211
- **15.2** The usual operators of the Morley space ............. 213
- **15.3** Lubin representation of first degree ................ 215
- **15.4** Comparing barycentric and Morley spaces ............ 217
- **15.5** Some examples of first degree ........................ 218
- **15.6** Lubin representation of second degree ............... 221
- **15.7** Poncelet representation .............................. 222
- **15.8** Poulbot’s points (using the Lubin-4 parametrization) 223
- **15.9** More about the foci of a conic ...................... 226
## 16 Collineations

16.1 Definition .................................................. 227
16.2 Involutory collineations ..................................... 228
16.3 Usual affine transforms as collineations ...................... 228
16.4 Barycentric multiplication as a collineation ................. 230
16.5 Complement and anticomplement as collineations ............ 231
16.6 Collineations and cevamul, cevadiv, crossmul, crossdiv ...... 231
16.7 Cevian conjugacies ............................................ 232
16.8 Miscellany .................................................... 234

16.8.1 Poles-of-lines and polar-of-points triangles .............. 234
16.8.2 Unary cofactor triangle, eigencenter ....................... 235

## 17 Cremona group and isoconjugacies

17.1 Homographic Cremona transforms of the projective plane .. 237
17.2 Defining the Cremona transforms ................................ 240
17.3 Working out some examples ................................... 241
17.4 Isoconjugacy and sqrtdiv operator ............................ 241
17.4.1 Some other constructions .................................. 243
17.4.2 Morley point of view ..................................... 244
17.4.3 The isogonal Morley formula ............................... 245
17.4.4 Isoconjkm .................................................. 245
17.5 Antigonal conjugacy .......................................... 246
17.5.1 Angular coordinates ..................................... 246
17.5.2 Antigonal conjugacy ..................................... 246
17.6 Isogonality and perspectivity ................................ 248

17.5.1 Angular coordinates ..................................... 246
17.5.2 Antigonal conjugacy ..................................... 246
17.6 Isogonality and perspectivity ................................ 248
17.7 Isogonality and perspectivity ................................ 248

## 18 Pencils of cycles in the complex plane

18.1 Pencil of cycles in the complex plane .......................... 249
18.1.1 Veronese map ............................................. 250
18.1.2 Circular group acting over the cycles' space .............. 252
18.2 Revisiting the Euler pencil .................................... 254
18.3 Revisiting the Brocard-Lemoine pencil ....................... 256
18.3.1 Isodynamic points ....................................... 257
18.4 The Pedoe formalism ......................................... 258
18.4.1 The Pedoe map ............................................ 258
18.4.2 Pedoe version of the homographic action ................ 260
18.5 The Spherical formalism ...................................... 261
18.5.1 The Spherical map ....................................... 261
18.5.2 Spherical version of the homographic action ............ 263
18.6 Stereographic projection ...................................... 264
18.7 Quaternary .................................................... 266
18.8 Stereographic formalism ..................................... 267
18.9 Comparison with Cartesian and Artinian metrics ............. 268

## 19 Hyperbolic geometry

19.1 The Poincaré plane ............................................ 271
19.2 The Klein plane ............................................... 273
19.3 From a model to the other .................................... 274
19.4 More about hyperbolic distance ............................... 276
19.5 Hyperbolic triangle .......................................... 277
19.5.1 Sideline .................................................. 277
19.5.2 Line-bisectors ........................................... 278
19.5.3 Altitudes ................................................. 278
19.5.4 Trigonometry ............................................. 278
19.6 The Upper-half plane ......................................... 279
19.7 Teaching tensors to a computer ................................ 279
19.8 The sphere: dealing with an example ........................... 281
19.8.1 External and internal coordinates ......................... 281
19.8.2 Jacobians ................................................ 283
<table>
<thead>
<tr>
<th>Section</th>
<th>Page</th>
</tr>
</thead>
<tbody>
<tr>
<td>19.8.1 Internal versus another internal</td>
<td>283</td>
</tr>
<tr>
<td>19.8.2 Internal versus external</td>
<td>284</td>
</tr>
<tr>
<td>19.8.3 More about the projectors</td>
<td>284</td>
</tr>
<tr>
<td>19.8.4 The metric tensor</td>
<td>285</td>
</tr>
<tr>
<td>19.9 Christoffel symbols</td>
<td>286</td>
</tr>
<tr>
<td>19.9.1 Covariance</td>
<td>286</td>
</tr>
<tr>
<td>19.9.2 Taking the steepest line as an example</td>
<td>286</td>
</tr>
<tr>
<td>19.9.3 Computing the Christoffels</td>
<td>287</td>
</tr>
<tr>
<td>19.9.4 Defining the Christoffels</td>
<td>288</td>
</tr>
<tr>
<td>19.9.5 Moving</td>
<td>289</td>
</tr>
<tr>
<td>19.10 Curvature</td>
<td>290</td>
</tr>
<tr>
<td>19.11 Back to Poincaré and Klein</td>
<td>292</td>
</tr>
<tr>
<td>20 About cubics</td>
<td>295</td>
</tr>
<tr>
<td>20.1 Characterisation of a cubic</td>
<td>295</td>
</tr>
<tr>
<td>20.1.1 More about the folium</td>
<td>295</td>
</tr>
<tr>
<td>20.1.2 Pascal’s theorem</td>
<td>296</td>
</tr>
<tr>
<td>20.2 Group structure of a cubic</td>
<td>296</td>
</tr>
<tr>
<td>20.3 Isocubics</td>
<td>298</td>
</tr>
<tr>
<td>20.4 Pivotal isocubics pK(P,U)</td>
<td>298</td>
</tr>
<tr>
<td>20.4.1 Another description</td>
<td>302</td>
</tr>
<tr>
<td>20.4.2 Group structure (pivotal cubics)</td>
<td>302</td>
</tr>
<tr>
<td>20.4.3 ABCIJKL cubics: the Lubin(2) point of view</td>
<td>303</td>
</tr>
<tr>
<td>20.4.4 Using a more handy basis</td>
<td>304</td>
</tr>
<tr>
<td>20.4.5 Darboux and Lucas cubics</td>
<td>306</td>
</tr>
<tr>
<td>20.4.5.1 Presentation of K004 and K007</td>
<td>306</td>
</tr>
<tr>
<td>20.4.5.2 The Orion bundle</td>
<td>308</td>
</tr>
<tr>
<td>20.4.6 Equal areas (second) cevian cubic aka K155</td>
<td>310</td>
</tr>
<tr>
<td>20.4.7 The cubic K060</td>
<td>311</td>
</tr>
<tr>
<td>20.4.8 Eigentransform</td>
<td>312</td>
</tr>
<tr>
<td>20.5 Non pivotal isocubics nK(P,U,k) and nK0(P,U)</td>
<td>313</td>
</tr>
<tr>
<td>20.5.1 Conicopivotal isocubics cK(#F,U)</td>
<td>314</td>
</tr>
<tr>
<td>20.5.2 Simpson cubic, aka K010</td>
<td>315</td>
</tr>
<tr>
<td>20.5.3 Brocard second cubic aka K018</td>
<td>317</td>
</tr>
<tr>
<td>21 Tripolar curves</td>
<td>319</td>
</tr>
<tr>
<td>21.1 The bicircular space</td>
<td>319</td>
</tr>
<tr>
<td>21.2 Define and draw</td>
<td>322</td>
</tr>
<tr>
<td>21.3 The generic case</td>
<td>322</td>
</tr>
<tr>
<td>21.4 Cross-ratios</td>
<td>326</td>
</tr>
<tr>
<td>21.5 Introducing the cut parameter</td>
<td>326</td>
</tr>
<tr>
<td>21.6 Barycentrics wrt the diagonal triangle</td>
<td>328</td>
</tr>
<tr>
<td>21.7 When $E$ is on the curve</td>
<td>329</td>
</tr>
<tr>
<td>21.8 The tripolar circular cubics</td>
<td>329</td>
</tr>
<tr>
<td>21.8.1 When the fourth focus is moving</td>
<td>331</td>
</tr>
<tr>
<td>22 Special Triangles</td>
<td>335</td>
</tr>
<tr>
<td>22.1 Changing coordinates, functions and equations</td>
<td>335</td>
</tr>
<tr>
<td>22.2 Residual triangles</td>
<td>336</td>
</tr>
<tr>
<td>22.3 Incentral triangle</td>
<td>336</td>
</tr>
<tr>
<td>22.4 Excentral triangle</td>
<td>336</td>
</tr>
<tr>
<td>22.5 Medial triangle</td>
<td>337</td>
</tr>
<tr>
<td>22.6 Antimedial triangle</td>
<td>337</td>
</tr>
<tr>
<td>22.7 Orthic triangle</td>
<td>337</td>
</tr>
<tr>
<td>22.8 Tangential triangle</td>
<td>338</td>
</tr>
<tr>
<td>22.9 Brocard triangle (first)</td>
<td>338</td>
</tr>
<tr>
<td>22.10 Brocard triangle (second)</td>
<td>338</td>
</tr>
<tr>
<td>22.11 Brocard triangle (third)</td>
<td>338</td>
</tr>
<tr>
<td>22.12 Intouch triangle (contact triangle)</td>
<td>339</td>
</tr>
</tbody>
</table>
25.12.4 Cevian triangles in a LFIT ................................................. 385
25.13 Families with constant area ................................................. 386
  25.13.1 Three non concurrent lines (rewritten) ................................. 386
  25.13.2 Three concurrent lines ................................................. 387
25.14 Concurrent hexagonal graphs ............................................... 388
  25.14.1 Assuming that $S$ is known ............................................. 388
  25.14.2 Assuming that $K$ is known ............................................. 388
  25.14.3 Assuming that $K$ is the center of gravity ............................. 388
25.15 When the graphs are given ................................................. 389
  25.15.1 Catalan graphs .......................................................... 389
  25.15.2 Hexagonal graphs ...................................................... 389
  25.15.3 The marvelous formula ................................................. 390
25.16 Observers (about perspectivities) ....................................... 391
  25.16.1 Metric observer ........................................................ 391
    25.16.1.1 Forcing the orthocenter ......................................... 392
    25.16.1.2 Forcing the isogonal center ..................................... 392
  25.16.2 Cevenol graphs ...................................................... 393
  25.16.3 Poulbot observers .................................................... 394
  25.16.4 Singular observers ................................................... 394
  25.16.5 Proof of the theorem ................................................. 394
  25.16.6 Reciprocal ............................................................ 396
25.17 Orthojoin ................................................................. 396
26 Quadrilaterals ........................................................................ 397
  26.1 Immortal glory of our ancestors ........................................... 397
  26.2 Lines only ........................................................................ 398
  26.3 Newton stuff ..................................................................... 399
  26.4 Steiner stuff ..................................................................... 400
  26.5 The Lubin cookbook for quadrilaterals .................................. 402
  26.6 Van Rees cubic ................................................................ 404
    26.6.1 Morley point of view ..................................................... 404
    26.6.2 Cartesian version ........................................................ 405
    26.6.3 Barycentric version ...................................................... 406
  26.7 Another cookbook ............................................................ 407
  26.8 Simson lines (using barycentrics) ......................................... 408
  26.9 The Steiner deltoid (using Lubin-1) ....................................... 408
  26.10 Pedal LFIT and orthopole .................................................. 410
    26.10.1 Starting by some computations ..................................... 410
    26.10.2 Using Simson lines ...................................................... 411
  26.11 Sister Marie Cordia Karl .................................................... 412
  26.12 The four pedal LFIT of a quadrilateral ................................ 414
  26.13 Exercises ....................................................................... 415
  26.14 Diagonal triangle ............................................................. 416
  26.15 Rigby points .................................................................... 416
27 Combs .................................................................................. 419
  27.0.1 More-more combos .......................................................... 419
  27.0.2 Euler triangles ............................................................... 419
28 Using Kimberling’s database into Maple ................................... 421
  28.1 Sparse version (2019 and after) ............................................ 421
    28.1.1 How-to update ............................................................. 421
    28.1.2 Duplicates .................................................................. 421
    28.1.3 la table barita ............................................................. 422
    28.1.4 Description ............................................................... 422
    28.1.5 Rationales ................................................................. 422
    28.1.6 Usage: the new ency procedure ................................... 423
  28.2 Older versions (2017 and before) ........................................ 423
  28.3 Obtaining formal barycentrics compliant with the search keys .... 424
Chapter 1

Introduction

Many changes have occurred in the way we are doing geometry, from the old ancient times of Euclid and Apollonius. Most of them are related to yet another way of performing "automated" computing of properties, rather than relying on intuition to find "beautiful geometric proofs". Many individuals have contributed to this long process, and attributing a given discovery to a given individual is not an easy task (Coolidge, 1940).

The most eminent milestones along this long road are the individuals that have summarized the discoveries of their time into an efficient way of writing down the questions to solve. Each time, the new way of writing was appearing as doing the job by itself and providing the required answers through something like a least action trajectory. Heroes are no more required, replaced by computing power.

Writing numbers and calculations in a tractable manner is associated with Al-Khwarizmi and his \textit{Algebra} (825). Using coordinates \((x, y)\) to describe points and compute their geometrical properties (as well as the exponent notation for polynomials) is associated with Descartes and his \textit{Géometrie} (1637). Using homogeneous coordinates \(x : y : z\) to implement the principle of continuity when dealing with objects that escape to infinity is associated with Moebius and his \textit{barizentrische Calcul} (1827).

More recently, the very idea of stamping an hash-code on each noteworthy point involved in Triangle Geometry and then practice some kind of computer aided inventory management (Kimberling, 1998-2021) has changed the practice of geometers. This idea has emerged from a more general trend, where barycentrics are understood to \textbf{define} points, lines, circles, triangle centers, etc., and zero determinants are understood to \textbf{define} collinearity and concurrence. Doing that way, triangle geometry, formally speaking, is much more general than the study of a single Euclidean triangle. In the formal treatment, sometimes called \textit{transfigured triangle geometry}, the symbols \(a, b, c\) are regarded as algebraic unknowns, so that points, defined as functions of \(a, b, c\), are not the usual points of a two-dimensional plane. When \((a, b, c)\) are real numbers restricted by the "triangle inequalities" for sidelengths, the resulting geometry is traditional triangle geometry (Kimberling, 1998).

When possible, computed proofs are given that use formal computing tools. This kind of proof is deprecated by several authors. Nevertheless, these proofs are the easiest since all the messy job is done by a computer and are also the safest. A construction that sounds like a "beautiful geometrical proof" is too often invalid due to some hidden exception. During a computerized proof, exceptions are appearing as multiplicative factors, according to the polynomial model:

\[
\text{conclusion} \times \text{exceptions} = \text{hypothesis}
\]

To quote the Knuth’s foreword to Petkovsek et al. (1996):

Science is what we understand well enough to explain to a computer. Art is everything else we do. During the past several years important parts of mathematics has been transformed from an Art to a Science.
1.1 Special remark for French natives

Lorsque vous écrivez pour un public américain, mieux vaut commencer par montrer que le sujet est suffisamment intéressant pour mériter du temps et de la peine. Lorsque vous écrivez pour un public français, mieux vaut commencer par montrer que l’auteur dispose d’une hauteur de vue suffisante. Si vous percevez les choses de cette façon, le Chapitre 4 est le bon endroit par où commencer.

1.2 Basic objects: points and lines

The abbreviation "etc." will stand for "et cyclically". When an "A-object" has been defined as \( F(A, B, C) \), then the B-object is \( F(B, C, A) \), obtained by the cyclic permutation \( ABC \mapsto BCA \). Example: If \( A' \) is the point where lines \( AP \) and \( BC \) meet, and \( B' \) and \( C' \) are defined cyclically, then \( B' \) is where lines \( BP \) and \( CA \) meet, and \( C' \) is where lines \( CP \) and \( AB \) meet.

Definition 1.2.1. Equality. We will use the \textit{simeq} sign (i.e. \( \simeq \)) to denote an equality up to a non vanishing multiplier, and restrict the use of the \textit{equal} sign (i.e. \( = \)) to a strong equality. In other words

\[
x = y \text{ means } x - y = 0 \text{ while } x \simeq y \text{ means } (\exists k \neq 0) (k x - y = 0)
\]

Moreover we will use the special signs \textit{doteq} (i.e. \( \dot{=} \)) and \textit{simdoteq} (i.e. \( \simdot{=} \)) to denote "definitional equalities", in order to emphasize the fact that these equalities aren’t equations: they are introducing new objects.

Definition 1.2.2. A point is an element of \( \mathbb{P}_2(\mathbb{R}^3) \). To tell the same thing more simply, a point is represented by a column of three numbers (the barycentrics of the point), not all of them being zero, and such a column is dealt "in a projective manner", i.e. up to a proportionality factor. An efficient way to write such a projective column is the colon notation:

\[
P \simeq p : q : r \quad \text{meaning that} \quad P \simeq \begin{pmatrix} p \\ q \\ r \end{pmatrix} \simeq \begin{pmatrix} kp \\ kq \\ kr \end{pmatrix} \forall k \in \mathbb{R} \setminus \{0\}
\]

Definition 1.2.3. A line is an element of the dual of the point space. To tell the same thing more simply, a line is represented by a row of three numbers (the barycentrics of the line), not all of them being zero, and such a row is dealt "in a projective manner", i.e. up to a proportionality factor. A line will be described as:

\[
\Delta \simeq \left( \begin{array}{ccc} \rho & \sigma & \tau \end{array} \right) \simeq \left( \begin{array}{ccc} \lambda \rho & \lambda \sigma & \lambda \tau \end{array} \right) \forall \lambda \in \mathbb{R} \setminus \{0\}
\]

Notation 1.2.4. In all these definitions, property \( p^2 + q^2 + r^2 \neq 0 \) and \( \rho^2 + \sigma^2 + \tau^2 \neq 0 \) are ever intended. Colon notation will *ever* be restricted to inline equations describing columns, and *neve*r be used for rows. Anyway, such a notation would be hopeless when dealing with matrices.

Definition 1.2.5. Incidence relations. We will say that a line \( \Delta \simeq (p, q, r) \) contains a point \( U \simeq u : v : w \), or that \( \Delta \) goes through \( U \), or that \( U \) belongs to \( \Delta \) when their dot product vanishes, i.e.:

\[
U \in \Delta \iff pu + qv + rw = 0
\]

Remark 1.2.6. It is clear that the incidence relation is projective, i.e. holds for any choice of the proportionality factors.

Definition 1.2.7. The collinearity of three points \( P \simeq p : q : r \), \( U \simeq u : v : w \), \( X \simeq x : y : z \) is defined by the following determinant equation:

\[
\begin{vmatrix}
x & p & u \\
y & q & v \\
z & r & w \\
\end{vmatrix} = 0 \quad (1.1)
\]

When \( P \neq U \), the set of all the \( X \) that satisfies (1.1) is what is usually called the line \( PU \).
Definition 1.2.8. A triangle is an ordered set of three non collinear points. Its natural representation is an invertible square "matrix of columns", where each column is defined up to a proportionality factor (right action of a diagonal matrix). On the contrary, a "may be degenerate triangle" is a matrix such that (i) columns are not proportional to each other and (ii) rank is at least two. When at least two points are equal, the triangle is "totally degenerate".

Remark 1.2.9. Without explicit permission, a triangle is not allowed to be degenerate, while totally degenerate triangles are (quite ever) to be avoided.

Definition 1.2.10. The concurrence of three lines $\Delta_1 \simeq (d, e, f)$, $\Delta_2 \simeq (p, q, r)$ and $\Delta_3 \simeq (u, v, w)$ is defined by the following determinant equation:

$$\begin{vmatrix} d & e & f \\ r & s & t \\ u & v & w \end{vmatrix} = 0 \quad (1.2)$$

When $\Delta_1 \neq \Delta_2$, the set of all the $\Delta_3$ that satisfies (1.2) is usually called the pencil generated by the two lines.

Definition 1.2.11. A trigone is a set of ordered three non concurrent lines. Its natural representation is an invertible square "matrix of rows", where each row is defined up to a proportionality factor (left action of a diagonal matrix).

Proposition 1.2.12. The reciprocal matrix of a triangle is a trigone and conversely. Adjoint matrices can be used instead of inverses due to proportionality. Relation $T^{-1} \cdot T = Id$ is nothing but the incidence relations : $A' \notin B'C'$, $A' \in A'B'$, $A' \in A'C'$ and cyclically.

Remark 1.2.13. It should be noticed that a tetra-angle defines an hexa-gone, while an tetra-gone (quadrilateral) defines an hexa-angle : $n = n (n-1)/2$ holds only when $n = 3$.

Definition 1.2.14. The line at infinity $L_\infty$ is the locus of points $x : y : z$ such that $x + y + z = 0$, so that any point out of $L_\infty$ can be described by a triple such that $x + y + z = 1$. Using only this representation would discard $L_\infty$ and is nothing but the usual affine geometry.

Definition 1.2.15. Barycentric basis. Special points $A = 1 : 0 : 0$, $B = 0 : 1 : 0$, $C = 0 : 0 : 1$ are usually identified with the vertices of a triangle in the euclidian plane, so that all (barycentric) points can be mapped onto the euclidian plane (completed with the appropriate line at infinity). The side lengths of this triangle are denoted $BC = a$, $CA = b$ and $AB = c$. Since a triangle is not a two-angle, none of the $a$, $b$, $c$ are allowed to vanish.

1.3 Figures using geogebra

1.3.1 Alternatives

The very idea to pay something for using Pythagoras theorem seems terrific, and therefore using a free software like geogebra appears as a requirement. The following remarks may nevertheless have some historical value.

ps, pstricks PostScript was a proprietary language. But it becomes a de facto standard for talking to printers of any brand, and was finally released to the open source status.

maple proprietary. Moreover not sufficiently versatile.

kseg (KSEG, 1999-2006) was our initial best choice, providing *.ps images. But the maintenance stopped near 2006. Figures were fine in the Linux version, but macro-constructions were badly saved, requiring the use of the win$ version (through wine) for writing the macros.

This kind of plane is to be named as euclidian rather than as Euclidean, since the very idea to compute figures instead of drawing computations is as far as possible from the thoughts and practices of the historical figure who wrote the celebrated Elements.
1.3.2 Various versions of geogebra


2. /opt/geogebra-old/geogebra/geogebra
   --v GeoGebra 5.0.309.0 20 December 2016 Java 1.8.0_121-64bit

3. /usr/share/geogebra-classic/GeoGebra
   --v error!!!
   Version: 6.0.666.0-offline (21 September 2021)

4. /usr/share/geogebra/geogebra
   --v GeoGebra 5.0.755.0 17 January 2023 Java 1.8.0_121-64bit
   213686 Dec 2016 geogebra_cas.jar
   054109 Jan 2023 geogebra_cas.jar si court !!! surprise !!!

1.3.3 Using Geogebra

From 2014, more and more figures of the present Glossary were drawn using Geogebra.

1. Nowadays, exporting a graphical output is easy. In the old ancient times, the best output was obtained as a *.pdf file, resulting into a page, that required a manual `pdfcrop` to its bounding box.

2. Obtaining the second intersection of two objects requires the following incantation:
   ```
   Element[Remove[{Intersect[BBB,CCC]}, {A}], 1]
   ```

3. A command like
   ```
   X186 = TriangleCenter[A, B, C, 186]
   ```
   can be used when \( n < 3053 \) (Geogebra 5.0.309.0-3D, tested 2016-12-12). Moreover
   ```
   indexof(P, sequence(Trianglecenter(A,B,C,j),j,1,10))
   ```
   allows to identify \( P \) among the points \( X(j) \) known to geogebra (the other points generate (?,?) and everything stands at the right place).

1.3.3.1 Construction protocol

Export the construction protocol in *.html, with nocolor, nopicture, only three columns (name&type,definition,value). And then macro `extract.sh` will display something to insert into a LyX equation.

1.3.3.2 3D geometry

<table>
<thead>
<tr>
<th>Triangle</th>
<th>ABC</th>
<th>Polygon(A, B, C)</th>
</tr>
</thead>
<tbody>
<tr>
<td>Segment</td>
<td>AB</td>
<td>Segment[A, B, ABC]</td>
</tr>
<tr>
<td>Circle</td>
<td>f</td>
<td>Circle(A, 4.5, ABC)</td>
</tr>
<tr>
<td>Point</td>
<td>D</td>
<td>((x(H), y(H), 5))</td>
</tr>
<tr>
<td>Pyramid</td>
<td>ABCD</td>
<td>Pyramid(ABC, D)</td>
</tr>
<tr>
<td>Segment</td>
<td>BD</td>
<td>Segment[B, D, ABCD]</td>
</tr>
<tr>
<td>Triangle</td>
<td>ABD</td>
<td>Polygon(A, B, D, ABCD)</td>
</tr>
<tr>
<td>Line</td>
<td>j</td>
<td>PerpendicularLine[P, AB, ABC]</td>
</tr>
<tr>
<td>Line</td>
<td>k</td>
<td>PerpendicularLine[P, ABC]</td>
</tr>
<tr>
<td>Plane</td>
<td>o</td>
<td>PerpendicularPlane[P, BD]</td>
</tr>
</tbody>
</table>

- circle \( f \) is drawn in the \( ABC \) plane
- line \( j \) is parallel to plane \( ABC \), through point \( P \) and perpendicular to line \( AB \)
- line \( k \) is perpendicular to plane \( ABC \)
- segment \( AB \) is an element of \( ABC \), not an "independent" object.
1.3.3.3 Macros

1. Dealing with macros is not so easy. As a rule: ever suppress any non-required sub-macro when archiving a macro from its source file. Sometimes, you have to unzip the *.ggb file and modify it with a text editor.

2. When transmitting a complex point $P$ to a macro, a safe method is to create, inside the macro, a copy of this point using Tocomplex.

1.4 Type-keeping and type-crossing functions

Remark 1.4.1. The type-keeping/type-crossing properties are better understood when they are described in terms of collineations. Chapter 16 will be devoted to this topic. The aim of the current Section is only to provide some useful tools as soon as possible.

Definition 1.4.2. Trilinears and barycentrics. Triangle people splits into a barycentric tribe and a trilinear tribe. The trilinear tribe thinks that trilinears, i.e. $p : q : r^2$ are better looking than barycentrics and redefine everything according to their preferences. The barycentric tribe thinks that barycentrics, i.e. $p : q : r^3$ are better looking than trilinears and redefine everything according to their preferences.

Remark 1.4.3. Trilinears can be measured directly on the figure, since they are the directed distances to the sidelines. When compasses were actual compasses and not a button to click over, using trilinears was a must. Nowadays, the existence –and the persistence– of both systems can be used for an interesting renewal of the Capulet against Montague story, as in http://mathforum.org/kb/message.jspa?messageID=1091956. But this could also be used to gain a better insight over many point-transforms used in the Triangle Geometry.

Definition 1.4.4. Vectors are covariant, while forms are contravariant. Therefore, coordinates that measure a vector are forms and are contravariant. At the same time, coordinates that measure a form are covariant. In other words, $p : q : r$ is contravariant, while $[p, q, r]$ is covariant.

Definition 1.4.5. We will say that a function $P \mapsto f(P) : p : q : r \mapsto u : v : w$ is type-keeping or type-crossing or type scrambling according to:

- **type keeping** when $f(\alpha p : \beta q : \gamma r) = \alpha u : \beta v : \gamma w$
- **type crossing** when $f(\alpha p : \beta q : \gamma r) = u : v : w$
- **type scrambling** otherwise

For a function of several variables, global type-keeping means:

$$f(\alpha p : \beta q : \gamma r, \alpha u : \beta v : \gamma w) = \alpha x : \beta y : \gamma z \quad \text{when} \quad f(p : q : r, u : v : w) = x : y : z$$

Remark 1.4.6. An object that is intended to describe a point has to be contravariant. An object that is intended to describe a line has to be covariant, while relationships like collinearity (1.1) and concurrence (1.2) have to be invariant. Therefore a function whose input and output are points has to be type-keeping. In the same way, a function whose input and output are lines has to be type-keeping. On the contrary, a function whose entries are points and output are lines has to be type-crossing. In the same way, a function whose entries are lines and output are points has to be type-crossing. These facts are the reasons why both tribes, using barycentrics or trilinears, are proceeding to the same geometry.

Definition 1.4.7. Barycentric multiplication is the multiplication component by component of the barycentrics of two points. This operation is denoted:

$$P \ast_b X \simeq p x : q y : r z$$

Component-wise multiplication of trilinears would be another possibility. This is the way of doing of the trilinear tribe.
1.5 Duality between point and lines

**Definition 1.4.8. Barycentric division** is the division component by component of the barycentrics of two points. This operation is denoted:

\[ P \div_b X \simeq \frac{p}{x} : \frac{q}{y} : \frac{r}{z} \]  

(1.4)

Component-wise division of trilinears would be another possibility. This is the way of doing of the trilinear tribe.

**Remark 1.4.9.** These transforms are introduced here to provide an easy description of some other transforms. The study of their geometrical meaning is postponed to Chapter 17. For our present needs, we only need to remark that both:

\[ X \mapsto X \ast P \div_b U \quad \text{and} \quad X \mapsto P \ast U \div_b X \]

are globally type-keeping transforms, that can be used to obtain points from points, or lines from lines.

**Definition 1.4.10. Sqrtdiv.** Let \( F \simeq f : g : h \) be a fixed point, and \( U \) a moving point, restricted to avoid the sidelines of \( ABC \). The mapping defined by:

\[ \sqrt{\text{div}}_F (U) \triangleq U \ast F \simeq \frac{f^2}{u} : \frac{g^2}{v} : \frac{h^2}{w} \]  

(1.5)

is globally type-keeping and describes a pointwise transform, whose fixed points are the four \( \pm f : \pm g : \pm h \). This map \( U \mapsto U \ast F \) is exactly the same as \( U \mapsto P \ast U \div_b F \).

The second form is often used, introducing a fictitious point \( P \simeq F \ast B \). This will be studied at length at Section 17.4.

**Remark 1.4.11.** Using \( \# \) instead of \( \ast \) in this context is already the way of doing of the cubics’ people (Ehrmann and Gibert, 2005): fixed points of the transform (\( F \) and its relatives) have a clearer geometrical meaning than \( P \). On the other hand, when \( P \) crosses the borders of \( ABC \), the coordinates of point \( F \) become imaginary, and the configuration is less visual.

**Remark 1.4.12.** The converse operation of \( \sqrt{\text{div}} \) would be \( \sqrt{\text{mul}} \) defined as \( (U, X) \mapsto \sqrt{uv} : \sqrt{wz} \) but this map is multivalued. When \( U, X \) are triangle centers and \( uv, vz \) are perfect squares, it makes sense to fix signs so that \( F \) is also a triangle center.

## 1.5 Duality between point and lines

Does equation \( pu + qv + rw = 0 \) means \( P \in \Delta_U \) or \( U \in \Delta_P \)? Without any further indication, one cannot decide which is the point and which is the line. This is called duality. If you want to be specific, you have to say:

\[
\begin{bmatrix}
u & v & w \\
p & q & r
\end{bmatrix} \cdot 
\begin{bmatrix}
u \\
p \\\nq \\\nr
\end{bmatrix} = 0 \quad \text{or} \quad 
\begin{bmatrix}
u & v & w \\
p & q & r
\end{bmatrix} \cdot 
\begin{bmatrix}
u \\
p \\\nq \\\nr
\end{bmatrix} = 0
\]

and remember how points/lines are mapped into columns/rows. In any case, points aren’t lines and columns aren’t rows. An efficient formulation of incidence axioms must recognize this elementary fact.

**Definition 1.5.1.** The wedge operator is the universal factorization of the determinant. This means that wedge of two columns is a row, while wedge of two rows is a column. One has:

\[
\begin{bmatrix}
p \\
q \\
r
\end{bmatrix} \wedge 
\begin{bmatrix}
u \\
p \\\nq \\\nr
\end{bmatrix} \simeq (qw - rv, ru - pw, pv - qu)
\]

\[
(p, q, r) \wedge (u, v, w) \simeq 
\begin{bmatrix}
wv - rv \\
ru - pw \\
pv - qu
\end{bmatrix}
\]
Proposition 1.5.2. When \( P \neq U \), the barycentrics of the line \( PU \) are provided by operation \( P \wedge U \). As it should be, this operation is commutative and is type-crossing.

Proof. The wedge of two points cannot be \( 0 : 0 : 0 \) when the points are different, therefore \( \Delta = P \wedge U \) defines a line. By definition, we have:

\[
\begin{pmatrix} p \\ q \\ r \\ \alpha \\ \beta \\ \gamma \\ 1 \end{pmatrix} \wedge \begin{pmatrix} u \\ v \\ w \\ \alpha \\ \beta \\ \gamma \\ 1 \end{pmatrix} = \begin{pmatrix} x \\ y \\ z \\ \alpha \\ \beta \\ \gamma \\ 1 \end{pmatrix}
\]

and the conclusion comes from the fact that inclusion of a line into another implies equality. Type-crossing is obvious from stratospheric reasons... but can also be checked on the components (up to a global \( \alpha \beta \gamma \) factor). When dealing with lines, the same argument shows that "line wedge line" is a point. \( \square \)

Proposition 1.5.3. When line \( \Delta_{12} \) is given by points \( P_1 \) and \( P_2 \) (with \( P_1 \neq P_2 \)) and line \( \Delta_3 \) is given by its barycentrics then either both lines are equal or their intersection \( M \) is given by:

\[
\Delta_{12} \cap \Delta_3 \simeq (\Delta_3 \cdot P_1) \ P_2 - (\Delta_3 \cdot P_2) \ P_1
\]

Proof. Call \( M \) this object. It is clear that \( M \in P_1 P_2 \). And we can check that \( \Delta_3 \cdot M = 0 \). Another proof is that this \( \simeq \) is in fact a component-wise identity. \( \square \)

Proposition 1.5.4. Suppose that lines \( \Delta_{12} \) and \( \Delta_{34} \) are respectively defined by points \( P_1, P_2 \) and points \( P_3, P_4 \). Then either both lines are equal or their intersection \( M \) is given by:

\[
M \doteq (P_1 \wedge P_2) \wedge (P_3 \wedge P_4) \simeq P_2 \det [P_1 P_3 P_4] - P_1 \det [P_2 P_3 P_4]
\]

Proof. Obvious from the previous proposition and the definition \( \det [P_1 P_3 P_4] = (P_3 \wedge P_4) \cdot P_1 \). Another proof is that this \( \simeq \) is in fact a component-wise identity. \( \square \)

Definition 1.5.5. The wedge point \( X_\Delta \) of a line is what is obtained by a simple transposition of the barycentrics. This way of doing is based on a misperception of the wedge operation since \( PU \doteq (P \wedge U) \) is a line (row) and not a point (column). When written in trilinears, this object don't look good. Not without reason.

Definition 1.5.6. The Weisstein point \( X_W \) of a line is what is obtained when applying the same misperception to trilinears. Applied to line \( PU \), this leads to the crossdifference of \( P \) and \( U \). When written in barycentrics:

\[
X_W \doteq \text{crossdiff} (P, U) \simeq a^2 (qw - rv) : b^2 (ru - pw) : c^2 (pv - qu)
\]

this object don't look good. Not without reason.

A better founded concept must lead to a type-crossing transform.

Definition 1.5.7. The tripole of a line and the tripolar of a point is what is obtained by "transpose and reciprocate". Clearly, the one-to-one correspondence between pole and polar is lost when a coordinate vanishes (line through a vertex, or point on a sideline).

Remark 1.5.8. A less stratospheric definition of the tripolar is given in Definition 3.4.3.

Remark 1.5.9. When applying "transpose and reciprocate", both tribes are thinking they are acting "their way", and are talking about "trilinear pole" and "barycentric pole". But the result is the same since reciprocation of barycentrics (aka isotomic conjugacy, Section 3.4) acts over \( X_\Lambda \) while reciprocation of trilinears (aka isogonal conjugacy, Section 9.2) acts over \( X_W \). To summarize (using later introduced concepts):

\[
X_\Lambda = \dagger \Delta ; \ X_W = \dagger \Delta \ast X_6 ; \ \text{tripole} = \text{isotom} (X_\Lambda) = \text{isogon} (X_K)
\]

Remark 1.5.10. When tripole is at infinity, the line is tangent to the Steiner in-ellipse (cf Example 12.11.1).
Example 1.5.11. Table 1.1 describes some well-known lines. For example, the Euler line goes through \( X_2 \) (centroid) and \( X_3 \) (circumcenter). Its equation is \( \sum x (b^2 - c^2)(b^2 + c^2 - a^2) = 0 \). Formally, center \( (b^2 - c^2)(b^2 + c^2 - a^2) \) is \( X_{647} = t \Delta \), while center \( a^2 (b^2 - c^2)(b^2 + c^2 - a^2) \) is \( X_{519} = L_{649} \). This center has been used sometimes to describe lines, leading to \( \text{Euler} = L_{647} \). The next column (\( \infty \)) gives the infinity point while the remaining two columns give the later defined orthopoint (\( \infty \perp \)) and orthopole (\( \perp \)).

Proposition 1.5.12. Tripole and tripolar, being correctly typed, are constructible (Figure 1.1). Start from \( P \). Draw \( AP \) and obtain \( A' = AP \cap BC \). Construct \( A'' \in BC \) so that division \( BCA'A'' \) is harmonic (Section 3.2). Act cyclically and obtain \( B'' \) and \( C'' \). Then \( A'B'C'' \) are collinear, and the line they define is nothing but the tripolar of \( P \). (and are named tripo in Table 1.1).

![Figure 1.1: Point P and line A"B"C" are the tripolars of each other.](image)
Remark 1.5.13. One of the most important consequence of all these duality formulas is the rock-solid equality giving the intersection of two lines each of them defined by two points:

\[(PQ \cap RS) = (P \wedge Q) \wedge (R \wedge S)\]

**Proposition 1.5.14.** A symmetric parametrization in \(\rho, \sigma, \tau\) of the points \(U \simeq u : v : w\) that lie on line \(\Delta \simeq [p, q, r]\) is:

\[U \simeq q\tau - r\sigma : r\rho - p\tau : p\sigma - q\rho\]  \hspace{1cm} (1.7)

**Proof.** The first formula is \(U \simeq \Delta \wedge [\rho, \sigma, \tau]\), defining a point on a line as the (projective) intersection of this line with another one. The second one is obvious. \(\Box\)

**Proposition 1.5.15.** (Spoiler) Line tripolar (\(P\)) is the locus of the tripoles \(U\) of the tangents to the inconic \(IC(P)\). Moreover, the contact point of tripolar (\(U\)) is \(Q \simeq U \ast U \ast b\).

**Proof.** This is the right place for the assertion, but not for its proof... Postponed to Proposition 12.8.10. \(\Box\)

**Remark 1.5.16.** Caveat: A triangle is not a conic. When \(U\) lies on the polar of \(P\) wrt a conic, then \(P\) lies on the polar of \(U\) wrt the same conic. When \(U\) lies on tripolar (\(P\)) (the polar of \(P\) wrt triangle \(ABC\)) we have:

\[\frac{u}{p} + \frac{v}{q} + \frac{w}{r} = 0\]

and this isn’t a commutative relation.

### 1.6 Isoconjugacy has moved

See Chapter 17, chiefly Section 17.4.
Chapter 2

Central objects

Centrality is a key notion. Emphasis on this concept was put by the founding paper of Kimberling (1998). The corresponding definitions have been tailored so that a central point is something like $I = X(1)$ or $G = X(2)$ or $O = X(3)$ etc., while a central triangle is something like $ABC$ itself or $JKL$ (the triangle of the excenters).

2.1 Triangle centers

When a concept emerges, various attempts are tested. Twenty years later, the most efficient ones take place in the "favored" list, while the others have to be deprecated. No criticism is implied here.

2.1.1 Favored concepts

Three points defines six triangles when taking the order into account. But there exists only one orthocenter. A central point is something that behaves like that. Moreover, geometrical theorems are not supposed to change when the king’s foot shortens (not to speak of what happens when the king itself is shortened): barycentric functions have to be homogeneous. This leads to the following definition.

Definition 2.1.1. A triangle center (or a central line) is a point (or a line) of the form

\[
\begin{pmatrix}
  f(a, b, c) \\
  f(b, c, a) \\
  f(c, a, b)
\end{pmatrix}
\]

where $f$ is a nonzero function satisfying two conditions:

1. $f$ is homogeneous in $a, b, c$; i.e., there is a real number $h$ such that $f(\lambda a, \lambda b, \lambda c) = \lambda^h f(a, b, c)$ for all $(a, b, c)$ in the domain of $f$;

2. $f$ is symmetric in $b$ and $c$; i.e., $f(a, c, b) = f(a, b, c)$.

Definition 2.1.2. Bicentric points. When condition (2) is relaxed, we have to consider the two different points

\[
\begin{pmatrix}
  f(a, b, c) \\
  f(b, c, a) \\
  f(c, a, b)
\end{pmatrix}, \quad
\begin{pmatrix}
  f(a, c, b) \\
  f(b, a, c) \\
  f(c, b, a)
\end{pmatrix}
\]

They are called bicentric, together comprising a bicentric pair. Example: the Brocard points, $\omega^+ = a^2 b^2 : b^2 c^2 : c^2 a^2$ and $\omega^- = c^2 a^2 : a^2 b^2 : b^2 c^2$ (cf Proposition 7.7.1).

Definition 2.1.3. A strong center is a triangle center whose defining function belongs to $C[a^2, b^2, c^2]$. Coordinates of such a point only depend on the coordinates of the vertices.
Definition 2.1.4. A rational center is a triangle center whose defining function belongs to \( \mathbb{C}[a^2, b^2, c^2, S] \), where \( S \) is the area of triangle \( ABC \). When the coordinates of the vertices are rational, the coordinates of the center are made of rational quantities and fixed numbers.

Remark 2.1.5. Rational points that are not of the strong kind are in fact occurring by pairs, depending on the sign chosen for \( S \) (i.e. depending from the orientation of the triangle \( \text{wrt the orientation of the plane} \)). Example:

\[
X(15) \approx a^2 \left( S_a + 2/3 S \sqrt{3} \right) : b^2 \left( S_b + 2/3 S \sqrt{3} \right) : c^2 \left( S_c + 2/3 S \sqrt{3} \right)
\]

\[
x(16) \approx a^2 \left( S_a - 2/3 S \sqrt{3} \right) : b^2 \left( S_b - 2/3 S \sqrt{3} \right) : c^2 \left( S_c - 2/3 S \sqrt{3} \right)
\]

Definition 2.1.6. A weak center is a triangle center whose defining function belongs to \( \mathbb{C}[a, b, c, S] \) without being a rational center.

Theorem 2.1.7. Lemoine transforms. When using barycentrics, the identity together with the three transforms \( a \mapsto -a \), \( b \mapsto -b \) and \( c \mapsto -c \) form a Klein group. Nowadays, they are called the Lemoine transforms, while Lemoine himself called them the "transformations continues" (see Lemoine, 1891). Spoiler: when using Labin-2, these transforms are obtained by \( \alpha \mapsto -\alpha \) or \( \beta \mapsto -\beta \) or \( \gamma \mapsto -\gamma \).

Proof. The fact that \( L_a \circ L_b = L_c \) comes from the homogeneity required for the formulas of interest. Remember: a theorem is a proposition with the biggest consequences, not necessarily something difficult to prove.

Remark 2.1.8. Rational points (strong or not) are invariant by the Lemoine transforms, while the weak centers are replicated, leading to what is called a set of four extraversions. Obviously, the incenter and its three excenters were the pattern used to shape this concept.

Theorem 2.1.9. Klein transforms. When using barycentrics \( x : y : z \), the identity together with the three transforms \( x \mapsto -x \), \( y \mapsto -y \) and \( z \mapsto -z \) form a Klein group. They are called the Klein transforms (see Lemoine, 1891) and a group of four points \( \pm f : \pm g : \pm h \) is called a Klein quadrangle. Spoiler. Any three points of such a set are the vertices of the anticevian triangle of the remaining one, while triangle \( ABC \) is the diagonal triangle of the quadrangle (see Proposition 3.4.15).

Proof. Here, the fact that \( L_a \circ L_b = L_c \) comes from the very definition of a projective space.

Remark 2.1.10. When function \( f(a, b, c) \) is a times an even polynomial, then both transforms give the same result. A list of such points is given at the introduction of points \( X(7001) \) - \( X(7373) \).

2.1.2 Deprecated concepts

Definition 2.1.11. A transcendental center is a triangle center \( X \) that cannot be defined as \( X = f(a, b, c) : f(b, c, a) : f(c, a, b) \) using an algebraic function \( f \). Examples: \( X(359) \) and \( X(360) \).

Definition 2.1.12. A major center is a triangle center \( X \) for which there exists a function \( f \) of the angles such that \( X = f(A) : f(B) : f(C) \). Examples: \( X(1) \), \( X(2) \), \( X(3) \), \( X(4) \), \( X(6) \). Major centers solve certain problems in functional equations (Kimberling, 1993, 1997).

Consider two examples, \( X(9) \) and \( X(37) \), of which first trilinears are \( b + c - a \) and \( b + c \), respectively. It is not clear from these trilinears that \( X(9) \) is a major center, whereas \( X(37) \) is not. Indeed, \( X(9) \) also has first trilinear \( \cot(A/2) \), so that \( X(9) \) is a major center, but there remains this problem: how to establish that \( X(37) \) and others are not major. In April, 2008, Manol Iliev found a criterion for a triangle center to be not a major center (**reference missing**). He applied his test to the first 3236 triangle centers in ETC and found that exactly 292 of them are major, as listed Table 2.1.

Definition 2.1.13. Angular Lemoine transform.

\[
A \mapsto -A \ ; \ B \mapsto \pi - A \ ; \ C \mapsto \pi - C \ ; \ S \mapsto -S \ ; \ R \mapsto R \ ; \ r_0 \leftrightarrow r_a
\]
Table 2.1: Major centers

2.2 Central triangle

Definitions of this Section are tailored so that triangle $ABC$ itself as well as later defined (Section 3.4) cevian and anticevian triangles are central objects. The corresponding matrices are, columnwise, as follows:

$$
C_P \simeq \begin{pmatrix} 0 & p & p \\ q & 0 & q \\ r & r & 0 \end{pmatrix}, \quad A_P \simeq \begin{pmatrix} -p & p & p \\ q & -q & q \\ r & r & -r \end{pmatrix}
$$

**Definition 2.2.1.** Suppose that $f, g$ are two homogeneous functions having the same degree of homogeneity. One of them (but not both) can be the zero function. Then the $(f,g)$-central triangle is defined as:

$$
[A', B', C'] \simeq \begin{pmatrix} f(a, b, c) & g(a, c, b) & g(a, b, c) \\ g(b, c, a) & f(b, c, a) & g(b, a, c) \\ g(c, b, a) & g(c, a, b) & f(c, a, b) \end{pmatrix}
$$

**Example 2.2.2.** Triangle $ABC$ is $(1,0)$ while $C_P$ is $(0,f)$ and $A_P$ is $(-f,f)$ when point $P$ is defined by center function $f$.

**Proof.** We have $P = f(a,b,c) : f(b,c,a) : f(c,a,b)$, together with $f(a,b,c) = f(a,b,c)$.
Example 2.2.3. As a summary, we have:

$$T(bc, c^2) = \begin{pmatrix}
bc \\
b^2 + bc + c^2 \\
b^2 + bc + c^2 \\
b^2 + bc + c^2 \\
bc \\
b^2 + bc + c^2 \\
b^2 + bc + c^2 \\
b^2 + bc + c^2 \\
\end{pmatrix}
\begin{pmatrix}
c^2 \\
ar^2 + ca + c^2 \\
ar^2 + ca + c^2 \\
ar^2 + ca + c^2 \\
\frac{ab}{a^2 + ab + b^2} \\
\frac{ab}{a^2 + ab + b^2} \\
\frac{ab}{a^2 + ab + b^2} \\
\frac{ab}{a^2 + ab + b^2} \\
\end{pmatrix}
$$

Notation 2.2.4. Among the functions often used, we have:

- $S_{area}(ABC)$, Not twice the area, as used in ETC.
- $S_a, S_b, S_c$ the Conway symbols. $S_a = (b^2 + c^2 - a^2) / 2$
- $\cot \omega$ where $\omega$ is the Brocard angle, $\cot \omega = (a^2 + b^2 + c^2) / 2S$
- $s = (a + b + c) / 2$, so that $2s$ is the perimeter.

Proposition 2.2.5. Symmetric (metric) functions. An $(a, b, c, S)$ expression that is symmetric in $(a, b, c)$ can be expressed as a function of $s, R, r_0$ where $s$ is the semiperimeter, while $R, r_0$ are the oriented radiiuses of the circum- and in-scripted circles (as given in $r_0 = S/s$; $R = abc/4S$, so that $Rr_0 > 0$).

Proof. Use the symmetric functions $S = r_0 s$; $a + b + c = 2s$; $ab + bc + ca = D$; $abc = P$ and then substitute

$$P = 4s r_0 R; \quad D = 4R r_0 + r_0^2 + s^2$$

Caveat: quantities $R, r_0$ are often perceived as positive but, here, they have to carry the orientation of the triangle.

2.3 Symbolic substitution

Definition 2.3.1. Symbolic substitution. Suppose $p(a, b, c), q(a, b, c), r(a, b, c)$ are functions of $a, b, c$, all of the same degree of homogeneity. As the transfigured plane consists of all functions of the form $X = x(a, b, c) : y(a, b, c) : z(a, b, c)$, the substitution indicated by

$$a \mapsto p(a, b, c), b \mapsto q(a, b, c), c \mapsto r(a, b, c)$$

maps the transfigured plane into itself.

Remark 2.3.2. Such a substitution may have no clear geometric meaning, as suggested by the name, symbolic substitution. On the other hand, symbolic substitutions are of geometric interest because they map lines to lines, conics to conics, cubics to cubics, and they preserve incidence.

Example 2.3.3. The symbolic substitution $(a, b, c) \mapsto (1/a, 1/b, 1/c)$ maps every triangle center to a triangle center, every pair of bicentric points to a pair of bicentric points, every circumconic to a circumconic, etc. However, when $(a, b, c) = (3, 4, 5)$, for example, then $a, b, c$ are sidelengths of an euclidian triangle, but $1/a, 1/b, 1/c$ are not.

Symbolic substitutions were introduced in Kimberling (2007).
Chapter 3

Cevian stuff

3.1 Centroid stuff

Definition 3.1.1. The reflection of point \( U = u : v : w \) in point \( P = p : q : r \) (not at infinity) is the point \( X \) such that:

\[
X \simeq \begin{pmatrix}
(p - q - r)u + 2p(v + w) \\
(q - p - r)v + 2q(u + w) \\
(r - p - q)w + 2r(u + v)
\end{pmatrix}
\]  

(3.1)

Proof. We want to obtain \( X = 2P - U \) when \( P, U, X \) are finite and in normalized form. When \( P \) is finite and \( U \in \mathcal{L}_\infty \) then \( X = U \) (OK). Taking \( P \in \mathcal{L}_\infty \) would result into \( X = P \) for any value of \( U \), not an acceptable result.

Stratospheric proof. \( X \) is obtained under the action described by:

\[
2 \begin{pmatrix}
p \\
q \\
r
\end{pmatrix} \cdot \mathcal{L}_\infty - \begin{pmatrix}
p \\
q \\
r
\end{pmatrix} \cdot \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}
\]

Definition 3.1.2. Complement and anticomplement are inverse transforms defined so that the complement of a vertex of triangle \( ABC \) is the middle of the opposite side. In other words (using barycentrics),

\[
\text{complement} (U) \doteq (3G - U)/2 \simeq q + r : p + r : p + q
\]

\[
\text{anticomplement} (Q) \doteq (3G - 2Q) \simeq -p + q + r : p - q + r : p + q - r
\]

According to Court, p. 297, the term \textit{complementary point} dates from 1885, and the term \textit{anti-complementary point} dates from 1886.

Definition 3.1.3. The \textbf{medial triangle} \( C_2 \) is the complement of triangle \( ABC \). The A-vertex of \( C_2 \) is the middle of segment \( BC \), and cyclically. In other words:

\[
C_2 \simeq \begin{pmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{pmatrix}
\]

Definition 3.1.4. The \textbf{antimedial triangle} \( A_2 \) is the anticomplement of triangle \( ABC \) (and is also called the anticomplementary triangle). Each sideline of \( A_2 \) contains a vertex of \( ABC \) and corresponding sidelines of both triangles are parallel. In other words:

\[
A_2 \simeq \begin{pmatrix} -1 & 1 & 1 \\ 1 & -1 & 1 \\ 1 & 1 & -1 \end{pmatrix}
\]

These triangles are the model used to define the next coming cevian and anticevian triangles (they are the triangles related to the centroid \( X_2 \)).
### Definition 3.2.1.
The cross-ratio of four (different) parameters \( z_j \in \mathbb{C} \) is defined as
\[
\text{cross}_\text{ratio}_C(z_1, z_2, z_3, z_4) = \frac{(z_4 - z_2)(z_3 - z_1)}{(z_4 - z_1)(z_3 - z_2)}.
\]

### Remark 3.2.2.
This same quantity is introduced as
\[
\frac{z_1 - z_3}{z_1 - z_4} \div \frac{z_2 - z_3}{z_2 - z_4},
\]
i.e. as \((z_4, z_3, z_2, z_1)\) by (Pedoe, 1970, p 212) and (Schwerdtfeger, 1962, p. 35), the intent being to emphasize the definition as a ratio of ratios. Our intent in using this equivalent product form is to emphasize the choice made among the six possibilities described just below.

### Definition 3.2.3.
The cross-ratio of four elements of the Riemann sphere \( \mathbb{Z} : T \) is defined as:
\[
\text{cross}_\text{ratio}_C(z_1 : t_1; z_2 : t_2; z_3 : t_3; z_4 : t_4) = \text{cross}_\text{ratio}_C(z_1; z_2; z_3; z_4).
\]
with the usual rules to resolve indeterminacies.

### Proposition 3.2.4.
Cross-ratio is a projective quantity on the Riemann sphere and is invariant under the homographies, i.e. the group \( \text{PGL}_\mathbb{C}(\mathbb{C}^2) \).

**Proof.** well-known result. The general case is an identity about polynomials and the special cases come from continuity.

### Proposition 3.2.5.
Group of the cross-ratio. The cross ratio remains unchanged under the action of the bi-transpositions like \([a, b, c, z] \mapsto [b, a, z, c]\). Therefore the action of \( S_4 \) generates only 6 values for the cross-ratio. We have:

\[
\begin{align*}
& a & b & c & \quad z & a & a \\
& b & c & a & \quad b & z & b \\
& c & a & b & \quad c & c & z \\
& z & z & & \quad a & b & c \\
& k & k-1 & 1 & 1-k & k & 1-k
\end{align*}
\]

Special cases are \( \{\exp(+i\pi/3), \exp(-i\pi/3)\} \) (equilateral triangle and one of its centers) \( \{1, 0, \infty\} \) (two points are equal) and \( \{-1, 2, 1/2\} \) (harmonicity).

**Proof.** Direct examination.

### Definition 3.2.6.
Consider the linear projective family defined from two fixed rows (or columns) \( X, Y \) (where \( X \not\sqsubseteq Y \) is assumed)
\[
\Delta = \{zX + tY | (z,t) \neq (0,0)\}
\]
The cross-ratio of four elements of \( \Delta \) is defined as the cross-ratio of their parameters, i.e.
\[
\text{cross}_\text{ratio}_C(z_jX + t_jY) = \text{cross}_\text{ratio}_C(z_j/t_j)
\]

### Theorem 3.2.7.
The cross-ratio of four elements of a linear projective family (assuming at least three different elements) is intrinsic, i.e. doesn’t depend on the columns (or rows) chosen to describe the family. When four different collinear objects \( P, Q, R, S \) are given, then non zero multipliers \( p, q, r, s \) and a constant \( \lambda \) can be found that satisfy the hard equalities:
\[
\begin{align*}
rR &= pP + qQ \\
sS &= \lambda pP + qQ
\end{align*}
\]
Moreover quantity \( \lambda \) depends only on the four objects and their order. This quantity is the former defined cross-ratio of the 4-uple.

**Proof.** To prove the existence, we write \( P = z_1X + t_1Y \), etc and solve the resulting set of 4 equations. The value obtained for \( \lambda \) is the former defined cross-ratio "using \( X, Y \)." The uniqueness of \( \lambda \) comes from the uniqueness, up to a proportionality factor, of \( r : p : q \) and \( s : \lambda p : q \) since \( P, Q \) is yet another generating family.
Corollary 3.2.8. When the family is parametrized as \( P_j = k_j X + (1 - k_j) Y \), then \( \lambda = \text{cross\_ratio}(k_j) \).

Proof. By definition, \( \lambda = \text{cross\_ratio}(k_j/(1 - k_j)) \). But \( \text{cross\_ratio}(\cdot) \) is invariant under \( z \mapsto z/(1 - z) \).

Corollary 3.2.9. Let the \( P_j \) be given by their barycentrics \( p_j : q_j : r_j \). Assuming that two of the \( p_j \) are not 0, then \( \lambda = \text{cross\_ratio}(p_j/(p_j + q_j + r_j)) \).

Proof. This holds only if the points are on the same line, but not the infinity line!

Definition 3.2.10. Four points \( A, B, J, K \) on an ordinary straight line form an harmonic division when:

\[
\frac{AJ}{BJ} \div \frac{AK}{BK} = -1
\]

Since cross-ratio is a projective invariant, this relationship is carried along collineations.

Proposition 3.2.11. Let be given three members \( P, Q, R \) of a linear projective family with, at least \( P \neq Q \). Then it exists exactly one member \( S \) of the family such that \( \text{cross\_ratio}(P, Q, R, S) = -1 \). This object is called the fourth harmonic of the first three.

Proof. Cross-ratio is an homographic function of parameter \( k_S \), and therefore bijective between \( \lambda \) and \( k_S \).

Construction 3.2.12. Let \( A, P, B \) be three aligned distinct points, and \( E \) be a point external to this green line (see Figure 3.1). Division \((A, B, P, Q)\) will be harmonic if the pencil \((EA, EB, EP, EQ)\) is harmonic. To this end, we draw the blue line \( \Delta \) i.e. the parallel to \( EA \) through \( B \). Then \( M \) is \( DP \cap \Delta \) and \( N \) the reflection of \( M \) into \( C \). The division \((\infty \Delta, B, M, N)\) is obviously harmonic. And we obtain \( Q \) as \( EN \cap AB \).

Remark 3.2.13. In the Cartesian plane, the fourth harmonic is also the reflection of the third point into the circle having the first two as diameter.

Remark 3.2.14. Conic cross-ratio is described in Section 12.10.

3.3 About combos

Definition 3.3.1. combos. When \( P \simeq p : q : r \) and \( U \simeq u : v : w \) are points at finite distance and \( f = f(a, b, c) \), \( g = g(a, b, c) \) are nonzero homogeneous functions having the same degree of homogeneity, then the \((f, g)\) combo of \( P \) and \( U \), denoted as \( f \times P + g \times U \), is:

\[
f \times P + g \times U \simeq \frac{P}{L_{\infty} \cdot P} f(a, b, c) + \frac{U}{L_{\infty} \cdot U} g(a, b, c)
\]
Remark 3.3.2. Written that way, one has not to discuss if barycentrics or trilinears are used.

**Proposition 3.3.3.** With the same hypotheses, points $P, U, fP + gU, fP + hU$ are collinear and the cross ratio $(P, U, fP + gU, fP + hU) = h/g$.

As a special case, $fP + gU$ and $hP$ are harmonic conjugate wrt $P, U$.

**Proposition 3.3.4.** When $f, g, h$ are homogeneous symmetric functions all of the same degree of homogeneity, and $X, X', X''$ are triangle centers, then $fX + gX' + hX''$ is a triangle center. Conversely, given three non collinear triangle centers, any fourth triangle center is a combo of the first three, using symmetric functions as coefficients.

**Proof.** This amounts to say that, in $\mathbb{R}^3$, any invertible matrix defines a basis of the space.

**Remark 3.3.5.** Part of the time, normalization is useless. Knowing that $X(482) \simeq s \times nX(1) + (r + 4R) \times nX(7)$ can be required, but using $X(482) \simeq vX(1) + 4S vX(7)$ (where $vX$ is what is given in the table) can be sufficient.

**Proposition 3.3.6.** The columns of a matrix $\begin{pmatrix} T \end{pmatrix}$ describing the vertices of a central triangle have to be normalized in such a way that $L_\infty \cdot \begin{pmatrix} T \end{pmatrix} \simeq L_\infty$. Such matrices form a group under multiplication. And then $\begin{pmatrix} T \end{pmatrix} \cdot P$ is (another) triangle center when $P$ is a triangle center.

### 3.4 Cevian, anticevian, cocevian triangles

**Definition 3.4.1.** Cevian triangle. Let $P$ be a point not on a sideline of $ABC$. The lines $AP, BP, CP$ are the *cevians* of $P$. Let $A_p = AP \cap BC$. Define $B_p$ and $C_p$ cyclically. Triangle $A_pB_pC_p$ is called the cevian triangle of triangle $ABC$.

$$\text{cevian} \begin{pmatrix} p \\ q \\ r \end{pmatrix} = \begin{pmatrix} 0 & p & p \\ q & 0 & q \\ r & r & 0 \end{pmatrix} \tag{3.2}$$

**Example 3.4.2.** Examples of cevian triangles are given in Table 3.1.

**Proposition 3.4.3.** Cocevian triangle. Let $P = p : q : r$ be a point not on a sideline of $ABC$, let $A_PB_PC_P$ be its cevian triangle and define $T_A$ as $BC \cap BPC_P$, etc. Then points $T_A, T_B, T_C$ are aligned on a line which is called the *tripolar line* of $P$, while the (degenerate) triangle $T_A, T_B, T_C$ itself is called the cocevian triangle of $P$. One has:

$$\text{cocevian} \begin{pmatrix} p \\ q \\ r \end{pmatrix} \simeq \begin{pmatrix} 0 & p & -p \\ -q & 0 & q \\ r & -r & 0 \end{pmatrix} \tag{3.3}$$

$$\text{tripolar} \begin{pmatrix} p \\ q \\ r \end{pmatrix} \simeq \begin{pmatrix} 1 & 1 & 1 \\ p & q & r \end{pmatrix} \simeq [qr, rp, pq] \tag{3.4}$$

**Proof.** Direct computation. □

**Remark 3.4.4.** (Spoiler) Point $P$ is the *perspector* of triangles $ABC$ and $A_PB_PC_P$, i.e. their (degenerate) vertex triangle (see Definition 3.8.1). While tripolar $(P)$ is the *perspectrix* of $ABC$ and $A_PB_PC_P$, i.e. their (degenerate) sideline trigone (see Definition 3.8.4).

**Definition 3.4.5.** Anticevian triangle. Let $P$ be a point not on a sideline of $ABC$. The anticevian of $P$ is the triangle $P_AP_BP_C$ such that $ABC$ is the cevian triangle of $P$ wrt $P_AP_BP_C$.

**Example 3.4.6.** Examples of anticevian triangles are given in Table 3.1.

**Construction 3.4.7.** Let $P$ be a point not on the sidelines of $ABC$. Draw $A_p = AP \cap BC$, etc (the cevian triangle of $P$) and then draw $T_A \equiv BC \cap BPC_P$, etc (the cocevian triangle of $P$). Then we have $P_A = BT_B \cap CT_C$, etc (the anticevian triangle of $P$)
3. Cevian stuff

Proof. Compute all these points and obtain

\[
\begin{pmatrix}
p \\
q \\
r
\end{pmatrix} \text{; cev} = \begin{pmatrix}
p & p & p \\
q & 0 & q \\
r & r & 0
\end{pmatrix}; \quad \text{cocev} \simeq \begin{pmatrix}
0 & p & \ -p \\
-q & 0 & q \\
r & -r & 0
\end{pmatrix}; \quad \text{anticev} \simeq \begin{pmatrix}
-p & p & p \\
q & -q & q \\
r & r & -r
\end{pmatrix}
\]

(3.5)

Then one can verify the relations \( A \in P_B P_C \), etc (\( ABC \) is inscribed in \( P_A P_B P_C \)), \( P \in AP_A \), etc (\( P \) is the perspector of \( ABC \) and \( P_A P_B P_C \)), \( T_A = BC \cap P_B P_C \), etc (\( T_A T_B T_C \) is the perspectrix of \( ABC \) and \( P_A P_B P_C \)). □

Remark 3.4.8. (Spoiler) Point \( P \) is the perspector of triangles \( ABC \) and \( P_A P_B P_C \), i.e. their (degenerate) vertex triangle (see Definition 3.8.1). While tripolar (\( P \)) is the perspectrix of \( ABC \) and \( P_A P_B P_C \), i.e. their (degenerate) sideline trigone (see Definition 3.8.4).

Proposition 3.4.9. We have three sets of aligned points whose divisions are harmonic, namely

\[
\begin{align*}
\text{cross}_\text{ratio} \ (A, A P, P, P A) &= -1 \\
\text{cross}_\text{ratio} \ (B, C, A P, T_a) &= -1 \\
\text{cross}_\text{ratio} \ (P_B, P_C, A, T_a) &= -1
\end{align*}
\]

Proof. We have formally:

\[
\begin{align*}
0 : q : r \pm p : 0 : 0 &= (p : q : r), \ (-p : q : r) \quad \text{while} \\
0 : q : r \pm 0 : 0 : r &= (0 : q : r), (0 : q : -r) \quad \text{and} \\
( p : -q : r) \pm ( p : q : -r) &= (2p : 0 : 0), (0 : -2q : +2r)
\end{align*}
\]

3.4.1 Well-known triangles

<table>
<thead>
<tr>
<th>cevian</th>
<th>bary (p)</th>
<th>G</th>
<th>O</th>
<th>I</th>
</tr>
</thead>
<tbody>
<tr>
<td>incentral</td>
<td>22.3</td>
<td>G(1)</td>
<td>a</td>
<td>X(5)</td>
</tr>
<tr>
<td>medial</td>
<td>22.5</td>
<td>G(2)</td>
<td>1</td>
<td>X(51)</td>
</tr>
<tr>
<td>orthic</td>
<td>22.7</td>
<td>X(4)</td>
<td>tanA</td>
<td>X(154)</td>
</tr>
<tr>
<td>intouch</td>
<td>22.12</td>
<td>X(7)</td>
<td>((b + c - a)^{-1})</td>
<td></td>
</tr>
<tr>
<td>extouch</td>
<td>22.13</td>
<td>X(8)</td>
<td>(b + c - a)</td>
<td>X(210)</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>anticevian</th>
<th>bary</th>
<th>G</th>
<th>O</th>
<th>I</th>
</tr>
</thead>
<tbody>
<tr>
<td>excentral</td>
<td>22.4</td>
<td>G(1)</td>
<td>a</td>
<td>X(165)</td>
</tr>
<tr>
<td>antimedial</td>
<td>22.6</td>
<td>G(2)</td>
<td>1</td>
<td>X(2)</td>
</tr>
<tr>
<td>tangential</td>
<td>22.8</td>
<td>X(6)</td>
<td>(a^2)</td>
<td>X(154)</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>other</th>
<th></th>
<th></th>
<th></th>
<th></th>
</tr>
</thead>
<tbody>
<tr>
<td>Fuhrmann</td>
<td>22.15</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Brocard triangles</td>
<td>22.9</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Hexyl</td>
<td>22.14</td>
<td>X(3576)</td>
<td>X(1)</td>
<td></td>
</tr>
<tr>
<td>star</td>
<td>22.16</td>
<td>X(3817)</td>
<td>X(946)</td>
<td>?</td>
</tr>
</tbody>
</table>

Table 3.1: Some well-known triangles

3.4.2 Isotomic and reciprocal conjugacies

Proposition 3.4.10. Equation 3.6 gives the condition for an inscribed triangle to be the Cevian triangle of some point \( P \).

\[
\begin{pmatrix}
0 & p_2 & p_3 \\
q_1 & 0 & q_3 \\
r_1 & r_2 & 0
\end{pmatrix} \text{ is Cevian } \iff p_2 q_3 r_1 - p_3 q_1 r_2 = 0
\]

and then \( p : q : r = r_1 p_2 : q_1 r_2 : r_1 r_2 \)

— pdlx : Translation of the Kimberling’s Glossary into barycentrics ——
An absolutely hopeless formula, but nevertheless more "obviously symmetrical" is:

\[ p : q : r = \sqrt[3]{\frac{p^2}{2} p_3 q_1 r_1 : \sqrt[3]{\frac{q^2}{2} q_2 r_2 : \sqrt[3]{\frac{r^2}{2} r_3 p_3}} q_3 \]

Proof. Line \( AP_1 \) is \((0, r_1, -q_1)\) and cyclically. These lines are concurrent when their determinant vanishes. Their common point is then given by any column of the adjoint matrix, or the cubic root of their product.

Proposition 3.4.11. *Isotomic conjugacy.* Suppose \( U = u : v : w \) is a point not on a sideline of \( ABC \). Take the cevians \( A_U = AU \cap BC \), etc and then reflect \( A_U B_U C_U \) about the midpoints of sides \( BC, CA, AB \), respectively, to obtain points \( A', B', C' \). Then lines \( AA', BB', CC' \) are concurrent. Their common intersection is called the isotomic conjugate of \( U \). The corresponding barycentrics are:

\[ \text{isotom}(u : v : w) = \frac{1}{u} : \frac{1}{v} : \frac{1}{w} \quad (3.7) \]

Proof. Immediate computation.

Remark 3.4.12. The fixed points of this transform are the gravity center and its relatives, so that \( \text{isot}(U) = U^* \) while \( U \neq \text{isot}(U) = X(2) \). It should be noticed that \( X(2) \) plays together the role of points \( F \) and a of Definition 1.4.10.

Proposition 3.4.13. *Reciprocal conjugacy.* Suppose \( \Delta = [\rho, \sigma, \tau] \) is a line not through a vertex of \( ABC \). Let \( A_\Delta = BC \cap \Delta \), etc the traces of \( \Delta \) on the sidelines. Reflect them about the corresponding midpoint and obtain \( T_A = B + C - A_\Delta \), etc. The points \( T_A, T_B, T_C \) are aligned, defining a line called the reciprocal of the given one since we have:

\[ \text{recip}([\rho, \sigma, \tau]) = \left[ \frac{1}{\rho}, \frac{1}{\sigma}, \frac{1}{\tau} \right] \]

Proof. Immediate computation.

Remark 3.4.14. For a given \( U \simeq u : v : w \), we have

\[ (\text{tripolar} \circ \text{isotom})(U) = [u, v, w] = (\text{recip} \circ \text{tripolar})(U) \]

Proposition 3.4.15. *Anticevian quadrangle.* (1) Triangle \( ABC \) is inscribed in triangle \( P_A P_B P_C \).
(2) \( ABC \) is the cevian triangle of \( P \) wrt the anticevian triangle. (3) Anticevian triangle of point \( P \) wrt \( ABC \) is \( PP_C P_B \) (the two remaining points are permuted). See also Theorem 2.1.9.

### 3.5 Transversal lines, Menelaus and Miquel theorems

**Theorem 3.5.1 (Ceva).** Let \( A' \in BC, B' \in CA \) and \( C' \in AB \) be three points on the sidelines of triangle \( ABC \), but different from the vertices. Then lines \( AA', BB', CC' \) are concurrent if and only if:

\[ \frac{AB}{BC'} \cdot \frac{BC}{CA'} \cdot \frac{CA}{BA'} = -1 \]

Proof. The usual proof uses Menelaus theorem. Another proof, using determinants, is given below.

**Proposition 3.5.2 (Genuine Menelaus theorem).** Let \( A' \in BC, B' \in CA \) and \( C' \in AB \) be three points on the sidelines of triangle \( ABC \), but different from the vertices. Then \( A', B', C' \) are collinear if and only if:

\[ \frac{AB'}{BC'} \cdot \frac{BC'}{CA'} \cdot \frac{CA'}{BA'} = +1 \]

Proof. Let us parametrize the situation by \( A' = k_a B + (1 - k_a) C \), etc. Alignment is described by:

\[
\begin{vmatrix}
0 & 1 - k_b & k_c \\
ka & 0 & 1 - kc \\
1 - ka & kb & 0 \\
\end{vmatrix} = 0
\]

while \( A' - B = (1 - k_a) (C - B) \) and \( A' - C = k_a (B - C) \).
Example 3.5.3. The cevian of \( P \) verify this formula.

Proposition 3.5.4 (Miquel theorem). Let \( A' \in BC, B' \in CA \) and \( C' \in AB \) be three points on the sidelines of triangle \( ABC \), but different from the vertices. Then circles \( AB'C', A'BC', A'B'C' \) are passing through a same point \( M \), the Miquel point of \( A'B'C' \) wrt \( ABC \).

Proof. Equation of circle \( AB'C' \) is:

\[
a^2yz + b^2zx + c^2xy - (x + y + z)\left( k_ac^2 + z(1 - k_b) b^2 \right)
\]

Therefore their last common point is:

\[
M \simeq \begin{pmatrix}
a^2 (k_a (k_a - 1) a^2 + (k_a - 1) (k_b - 1) b^2 + c^2 k_a)  
b^2 (k_a k_b a^2 + k_b (k_b - 1) b^2 + (k_b - 1) (k_c - 1) c^2)  
c^2 ((k_c - 1) (k_a - 1) a^2 + b^2 k_b k_c + k_c (k_c - 1) c^2)
\end{pmatrix}
\]

Since this expression is symmetric, the point \( M \) is also on the third circle.

Theorem 3.5.5 (Extended Menelaus theorem). Let \( A' \in BC, B' \in CA \) and \( C' \in AB \) be three points on the sidelines of triangle \( ABC \), but different from the vertices. All the following are necessary and sufficient conditions for \( A', B', C' \) to be collinear:

(i) the Menelaus condition : \((1 - k_a) (1 - k_b) (1 - k_c) + k_a k_b k_c = 0 \)
(ii) the midpoints \( M_a = (A + A')/2, M_b = (B + B')/2, M_c = (C + C')/2 \) are on the same line (the so-called Newton line of the quadrilateral)
(iii) the Miquel point of \( A'B'C' \) is on the circumcircle of \( ABC \).
(iv) –spoiler– the homographic application \( \Psi \) defined in \( \mathbb{P}_C(\mathbb{C}^2) \) by \( A \mapsto A', B \mapsto B', C \mapsto C' \) is involutory.

Proof. (i) obvious ; (ii) determinant ; (iii) condition for \( M \in \Gamma \) is the Menelaus condition times an ugly factor that can be written as:

\[
(2 a^2 k_a + 2 b^2 k_b + 2 c^2 k_c - c^2 - b^2 - a^2)^2 + 16 S^2
\]

(iv) Condition for \( \Psi \) to be involutory is \( \det_3 [1, z_A + z'_A, z_A z'_A] = 0 \). This results into \( I_{s_4} \) times the Menelaus condition (see Section 15.3 for notations and more details).

— pldx : Translation of the Kimberling’s Glossary into barycentrics —
3.6 Tripolar centroid

**Proposition 3.6.1.** When \( P \) is neither a vertex nor \( X(2) \), the centroid of the cocevian triangle is well defined (perhaps on \( \mathcal{L}_\infty \)), and called the tripolar centroid of \( P \) (Stothers, 2003b).

\[
TG(P) = p(q-r)(2p-q-r) : q(p-r)(p-2q+r) : r(q-p)(p+q-2r)
\]

**Remark 3.6.2.** When \( q = r \) then \( TG(P) = 0 : 1 : -1 \) (at infinity on \( BC \)). When all \( p, q, r \) are different, \( TG(P) \) is a finite point.

**Proposition 3.6.3.** When all \( p, q, r \) are different, then it exists exactly another point that shares the same \( TG(P) \), namely:

\[
\text{other}(P) = \frac{q + r - 2p}{rp + qr - 2qr} : \frac{r + p - 2q}{qp + qr - 2rp} : \frac{p + q - 2r}{qr + rp - 2qp}
\]

**Proof.** Direct computation. When eliminating \( k, u, v \) in \( TG(P) = kTG(U) \), special cases are \( p, q - r, u, v - w, qw - rv, 2p - q - r \) and cyclically. For all named points \( X \), it happens that \( \text{other}(X) \) is not named.

**Example 3.6.4.** Points \( X(1635) \) to \( X(1651) \) are defined that way. Examples include:

<p>| | | | | | | |</p>
<table>
<thead>
<tr>
<th></th>
<th></th>
<th></th>
<th></th>
<th></th>
<th></th>
<th></th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>1635</td>
<td>98</td>
<td>1640</td>
<td>263</td>
<td>2491</td>
<td>525</td>
</tr>
<tr>
<td>2</td>
<td>1636</td>
<td>99</td>
<td>1641</td>
<td>512</td>
<td>1645</td>
<td>648</td>
</tr>
<tr>
<td>3</td>
<td>1637</td>
<td>100</td>
<td>1642</td>
<td>513</td>
<td>1646</td>
<td>957</td>
</tr>
<tr>
<td>4</td>
<td>351</td>
<td>105</td>
<td>1643</td>
<td>514</td>
<td>1647</td>
<td>1002</td>
</tr>
<tr>
<td>5</td>
<td>1638</td>
<td>190</td>
<td>1644</td>
<td>523</td>
<td>1648</td>
<td>1022</td>
</tr>
<tr>
<td>6</td>
<td>1639</td>
<td>262</td>
<td>3569</td>
<td>524</td>
<td>1649</td>
<td>2394</td>
</tr>
<tr>
<td>7</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>8</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

3.7 Cross-triangle

**Definition 3.7.1.** Cross-triangle. The cross-triangle of two given triangles \( T_1 = A_1B_1C_1 \) and \( T_2 = A_2B_2C_2 \) is defined as the triangle \( T_4 = B_1C_2 \cap B_2C_1, C_1A_2 \cap C_2A_1, A_1B_2 \cap A_2B_1 \). Its name comes from the fact we are crossing the vertices of the homologue sidelines.

3.8 Perspectivity

**Definition 3.8.1.** Vertex trigone, vertex triangle. Let \( T_1 = A_1B_1C_1 \) and \( T_2 = A_2B_2C_2 \) be two triangles (i.e. two ordered sets of three points). Their vertex trigone \( T^*_3 \) is the set of three lines \( A_3 = A_1A_2, B_3 = B_1B_2, C_3 = C_1C_2, \) while their vertex triangle is the set of points \( A_3 = B_3 \cap C_3 \), etc. The vertex triangle is the dual of the vertex trigone.
Remark 3.8.2. Exclude the case where all points are on the same line, the rank of the vertex trigone is either 3 or 2, while the rank of the vertex triangle is either 3 or 1 (adjoint matrix).

Definition 3.8.3. Perspector. Let $T_1 = A_1B_1C_1$ and $T_2 = A_2B_2C_2$ be two triangles. When the vertex trigone degenerates, i.e. when the lines $A_3, B_3, C_3$, concur at some point $P$, this point is called the perspector (replacing center of perspective) of the (ordered) triangles.

Definition 3.8.4. Sideline triangle, sideline trigone. Let $T_1 = A_1B_1C_1$ and $T_2 = A_2B_2C_2$ be two trigones (i.e. two ordered sets of three lines). Their sideline triangle $T_1^*$ is the set of three points $A_4 = A_1 \cap A_2$, $B_4 = B_1 \cap B_2$, $C_4 = C_1 \cap C_2$, while their sideline trigone is the set of lines $B_4C_4, C_4A_4, A_4B_4$ (i.e. the dual of the sideline triangle).

Remark 3.8.5. Excluding the case where all lines are through the same point, the rank of the sideline triangle is either 3 or 2, while the rank of the sideline trigone is either 3 or 1 (adjoint matrix).

Definition 3.8.6. Perspectrix. Let $T_1$ and $T_2$ be two triangles. When the sideline triangle of $T_1^*$ and $T_2^*$ degenerates, i.e. when points $A_4, B_4, C_4$ are on the same line, this line is called the perspectrix (replacing axis of perspective) of the triangles.

Theorem 3.8.7. [Desargues]. When none of the triangles $T_1$ and $T_2$ are degenerate, the existence of a perspector is equivalent to the existence of a perspectrix.

Proof. In this context, trigone $T_3$ is called the Desargues trigone and triangle $T_4$ is called the Desargues triangle. The result comes from

$$\det T_4 = \det T_1 \det T_2 \det T_3$$

and such a lack of symmetry is better understood when considering the dual formulas:

$$\det \homol(T_1^*, T_2^*) = \det T_1 \det T_2 \det \homol(T_1, T_2)$$

$$\det \homol(T_1^*, T_2^*) = \det T_1 \det T_2 \det \homol(T_1, T_2)$$

Example 3.8.8. Let $T_1$ be the reference triangle $ABC$ and $T_2$ the cevian triangle $APBPQ$ of a point $P$. Then

1. $T_3 = \begin{bmatrix} 0 & -r & q \\ -r & 0 & -p \\ -q & p & 0 \end{bmatrix}$ is the set of the three cevian lines $AP$, etc while $T_3^* = P^*P$: this triangle degenerates into three times the point $P$. Therefore $P$ is the perspector of both triangles.

2. $T_4 = \begin{bmatrix} 0 & p & -p \\ -q & 0 & q \\ -r & -r & 0 \end{bmatrix}$ is the cocevian triangle. Compared to $T_2 = \begin{bmatrix} 0 & p & p \\ q & 0 & q \\ r & r & 0 \end{bmatrix}$, this $T_4$ is the set of the associated cross-ratio points, while $T_4^*$ degenerates into three times the tripolar line tripolar $P \equal [qr, rp, pq]$. Therefore this line is the perspectrix of both triangles.

Example 3.8.9. Let $T_1$ be the triangle and $T_2$ the anticevian triangle $PAPBPQ$ of a point $P$. Let us recall that cross_ratio $(A, A_P, P; P_A) = -1$. Then again perspector is $P$ and perspectrix is tripolar $P$.

Exercise 3.8.10. Spoiler: for any $P$, the pedal triangle of $P$ is in perspective with $AM$, the triangle whose vertices are the directions of the altitudes $AH, BH, CH$. The perspector is $P$ itself. Since vertices of $AM$ are on $E_\infty$, the flatness of the sideline triangle was granted and carries no additional information.

Proposition 3.8.11. Perspectivity kit. Such a kit is defined as $p : q : r : u : v : w$, i.e. six numbers up to a common proportionality factor, none of them being 0. This amounts to give points $P \simeq p : q : r$, $U \simeq u : v : w$ (none on an $ABC$- sideline) together with a synchronization factor.

— pldx : Translation of the Kimberling’s Glossary into barycentrics ——
k = (p + q + r) / (u + v + w). Exchanging P and U changes k in 1/k. We define triangle $\mathcal{T}_0$ as $ABC$ and triangle $\mathcal{T}_1$ and $\mathcal{T}_2$ by:

$$\mathcal{T}_1 \simeq \begin{pmatrix} u & p & p \\ q & v & q \\ r & r & w \end{pmatrix}; \quad \mathcal{T}_2 \simeq \begin{pmatrix} p & u & u \\ v & q & v \\ w & w & r \end{pmatrix}$$

1. $\mathcal{T}_0$ and $\mathcal{T}_1$ are perspective wrt to P, $\mathcal{T}_0$ and $\mathcal{T}_2$ are perspective wrt U, while $\mathcal{T}_2, \mathcal{T}_3$ admit $P + U$ as perspector.

2. Each triangle $\mathcal{T}_1$, $\mathcal{T}_2, \mathcal{T}_3$ is the cross-triangle of the other two.

3. All pair of triangles have the same Desargues triangle, defining the same perspectrix:

$$\Delta \simeq \left[ 1 : \frac{1}{p-u} ; \frac{1}{q-v} ; \frac{1}{r-w} \right]$$

4. Reciprocally, any triangle in perspective with $ABC$ can be parametrized that way.

**Proposition 3.8.12.** When triangle $\mathcal{T}_0 = ABC$ and triangle $\mathcal{T}$ admit $P \simeq p : q : r$ and tripolar $\Delta \simeq [qr,rp,pq]$ as perspector and perspectrix, then it exists $k \in \mathbb{R} \cup \{\infty\}$ such that

$$\mathcal{T}_k \simeq \begin{pmatrix} kp & p & p \\ q & kq & q \\ r & r & kr \end{pmatrix}$$

**Proof.** Start from preceding proposition and write that $\Delta$ is tripolar P rather than another line.
Proposition 3.8.13. Suppose additionally that line $PU$ doesn’t go through any of the vertices $A, B, C$, define

$$Q \simeq \begin{pmatrix} (qw - rv) (p - u) p \\ (ru - pw) (q - v) q \\ (pv - qu) (r - w) r \end{pmatrix}; \quad V \simeq \begin{pmatrix} (qw - rv) (p - u) u \\ (ru - pw) (q - v) v \\ (pv - qu) (r - w) w \end{pmatrix}$$

and consider the triangles $T_0, T_1, T_2$, the cevian triangles of $P$: $T_{0P}, T_{1P}, T_{2P}$ and the cevian triangles of $U$: $T_{0U}, T_{1U}, T_{2U}$. Then: (1) $P$ is the perspector of each pair: $(T_0, T_1), (T_{0P}, T_{1P}), (T_0, T_{0P}), (T_1, T_{0P}), (T_1, T_{1P});$

(2) Perspector of $(T_1, T_2)$ is $p + u : q + v : r + w.$

(3) $Q \neq V$ and line $\Delta = QV$ is the perspectrix of each pair: $(T_0, T_1), (T_{0P}, T_{1P}), (T_1, T_2), (T_0, T_2), (T_{0U}, T_{2U});$

(4) Moreover, the perspectrices of $(T_0, T_{0P}), (T_0, T_{1P}), (T_1, T_{0P}), (T_1, T_{1P})$ are going through $Q$ (and symmetrically for $U$).

Proof. The perspectrix of $(T_0, T_{0P})$ is $\Delta_P = \begin{bmatrix} qr & pr & pq \end{bmatrix}$, the tripolar of $P$ and we have:

$$\begin{array}{c c c}
T_0 & T_{1P} & pqr \Delta_1 - \det T_2 \Delta_P \\
T_1 & T_{0P} & 2 pqr \Delta_1 - \det T_2 \Delta_P \\
T_1 & T_{1P} & 3 pqr \Delta_1 - \det T_2 \Delta_P
\end{array}$$


3.9 Cevian nests

Definition 3.9.1. Cevian nest. Suppose $T_1, T_2, T_3$ are triangles and that $T_1$ is inscribed in $T_2$ and $T_2$ is inscribed in $T_3$. If any two of the tree triangles are perspective, it is well-known that each is perspective to the third: $T_1$ is a cevian triangle of $T_2$ for some point $P$, $T_3$ is an anticevian triangle of $T_2$ for some point $U$ and $T_1, T_2$ are perspective wrt some point $X$. Such configuration is called a cevian nest.

Figure 3.5: P is cevamul(U,X)

Proposition 3.9.2. Map $P = P(U, X)$ giving the perspector of $T_1, T_2$ from the other two perspectors of a cevian nest is symmetric, while –for a given $P$– map $X = X(U)$ is involutory.

--- pldx : Translation of the Kimberling’s Glossary into barycentrics ---
Proof. Given the vertices \(S_i (i = 1, 2, 3)\) of triangle \(T_3\) and perspector \(U\), the vertices \(S_i (i = 4, 5, 6)\) of triangle \(T_2\) are obtained by \(S_i = (S_1 \cup U) \land (S_2 \land S_3)\) and cyclically. Process can be iterated, obtaining vertices \(S_i (i = 7, 8, 9)\) of \(T_1\). Then \(X\) is obtained as \(X = (S_1 \land S_7) \land (S_2 \land S_8)\). It can be checked that substituting \(U\) by \(X\) (and keeping everything else unchanged) leads back to \(U\), proving the second part. The first part follows immediately.

When triangle \(ABC\) belongs to such a nest, three possibilities can occur. The corresponding operations are summarized in Table 3.2, where "mul" stands for multiplication (giving \(P\)) and "div" for the converse operation. The Kimberling’s name is also given.

<table>
<thead>
<tr>
<th>(T_1)</th>
<th>(T_2)</th>
<th>(T_3)</th>
<th>(X, P)</th>
<th>Kimberling</th>
</tr>
</thead>
<tbody>
<tr>
<td>(C_P) wrt (T_2)</td>
<td>(C_U)</td>
<td>(ABC)</td>
<td>(X = \text{crossdiv}(P, U))</td>
<td>cross – conj</td>
</tr>
<tr>
<td>(C_P)</td>
<td>(ABC)</td>
<td>(A_U)</td>
<td>(P = \text{crossmul}(U, X))</td>
<td>cross – point</td>
</tr>
<tr>
<td>(ABC)</td>
<td>(A_P)</td>
<td>(A_U) wrt (T_2)</td>
<td>(X = \text{sqrtdiv}(P, U))</td>
<td>ceva – conj</td>
</tr>
<tr>
<td>(ABC)</td>
<td>(A_P)</td>
<td>(A_U) wrt (T_2)</td>
<td>(P = \text{sqrtmul}(U, X))</td>
<td>ceva – point</td>
</tr>
</tbody>
</table>

All these operations are (globally) type-keeping, since they transform points into constructible points.

Table 3.2: Three cases of cevian nets

3.10 The cross case (aka case I, cev of cev)

Definition 3.10.1. Crossmul\((U, X)\). As in Table 3.2 (I), let \(T_3\) (the biggest triangle) be \(ABC\), \(T_2 = A_U B_U C_U\) the (usual) cevian triangle of \(U\) and \(T_1 = A'B'C'\) the triangle inscribed in \(T_2\) obtained by \(A' = AX \cap B_U C_U\) and cyclically. Then \(T_1\) and \(T_2\) have a perspector \((P)\), and mapping \((U, X) \mapsto P\) is called cross-multiplication.

Definition 3.10.2. Crossdiv\((P, U)\). As in Table 3.2 (I), let \(T_3\) (the biggest triangle) be \(ABC\), \(T_2 = A_U B_U C_U\) the (usual) cevian triangle of \(U\) and \(T_1 = A'B'C'\) the cevian triangle of \(P\) wrt \(T_2\), obtained by \(A' = A_U P \cap B_U C_U\) and cyclically. Then \(T_1\) and \(T_3\) have a perspector \((X)\), and mapping \((P, U) \mapsto X\) is called cross-division.

Proposition 3.10.3. Computing rules of crossmul and crossdiv are given (using barycentrics) in Figure 3.6. Map \((U, X) \mapsto P\) is commutative and behaves like ordinary multiplication (eponymous property). Map \((P, U) \mapsto X\) behaves wrt crossmul like division behaves wrt ordinary multiplication (eponymous property).

Proof. Barycentrics \(p : q : r\) are defining point \(P'\) with respect to triangle \(T_3\). Call \(P : Q : R\) its barycentrics with respect to triangle \(T_2\), so that \((p : q : r) = \frac{T_2}{T_2}(P : Q : R)\). Then:

\[
\begin{bmatrix}
0 & u & u \\
v & 0 & 0 \\
w & 0 & 0
\end{bmatrix}
\begin{bmatrix}
0 & P & P \\
Q & 0 & Q \\
R & R & 0
\end{bmatrix}
\begin{bmatrix}
P & 0 & 0 \\
0 & Q & 0 \\
0 & 0 & R
\end{bmatrix}
\approx
\begin{bmatrix}
u(Q + R) \\
vQ \\
wR
\end{bmatrix}
\begin{bmatrix}
u \\
vP \\
wQ
\end{bmatrix}
\begin{bmatrix}
u \\
v \\
w
\end{bmatrix}
\approx
\begin{bmatrix}
u(P + R) \\
Q \\
R
\end{bmatrix}
\begin{bmatrix}
Q \\
P \\
R
\end{bmatrix}
\]

where the diagonal matrix had been chosen to "synchronize" the columns of triangle \(T_2\). Then Proposition 3.8.11 shows that \(T_3\) and \(T_7\) are perspective wrt \(P : v/Q : w/R\).

Remark 3.10.4. The cross-multiplication and cross-division were introduced in Kimberling (1998), using the names crosspoint and "cross conjugacy", together with the notation \(X = C(P, U)\) and also \(X = PeU\). In our opinion, a name that emphasizes the properties is more efficient.

The factorization given can be interpreted in terms of isoconjugacies (see Chapter 17). First compute what happens when \(U = X(2)\) and obtain complem \(\circ\) isotom. Then transpose this map by the transform \(U^*\) that fixes \(A, B, C\) and sends \(G = X(2)\) onto \(U\).

January 3, 2024 21:08 published under the GNU Free Documentation License
Proposition 3.10.5. The crossmul $P$ of $U, X$ is also the point of concurrence of (1) the line through points $AX \cap BU$ and $AU \cap BX$, (2) the line through points $BX \cap CU$ and $BU \cap CX$, (3) the line through points $CX \cap AU$ and $CU \cap AX$.

Definition 3.11.1. Cevamul$(U, X)$. As in Table 3.2 (II), let $T_2$ (the middle triangle) be $ABC$, $T_3 = U_A U_B U_C$ the anticevian triangle of $U$ and $T_1 = A'B'C'$ be the triangle inscribed in $T_2 = ABC$ obtained by $A' = U_A X \cap BC$ and cyclically. Then $T_1$ and $T_2$ have a perspector (i.e. $T_1$ is the cevian triangle of some point $P$), and mapping $(U, X) \mapsto P$ is called ceva-multiplication.

Definition 3.11.2. Cevadiv$(P, U)$. As in Table 3.2 (II), let $T_2$ (the middle triangle) be $ABC$, $T_3 = U_A U_B U_C$ the anticevian triangle of $U$ and $T_1 = A_P B_P C_P$ be the cevian triangle of $P$, 
obtained (as usual) by \( AP = AP \cap BC \) and cyclically. Then \( T_1 \) and \( T_3 \) have a perspector \((X)\), and mapping \((P, U) \mapsto X\) is called ceva-division.

**Proposition 3.11.3.** Computing rules of cevamul and cevadiv are given (using barycentrics) in Figure 3.8. Map \((U, X) \mapsto P\) is commutative i.e.:

\[
\begin{align*}
X & \text{ is the perspector of } C_P \text{ and } A_U \\
U & \text{ is the perspector of } C_P \text{ and } A_X
\end{align*}
\]

and behaves like ordinary multiplication (eponymous property). Map \((P, U) \mapsto X\) behaves wrt crossmul like division behaves wrt ordinary multiplication (eponymous property).

\[
\begin{array}{c}
P/U \\ \text{isotom} \circ \text{complem}
\end{array}
\begin{array}{c}
P/X \\ \text{isotom} \circ \text{complem}
\end{array}
\begin{array}{c}
P \\ \text{cevamul} \\ (U, X)
\end{array}
\begin{array}{c}
P \\ \text{cevadiv}
\end{array}
\begin{array}{c}
P/U \\ \text{isotom} \circ \text{complem}
\end{array}
\begin{array}{c}
P/X \\ \text{isotom} \circ \text{complem}
\end{array}
\begin{array}{c}
X/U \\ \text{isotom} \circ \text{complem}
\end{array}
\begin{array}{c}
X \\ \text{cevamul} \\ (U, X)
\end{array}
\begin{array}{c}
X \\ \text{cevadiv}
\end{array}
\begin{array}{c}
X
\end{array}
\]

cevamul \((u : v : w, \ x : y : z) = (uz + wx)(uy + vx) : (uz + wy)(uy + vx) : (uz + wy)(uz + wx)\) \hspace{1cm} (3.11)

cevadiv \((p : q : r, \ u : v : w) = \ u(-qr u + rp v + pq w) : v(qru - rpv + pq w) : w(qru + rpv - pq w)\) \hspace{1cm} (3.12)

Figure 3.8: cevamul, cevadiv

**Proposition 3.11.4.** (Spoiler) Seen as an \( V \mapsto X \) map, \( U \mapsto \text{cevamul}(U, V) \) is a Cremona transform, whose points of indeterminacy are the anticevian vertices of \( U \) and the exceptional locus is the union of the \( U \)-anticevian sidelines. Seen as an \( U \mapsto X \) map, \( U \mapsto \text{cevadiv}(P, U) \) is a Cremona transform, whose points of indeterminacy are the cevian vertices of \( P \) and the exceptional locus is the union of the \( P \)-cevian sidelines. Seen as an \( P \mapsto X \) map, \( P \mapsto \text{cevadiv}(P, U) \) is a Cremona transform, whose points of indeterminacy are the \( ABC \) vertices and the exceptional locus is the union of the \( ABC \) sidelines.

**Proof.** Direct examination. \(\square\)

**Remark 3.11.5.** The ceva-multiplication and ceva-division were introduced in Kimberling (1998), using the names **cevapoint** and "**ceva conjugacy**", together with the notation \( X = P \circ U \). In our opinion, a name that emphasizes the properties is more efficient.

**Proposition 3.11.6.** Suppose \( U = u : v : w \) and \( X = x : y : z \) are distinct points, neither lying on a sideline of \( ABC \). Let \( A_u = X_A X_B X_C \) and \( A_u = U_A U_B U_C \) be the anticevian triangles of \( X \) and \( U \) (wrt \( ABC \)). Define \( A' \) as \( U_A X \cap X_A U \) and \( B', C' \) cyclically (Figure 3.5). Then triangle \( A'B'C' \) is inscribed in \( ABC \) and is, in fact, the cevian triangle of \( P = \text{cevamul}(U, V) \).

**Proof.** Direct computation. \(\square\)

**Construction 3.11.7** (Floor Van Lamoen (2003/10/17)). Point \( \text{cevamul}(U, X) \) can be constructed from the cevian triangles: let \( A_U B_U C_U \) be the cevian triangle of \( U \), and \( A_X B_X C_X \) the cevian triangle of \( X \). Define:
Proposition 3.11.8. Then, as seen in Figure 3.9, triangle \(ABC\) is perspective to both triangles \(A_{ux}, B_{ux}, C_{ux}\) and \(A_{xu}, B_{xu}, C_{xu}\), and the perspector in both cases is the cevamul \((U,X)\).

Proof. Direct computation.

\[\]

Figure 3.9: Lamoen’s construction of \(P=\text{cevamul}(U,X)\)

3.12 The square case (aka case III, acev of acev)

Construction 3.12.1. sqrtdiv\((F,U)\). As in Table 3.2 (III), let \(T_1\) (the smallest triangle) be \(ABC\), \(T_2 = F_AF_BF_C\) the anticevian triangle of \(F\) and \(T_3\) be the anticevian triangle of \(U\) w.r.t \(T_2\). Then \(T_1\) and \(T_3\) have a perspector \(X\) and mapping \((F,U) \mapsto X\) is called sqrtdiv. We have formula :

\[
\text{sqrtdiv}_F (U) \doteq U_F^\# \doteq \frac{g^2}{v} : \frac{h^2}{w}
\]

Remark 3.12.2. This construction is not so easy as it seems to be. In fact, drawing an anticevian triangle requires a knowledge of the barycentrics (and this is no more a construction!), or some conics, or a forest of lines that are equivalent to the drawing of the conics themselves. One can also start from the cevian triangle and use some fourth harmonics, that are equivalent to inversions into some circles.

Definition 3.12.3. sqrtmul\((U,X)\). The inverse mapping of sqrtdiv should be the mapping sqrtmul \((U,X) \mapsto P\) "defined" by :

\[
p : q : r = \pm \sqrt{ux} : \pm \sqrt{vy} : \pm \sqrt{wz}
\]

... but (1) the solution is not unique and (2) in fact, the problem cannot even be stated clearly.

3.13 Danneels perspectors

Definition 3.13.1. First Danneels perspector. Let \(T_1 = A_U B_U C_U\) be the cevian triangle of a point \(U = u : v : w\). Let \(L_A\) be the line through \(A\) parallel to \(B_UC_U\), and define \(L_B\) and \(L_C\) cyclically. The lines \(L_A, L_B, L_C\) determine a triangle \(T_2\) perspective to \(T_1\) (and in fact homothetic, with factor 2). The corresponding perspector is \(DP_1 (U)\), the first Danneels perspector \((\#11037)\) of \(U\). Using barycentrics :

\[
DP_1 (U) = u^2(v+w) : v^2(w+u) : w^2(u+v)
\]
Proof. Compute (from left to right) the row \( B_U \cap C_U \cap \mathcal{L}_\infty \cap A \) and cyclically. Obtain a matrix describing a trigone and takes the adjoint to obtain \( T_2 \). Then compute the perspector. One can also remark that \( T_2 \) is the anticevian triangle of:

\[
X = u(v + w) : v(u + w) : w(u + v)
\]

and obtain \( \text{DP}_1(U) \) as \( \text{cevadiv}(U/X) \) (homothetic property is obvious... and useless to compute the perspector).

Remark 3.13.2. This point is named \( D(U) \) in ETC. No name of only one letter! Moreover this conflicts with the Maple’s derivation operator.

**Proposition 3.13.3.** Point \( G = X(2) \) is invariant under \( \text{DP}_1 \). Moreover, \( G, U \) and \( \text{DP}_1(U) \) are ever collinear. For example, Euler line is globally invariant.

**Proof.** Check that \( \det (X_2, U, \text{DP}_1(X_2 + \lambda U)) = 0 \).

**Proposition 3.13.4.** When \( \text{DP}_1(X) = G \) then either \( X = G \) or \( X \) lies on the Steiner circum-lipse.

**Proof.** Write \( \text{DP}_1(X) = kG \) and solve. Except from \( X = G, xy + yz + zx = 0 \) is obtained.

**Proposition 3.13.5.** When \( \text{DP}_1(U) \neq G \), i.e. when \( U \neq G \) and \( U \) not on the circumsteiner, it exists two other points that verify \( \text{DP}_1(X) = \text{DP}_1(U) \). Using barycentrics, we have:

\[
X \simeq \left( \begin{array}{c}
v + w - 2u - \frac{1}{u + w + v} \\
w + u - 2v - \frac{1}{u + v + w} \\
u + v - 2w - \frac{1}{u + w + v}
\end{array} \right) \text{ where}
\]

\[
W^2 = (u + v)(v + w)(w + u) \left( u^2(v + w) + v^2(w + u) + w^2(u + v) - 6uvw \right)
\]

**Proof.** Direct computation, assuming \( xy + yz + zx \neq 0 \). The main difficulty is to re-obtain a symmetric expression after elimination of \( k, z \) and resolution on \( y \).

**Example 3.13.6.** Here is a list of pairs \((I, J)\) of named points such that \( \text{DP}_1(X(I)) = X(J) \):

\[
\begin{array}{cccccccccc}
1 & 42 & 20 & 3079 & 189 & 1422 & 664 & 2 & 903 & 2 & 2481 & 2 \\
2 & 2 & 25 & 3080 & 190 & 2 & 666 & 2 & 1113 & 25 & 2966 & 2 \\
3 & 418 & 30 & 3081 & 264 & 324 & 668 & 2 & 1114 & 25 & 3225 & 2 \\
5 & 3078 & 75 & 321 & 366 & 367 & 671 & 2 & 1370 & 455 & 3227 & 2 \\
6 & 3051 & 99 & 2 & 648 & 2 & 886 & 2 & 1494 & 2 & 3228 & 2 \\
7 & 57 & 100 & 55 & 651 & 222 & 889 & 2 & 2479 & 2 \\
8 & 200 & 110 & 184 & 653 & 196 & 892 & 2 & 2480 & 2
\end{array}
\]

**Definition 3.13.7.** Second Danneels perspector. Suppose \( T_1 = A_U B_U C_U \) is the cevian triangle of a point \( U \). Let \( L_{AB} \) be the line through \( B \) parallel to \( A_U B_U \), and let \( L_{AC} \) be the line through \( C \) parallel to \( A_U C_U \). Define \( A' = L_{AB} \cap L_{AC} \) and \( B', C' \) cyclically. Finally obtain \( A'' = B'B' \cap C'C' \) and \( B'', C'' \) cyclically. It happens that triangle \( T_2 = A''B''C'' \) is perspective to \( T_1 = A_U B_U C_U \). The corresponding perspector is \( \text{DP}_2(U) \), the second Danneels perspector of \( U \) (Danneels, 2006). Using barycentrics:

\[
\text{DP}_2(U) = u(v-w)^2 : v(w-u)^2 : w(u-v)^2
\]

**Proof.** Compute (left to right) \( L_{AB} = A_U \cap B_U \cap \mathcal{L}_\infty \cap B, L_{AC} \) accordingly, then \( A' = L_{AB} \cap L_{AC} \) and \( B', C' \) cyclically. Obtain \( A'' = (B \cap B') \cap (C \cap C') \) and \( B'', C'' \) cyclically. See that \( T_2 = A''B''C'' \) is the anticevian triangle of point:

\[
X \simeq u(v-w) : v(w-u) : w(u-v)
\]

and obtain \( \text{DP}_2(U) \) as \( \text{cevadiv}(U, X) \).
Proposition 3.13.8. The circumconic through $A, B, C, U$ isoto $(U)$ admits $\text{DP}_2(U)$ as center and $u(v^2 - w^2) : v(w^2 - u^2) : w(u^2 - v^2)$ as perspector. And therefore isotomic conjugates have the same second Danneels’ perspector.

Proof. Immediate computation.

Example 3.13.9. List of $(U, U^*, \text{DP}_2(U)):$

<table>
<thead>
<tr>
<th>$U$</th>
<th>$U^*$</th>
<th>$\text{DP}$</th>
<th>$U$</th>
<th>$U^*$</th>
<th>$\text{DP}$</th>
<th>$U$</th>
<th>$U^*$</th>
<th>$\text{DP}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>75</td>
<td>244</td>
<td>37</td>
<td>274</td>
<td>3121</td>
<td>394</td>
<td>2052</td>
<td>3269</td>
</tr>
<tr>
<td>3</td>
<td>264</td>
<td>2972</td>
<td>42</td>
<td>310</td>
<td>3122</td>
<td>519</td>
<td>903</td>
<td>1647</td>
</tr>
<tr>
<td>4</td>
<td>69</td>
<td>125</td>
<td>57</td>
<td>312</td>
<td>2170</td>
<td>524</td>
<td>671</td>
<td>1648</td>
</tr>
<tr>
<td>6</td>
<td>76</td>
<td>3124</td>
<td>81</td>
<td>321</td>
<td>3125</td>
<td>536</td>
<td>3227</td>
<td>1646</td>
</tr>
<tr>
<td>7</td>
<td>8</td>
<td>11</td>
<td>94</td>
<td>323</td>
<td>2088</td>
<td>538</td>
<td>3228</td>
<td>1645</td>
</tr>
<tr>
<td>9</td>
<td>85</td>
<td>3119</td>
<td>98</td>
<td>325</td>
<td>868</td>
<td>2394</td>
<td>2407</td>
<td>1637</td>
</tr>
<tr>
<td>10</td>
<td>86</td>
<td>3120</td>
<td>99</td>
<td>523</td>
<td>1649</td>
<td>2395</td>
<td>2396</td>
<td>2491</td>
</tr>
<tr>
<td>20</td>
<td>253</td>
<td>122</td>
<td>100</td>
<td>693</td>
<td>3126</td>
<td>2397</td>
<td>2401</td>
<td>3310</td>
</tr>
<tr>
<td>30</td>
<td>1494</td>
<td>1650</td>
<td>200</td>
<td>1088</td>
<td>2310</td>
<td>2398</td>
<td>2400</td>
<td>676</td>
</tr>
</tbody>
</table>

and of points without named isotomic:

$[43, 3123], [88, 2087], [694, 2086], [1022, 1635], [1026, 2254], [2421, 3569]$
Chapter 4

The French Touch

This chapter is translated from an article originally published under the title "Géométrie projective pour agrégatifs" (Douillet, 2012). One cannot totally exclude that some provocative tone was used here or there.

Our friend keeps repeating that the current program of the recruitment competition for French Second Degree teachers (agrégation de Mathématiques) does not contain the elements necessary to understand the geometries of Moebius, Kimberling or Morley, that is to say the different versions of the plane projective geometry. I rather disagree with this point of view. Here are some elements to support my own opinion.

4.1 The "not so flying plane"¹

1. Definition. We call rantanplan the "quad ruled paper sheet" which is placed in front of a schoolchild. We make a (red) cross somewhere (at the intersection of two gridlines), and we write "Here you are". When this takes up too much space, we write "O". And we say "this is the origin".

2. Definition. You obtain the point "tchik, tchik, tchik, kling, tchouk, tchouk, bang, A" by placing yourself in O, with the margin behind you (the gaze is then directed along the horizontal lines of the papersheet). And you move a square forward (tchik), and another square forward (the second tchik), and again another square forward (the third tchik). Then you do a quarter turn (kling) and now, the gaze is directed along the vertical lines of the papersheet. And you move one square forward (tchouk), then another square forward (the second tchouk), you put down the pencil and make a cross (bang). And you write the name of the point, A.

3. Axiom (Archimedes + Thales + Cantor). The preceding definitional schema gives access to all the points of the rantanplan.

4. Definition, Borel notation. When there are "a lot" of tchiks and tchouks, we count them and we note $A = 3 + 2i$ . The "i" is used to denote the kling.

5. Foundational experiment. The candidate is at the blackboard and stands in front of the door. Then she makes a quarter turn, in the direction of the Jury. Then she makes another quarter turn, in the direction of the window. As a result, the candidate has made a half turn and all (horizontal) directions have been reversed. This is noted $i^2 = -1$ .

6. Scholie. Suppose the previous scene is observed by two Vice-Presidents of the Jury, one placed on the floor above, the other placed on the floor below, and further assume that the President of the Jury has a sufficient authority to impose on his assessors the use of watches whose hands turn in the same "clockwise" direction. Then one of them will see $A = 3 + 2i$ , and the other will see $A = 3 - 2i$ .

7. Hands out. The above experiment can be reproduced without such a grandiose staging. It is enough that the pupil looks at his rantanplan not from above, but from below, by transparency. So "everything starts turning the other way".

¹Tentative translation of the French joke: "le rantanplan"
8. Definition. Normalized coordinates. In order obtain an intrinsic object, we define the Normalized Coordinates of the point $A$ by:

$$\zeta_{\text{norm}}(A) = \begin{pmatrix} 3 + 2i \\ 1 \\ 3 - 2i \end{pmatrix}$$

We note above what is seen by the Vice president who is on the floor above and we note below what is seen by the Vice president who is on the floor below. And one can even imagine that the "1" between the two serves to unify the points of view of the two Vice-Presidents.

9. Scholie. When the two Vice-Presidents enter into a full open war (to become the next President of the Jury), the "unifying" point of view cannot be maintained, and it is advisable to replace this "1=unity" by a multiple, leading to the next definition.

10. Definition. Superior Coordinates. Coordinates proportional to the Normalized Coordinates are called Superior Coordinates$^2$. This will be noted using the sign $\simeq$ (simeq) and we have, for example, the relation

$$\zeta_A \simeq \begin{pmatrix} 6 + 4i \\ 2 \\ 6 - 4i \end{pmatrix}$$

In the wide outer world, where jokes about the ENS are likely to fail miserably, $\zeta_A$ is called the Morley coordinates of $A$.

11. Notation. The Morley coordinates of the current point of the plane are written $Z : T : Z$ (big zed, big tea, big zeta). Each candidate to the aggregation knows (or at least should know) that an algebraic variable is nothing else than a mark-a-place in the writing of polynomials which are finite series of multi-indexed coefficients. What could be the complex conjugate of a mark-a-place?

### 4.2 Algorithmic in the rantanplan

1. Theorem. The fraction field $K$ of an integral ring $A$ is constructed by identifying among themselves all the couples of $A \times A \setminus \{(0,0)\}$ which share the same alignment with the origin $(0,0)$. Whatever it has been said, this projectification $K = \mathbb{P}_A(A^2) \setminus \{\infty\}$ is and remains on the Agrégation program, and when you have that, you have quite everything else.

2. Softer version. In order to not frighten the school children, we can also say that when two points of the rantanplan are aligned with the origin, the coordinates are proportional, and the cross difference is zero, i.e. $x_{AB}y_B - x_{BY}A = 0$.

3. Thales theorem. When three points are three in number and aligned, the abscissa variations and ordinate variations are proportional. This can be written as:

$$(x_C - x_A)(y_B - y_A) - (x_B - x_A)(y_C - y_A) = 0$$

4. Definition (slope). The equation of a line is the condition for a third point $(x, y)$ to be aligned with two given points (two is two: the given points must be different). Once reorganized, the above expression can be written $y = px + m$, where $p$ is the slope, i.e. the proportionality ratio between the $\Delta y$ and the $\Delta x$.

5. Scholie. Beyond its limitations, the formula $y = px + m$ has the immense merit of characterizing the direction of a straight line by its slope, and even of characterizing straight lines as being the curves that keep going in a straight line, i.e. curves with a constant slope.

6. Algorithmic version of the Thales theorem. The stratospheric point of view consists in summarizing the Thales theorem by "we develop, we reorganize and we obtain $ax+by+c = 0$".

$^2$Alluding to the École Normale Supérieure
The algorithmic point of view consists in being interested in computations, to the point of trying to facilitate them. We have

\[ a = -yb + ya, \quad b = xB - xA, \quad c = xAyB - yAxB \]

In other words, the number \( c \) is the cross difference of the \( x \) and the \( y \). The result is known: the other two are also cross differences when using the "ever one" quantity. In other words, the equation of the line passing through two points is computed using:

\[
A \wedge B = \begin{pmatrix} x_A & x_B & x \\ y_A & y_B & y \\ 1 & 1 & 1 \end{pmatrix} = \begin{pmatrix} x_A \\ y_A \\ 1 \end{pmatrix} \wedge \begin{pmatrix} x_B \\ y_B \\ 1 \end{pmatrix} = [y_A \times 1 - y_B \times 1, x_B \times 1 - x_A \times 1, x_A y_B - y_A x_B]
\]

7. Duality. The cross-differences are put in a row because they characterize a straight line. It's not just a joke "line / row". We deliberately note the lines differently from the points ... quite simply because the lines are not points and the points are not lines\(^3\). If wanted, we could note the points in row, and the lines in column\(^4\). We could even exchange all the points with all the lines. We just have to find a way not to mix the two kinds of things.

8. From an advanced point of view, the equation of a line is written using a determinant and the wedge operator is the universal factorization of this multilinear operator:

\[
\begin{vmatrix} x_A & x_B & x \\ y_A & y_B & y \\ 1 & 1 & 1 \end{vmatrix} = \begin{pmatrix} x_A \\ y_A \\ 1 \end{pmatrix} \wedge \begin{pmatrix} x_B \\ y_B \\ 1 \end{pmatrix} = \begin{pmatrix} x \\ y \\ 1 \end{pmatrix}
\]

But, despite the possibility of such a stratospheric description, the fact remains that the cross difference method is a practical and fast way to calculate lines and intersections. This method can be taught and used well before learning any theory on large determinants. It is the same for the algorithm of the ordinary decimal division: it can be taught and used well before learning any theory on the limited expansions of a quotient of uniformly converging series.

9. Let us insist heavily on this fundamental point. The formula for subtracting fractions: \( \frac{a}{b} - \frac{c}{d} = \frac{ad - bc}{bd} \) is only saying that \((a, b)\) and \((ad, bd)\) are aligned with the origin. This is projective geometry. More precisely, projective geometry is nothing more than that: drawing the fractions that we want to subtract, then expand them to the same denominator. There is no reason to make a molehill out of such an elementary thing.

10. Example. We want to calculate \( E = AB \cap CD \) with \( A = 3 + 4i, B = 2 - 5i, C = -1 + i, D = 1 - i \). We have successively:

\[
\begin{pmatrix} 3 \\ 4 \\ 1 \end{pmatrix} \wedge \begin{pmatrix} 2 \\ -5 \\ 1 \end{pmatrix} = \begin{pmatrix} 9 & -1 & -23 \end{pmatrix}
\]

\[
\begin{pmatrix} -1 \\ 2 \\ 1 \end{pmatrix} \wedge \begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix} = \begin{pmatrix} 3 & 2 & -1 \end{pmatrix}
\]

\[
\begin{pmatrix} 9 & -1 & -23 \end{pmatrix} \wedge \begin{pmatrix} 3 & 2 & -1 \end{pmatrix} = \begin{pmatrix} 47 \\ -60 \\ 21 \end{pmatrix} \approx \begin{pmatrix} 47/21 \\ -20/7 \\ 1 \end{pmatrix}
\]

\(^3\)As the French joke says, les points ne sont point des droites, et les droites ne sont droite des points.

\(^4\)Cross over the Chanel, they behave like that.

— pdlx : Translation of the Kimberling’s Glossary into barycentrics ——
11. Example (continued). Using Morley’s affixes leads to:

\[
\begin{pmatrix} 3 + 4i \\ 1 \\ 3 - 4i \end{pmatrix} \wedge \begin{pmatrix} 2 - 5i \\ 1 \\ 2 + 5i \end{pmatrix} = \begin{pmatrix} -1 + 9i & -46i & 1 + 9i \\ 2 + 3i & -2i & -2 + 3i \end{pmatrix}
\]

Since complex conjugacy is an automorphism, the third component ends up being the conjugate of the first, since this was the case when starting. For now, the \( \mathbb{Z} \) component is merely a concession made to the fans of intrinsic constructions. Let’s show how to use it for a good reason.

4.3 Thales antiquadratic form

1. Fundamental formula for affine spaces. To arrive at \( B \), we start from some \( A \), then we follow the path that goes from \( A \) up to \( B \). We have:

\[
B = A + (B - A)
\]

2. Definition. The vector \( \overrightarrow{AB} \) is what we get by subtracting the normalized coordinates of \( A \) from the normalized coordinates of \( B \).

3. Scholie. Subtracting the upper coordinates would not give a well-defined object, since the upper coordinates of \( A \) can be multiplied by a factor different from that of the upper coordinates of \( B \).

4. Proposition. When we consider the vectors \( \overrightarrow{AB} \) for what they are, i.e. exactly defined objects (and not defined up to a factor), their set \( V \) forms a vector space of dimension 2.

5. Definition. If we assume that the vector \( \overrightarrow{AB} \) is non-zero, but if we disregard its size, only its direction remains, and we get a new kind of points, i.e. the points with \( T = 0 \). The set of the points which verify this equation is the line \([0; 1; 0]\). We write it \( L_z \).

6. Proposition. The point \( k \overrightarrow{AB} \) belongs to line \( AB \). In fact, this point is nothing else than \( AB \wedge L_z \). Proof: this result is obvious from the coordinates. The basic reason is that the subtraction of fractions also uses cross differences: don’t we have \( \mathbb{K} = \mathbb{P}_A (\mathbb{A}^2) \setminus \{\infty\} \)?

7. Operator \( W \). This is the operator which takes a straight line as input and gives its point at infinity as output. Using Morley’s affixes, we have:

\[
\Delta \wedge L_z = \left[ \begin{array}{c} W_z \\ 1 \end{array} \right] \cdot \Delta \quad \text{where} \quad \left[ \begin{array}{ccc} 0 & 0 & -1 \\ 0 & 0 & 0 \\ +1 & 0 & 0 \end{array} \right]
\]

8. Theorem (Thales). Two lines are parallel when the point at infinity of one of them belongs to the other. This is expressed by:

\[
\Delta_1 \parallel \Delta_2 \iff \Delta_1 \cdot \left[ \begin{array}{c} W_z \end{array} \right] \cdot \Delta_2 = 0
\]
4.4 Pythagoras quadratic form

1. Theorem (pons caballorum). The squared norm of a vector is computed using $|z|^2 = z \overline{z}$. For vectors of type $\overrightarrow{AB}$, this translates into:

$$|\overrightarrow{AB}|^2 = \overrightarrow{AB} \cdot \text{Pyth} \cdot \overrightarrow{AB}$$

where

$$\text{Pyth} = \frac{1}{2} \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix}$$

2. Proposition. Let $P$ a point at finite distance. Then there are two linear transformations $\psi$ such that:

(a) $\psi(P) = 0:0:0$

(b) $\psi(V) = V$

(c) for all $V \in V$, $\langle \psi(V) \mid V \rangle = 0$ while $\langle \psi(V) \mid \psi(V) \rangle = \langle V \mid V \rangle$.

Their characteristic polynomial is $\mu^3 + \mu$ and we have:

$$\psi = i \begin{pmatrix} +1 & -z/t & 0 \\ 0 & 0 & 0 \\ 0 & +z/t & -1 \end{pmatrix}$$

the other possibility being the opposite of the $\psi$ operator.

3. Definition. Operator $\text{OrtO}_z$. The action of one of the operators $\psi_P$ on the vector $1:0:1$ is the vector $+i:0:-i$. Therefore it corresponds to a direct rotation of a quarter turn for the observer from above (i.e. $Z$). The choice $P = O$ leads to the operator:

$$\text{OrtO}_z = i \begin{pmatrix} +1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -1 \end{pmatrix}$$

and we have $\text{OrtO}_z = 2i \text{orthodir}_z$.

4. Proposition. Considered as acting up to a factor, each of the operators $\psi$ and $\text{OrtO}_z$ sends a point at infinity on the point at infinity representing the orthogonal direction (orthopoint).

5. Remark. When $U \in L_z$, then $\Delta_U = 'U \cdot \text{Pyth}_z$ is the line of points $V$ such as $'U \cdot \text{Pyth}_z \cdot V = 0$. The point at infinity of $\Delta_U$ is $U' = W_z \cdot '\Delta_U = W_z \cdot \text{Pyth}_z \cdot U$. It is therefore convenient to choose $P$ in $O$, so that $\psi$ is proportional to $W_z \cdot \text{Pyth}_z$.

6. Operator $M$ (orthodir). This is the operator which takes a straight line as input and gives the orthopoint of its point at infinity as output. Using Morley’s affixes, we have:

$$\text{orthodir}(\Delta) = M_z \cdot '\Delta$$

where

$$M_z = \text{OrtO}_z \cdot W_z = \begin{pmatrix} 0 & 0 & -i \\ 0 & 0 & 0 \\ +i & 0 & 0 \end{pmatrix}$$

7. Theorem (Pythagoras). Two lines are perpendicular when the orthodir of one of them belongs to the other. This is expressed by:

$$\Delta_1 \perp \Delta_2 \iff \Delta_1 \cdot M_z \cdot \Delta_2 = 0$$
4.5 Tangent of an angle between two lines

1. Theorem. The tangent of the oriented angle determined by the lines $\Delta_1$ and $\Delta_2$ is obtained by dividing the antisymmetric form of Thales by the symmetric form of Pythagoras. In other words, we have:

$$\tan(\Delta_1, \Delta_2) = \frac{\Delta_1 \cdot [W] \cdot \Delta_2}{\Delta_1 \cdot [M] \cdot \Delta_2}$$

2. Remark. One of the interests of this formula is to provide a projective expression, i.e. an expression that resists to all these "definitions up to a factor". This is clear for the factors involving the $\Delta_j$. But this is also true for the transformation matrices. Indeed $[W]$ and $[M]$ are of the same type, namely "line to point", and are both transformed according to the same $X \mapsto aller \cdot X \cdot ^taller$ paradigm.

3. Elementary proof (the shorter, the better). We have:

$$(\begin{bmatrix} p + i; 2m; p - i \\ q + i; 2n; q - i \end{bmatrix}) \mapsto \frac{q - p}{1 + pq}$$

4. Everybody knows that tangents are making a group, which satisfies:

$$\tan(\Delta_1, \Delta_3) = \frac{\tan(\Delta_1, \Delta_2) + \tan(\Delta_2, \Delta_3)}{1 - \tan(\Delta_1, \Delta_2) \tan(\Delta_2, \Delta_3)}$$

with the usual homographic rules to manage the point $\infty \in \mathbb{C}$. And this applies even when the lines are not visible, and the tangents are complex numbers.

5. Note that the visible points at infinity of $\mathbb{P}_\mathbb{C}(\mathbb{C}^3)$, in other words the directions, are written $z : 0 : \overline{z}$ and therefore can be normalized to $\omega^2 : 0 : 1$. An angle between lines is represented by its double, which is an angle between vectors, so that the later can be represented by a point on the trigonometric circle.

6. Interpretation. When the President of the Jury raises his head to contemplate the infinity, he forms his opinion by dividing the points of view of his two Vice-Presidents, according to the formula $\omega^2 = z/\overline{z}$. And then, each of the quarter turns multiplies $\omega^2$ by $-1$. So intrinsic, says the President!
Chapter 5

Teaching Geometry to a computer

5.1 The random observer

When dealing with triangle geometry, we have to organize the coexistence of three kinds of objects. We have vectors, we have points and we have 3-tuples. A vector describes a translation of the "true plane". A vector has a direction but also a length and therefore is not defined "up to a proportionality factor". The set of all these vectors is a 2-dimensional vector space $V$ (more about it in what follows). A point is either an element of the "true plane", i.e. an ordinary point at finite distance, or a point at infinity describing the direction of some line. Such points have therefore to be described "up to a proportionality factor" by a column that belongs to a given copy of $P_{\mathbb{R}}(\mathbb{R}^3)$. In the same vein, lines are described "up to a proportionality factor" by a row that belongs to another copy $P_{\mathbb{R}}(\mathbb{R}^3)$.

And we need to talk with our computer in order to let it compute all the required results. These computations are done using 3-tuples and tools acting over 3-tuples so that computations must be described using $\mathbb{R}^3$ rather than using $V$ or $P_{\mathbb{R}}(\mathbb{R}^3)$. How to organize the coexistence of these three points of view is the object of a well-known theorem.

Theorem 5.1.1. Any affine space $E$ can be embedded into a vector space $\hat{E}$ in such a way that $E$ becomes an affine hyperplane of $\hat{E}$.

Proof. By definition, $E$ is not empty. Choose $\odot \in E$ (the random observer) and write $E = \odot + V$ where $V$ is the vector space associated with $E$. Choose $\odot \notin E$ and write $E = V \oplus \mathbb{R} \odot \odot$. Define $\zeta$ as the last coordinate in vector space $E$. Then $E = \{ m \in E \mid \zeta(m) = 1 \}$. Moreover the vector hyperplane $\{ m \in E \mid \zeta(m) = 0 \}$ is an isomorphic copy of $V$. ☐

Notation 5.1.2. An affine description of a point at finite distance $P$ is a 3-tuple $(\xi, \eta, \zeta)$ where $\zeta = 1$ is assumed. The semantic of these coordinates is the pre-existence of some random observer, that uses a Cartesian frame $(\xi, \eta)$ to describe what is occurring before her eyes.

5.2 Working out an example

Let us take an example and begin with an informal approach (cf http://www.les-mathematiques.net/phorum/read.php?78,585414,586289#msg-586289). We have a point $P$, given by a column, and two triangles $T_1$, $T_2$ given by the columns of their vertices:

\[
P = \frac{1}{2} \begin{pmatrix} 8 \\ 15 \\ 2 \end{pmatrix}; \quad T_1 = \begin{pmatrix} 7 & 3 & -2 \\ 9 & 9 & -3 \\ 1 & 1 & 1 \end{pmatrix}; \quad T_2 = \begin{pmatrix} -9 & 19 & 6 \\ 1 & -15 & 11 \\ 1 & 1 & 1 \end{pmatrix}
\]

As it should be, each column verifies $\zeta = 1$, which is the equation of the affine plane $E$ when seen as a subspace of $\mathbb{R}^3$.

We define $W$ as the matrix that transforms the matrix $T$ of a triangle $(P_j)$ into the matrix of the sideline vectors $(\overrightarrow{P_{j+1}P_{j+2}})$ of this triangle (indices are taken modulo 3 so that $P_4 = P_1$ etc).

And now, we compute $K = W \cdot T \cdot T \cdot W$ for both triangles. We have:
Describes any actor of the play. For example, we can take triangle \( L \) with the sides \( x, y, z \) of the ordinary affine euclidian plane. This embedding quadratic form depends on three arbitrary parameters. In fact, any other matrix that can be written as:

\[
\begin{pmatrix}
1 & 0 & 0 \\
0 & 1 & 0 \\
0 & 0 & 1
\end{pmatrix}
\]

defines the matrix \( M \) to belong to the kernel of matrix \( K \). The characteristic polynomial of this matrix is \( \chi(\mu) = \mu^3 - (a^2 + b^2 + c^2) \mu^2 + 12 S \mu^2 \). One eigenvalue is \( \mu = 0 \) and the other two are real (from symmetry of \( [K] \) and positive.

**Remark 5.2.1.** Matrix \( M \) used to compute the orthodir of a line is nothing but \( M = [K] \cdot 2S \).

A more "stratospheric", but nevertheless equivalent, definition for these Al-Kashi matrices would be:

\[
[K] \equiv [W] \cdot [\text{Pyth}_3] \cdot T \cdot [W]
\]

where \([\text{Pyth}_3]\) describes any \( \mathbb{R}^3 \)-quadratic form that embeds \((\xi, \eta) \mapsto \xi^2 + \eta^2\), the quadratic form of the ordinary affine euclidian plane. This embedding quadratic form depends on three arbitrary parameters since 6 coefficients are needed for dimension three, while only 3 are needed for dimension two.

### 5.3 An involved observer

Now, we will describe how things are looking when the observer is no more a random \( \mathcal{O} \) but rather an actor of the play. For example, we can take triangle \( T_1 \) as a new vector basis inside vector space \( \mathbb{R}^3 \) and calculate everything again using this new basis. From:

\[
\begin{pmatrix}
\xi \\
\eta \\
\zeta
\end{pmatrix} = T_1 \begin{pmatrix} x \\ y \\ z \end{pmatrix}
\]

it is clear that condition \( \zeta = 0 \) that shows that a 3-tuple belongs to \( \mathcal{V} \) becomes now \( x + y + z = 0 \). Defining \( \mathcal{L}_\infty = (1, 1, 1) \) this can be rewritten as \( \mathcal{L}_\infty \cdot [x, y, z] = 0 \). Seen "up to a proportionality factor" this will gives \( \mathcal{L}_\infty \cdot (x : y : z) = 0 \), i.e. the condition for \( x : y : z \in \mathbb{P}_2(\mathbb{R}^3) \) to belong to the line at infinity. But this is not our purpose for the moment.

The \( \mathbb{R}^3 \)-metric is now described by matrix \( [T_1 \cdot [\text{Pyth}_3] \cdot T_1] \). This matrix depends in turn on three arbitrary parameters. In fact, any other matrix that can be written as:

\[
[T_1 \cdot [\text{Pyth}_3] \cdot T_1 + \mathcal{L}_\infty + [U, \mathcal{L}_\infty]]
\]

where \([\text{Pyth}_3]\) is:

\[
\begin{pmatrix}
1 & 0 & 0 \\
0 & 1 & 0 \\
0 & 0 & 0
\end{pmatrix}
\]

using an arbitrary column \( U \) will be just as well to calculate the Pythagoras of \( \mathcal{V} \)-vector. A zero diagonal gives a more nice looking matrix, and is also more efficient for computing. Therefore we define the matrix \([\text{Pyth}]\) by this property and we obtain:

\[
[\text{Pyth}] = -\frac{1}{2} \begin{pmatrix}
0 & 16 & 225 \\
16 & 0 & 169 \\
225 & 169 & 0
\end{pmatrix} = -\frac{1}{2} \begin{pmatrix}
0 & c^2 & b^2 \\
c^2 & 0 & a^2 \\
b^2 & a^2 & 0
\end{pmatrix}
\]
Now, matrix $K_1$ can be computed using:

$$K_1 = W^T \cdot \text{Pyth} \cdot W = \begin{pmatrix} 169 & -189 & 20 \\ -189 & 225 & -36 \\ 20 & -36 & 16 \end{pmatrix}$$

while the coordinates of the other triangle and the extra point are transformed according to:

$$T_2 = T_1^{-1} \cdot T_2 = \frac{1}{24} \begin{pmatrix} -52 & 156 & 13 \\ 60 & -180 & 15 \\ 16 & 48 & -4 \end{pmatrix} ; P_{[1]} = T_1^{-1} \cdot P = \frac{1}{32} \begin{pmatrix} 13 \\ 15 \\ 4 \end{pmatrix}$$

And now, matrix $K_2$ can be computed using:

$$K_2 = W^T \cdot T_2 \cdot \text{Pythh} \cdot T_2 \cdot W = \begin{pmatrix} 845 & -65 & -780 \\ -65 & 325 & -260 \\ -780 & -260 & 1040 \end{pmatrix}$$

The quadratic form $\text{Pythh}_2$ can be obtained as above, or directly as:

$$\text{Pythh}_2 = -\frac{1}{2} \begin{pmatrix} 0 & \gamma^2 & \beta^2 \\ \gamma^2 & 0 & \alpha^2 \\ \beta^2 & \alpha^2 & 0 \end{pmatrix} = -\frac{1}{2} \begin{pmatrix} 0 & 1040 & 325 \\ 1040 & 0 & 845 \\ 325 & 845 & 0 \end{pmatrix}$$

### 5.4 From an involved observer to another one

All the preceding computations are standard ones. As long as points have no dependence relations all together, nothing else can be done. On the contrary, when points are constructed from each other, another point of view is more powerful. Taking $ABC$ as reference triangle, and describing all points by their barycentric coordinates, we obtain $P = 13 : 15 : 4, A' = -13 : 15 : 4, B' = 13 : -15 : 4, C' = 13 : 15 : -4$. If this is only a random coincidence, nothing more can be said.

On the contrary, if these points are really defined wrt $T_1$ as $P = a : b : c, A' = -a : b : c, B' = a : -b : c, C' = a : b : -c$, then things get more interesting. We have:

$$P_{[2]} = T_2^{-1} \cdot P = b + c - a : c + a - b : a + b - c$$

$$K_2 = \begin{pmatrix} \frac{4a^2bc}{(a-b+c)(b+a-c)} & -\frac{2abc}{b+a-c} & -\frac{2abc}{a-b+c} \\ -\frac{2abc}{b+a-c} & \frac{4ab^2c}{(b+c-a)(b+a-c)} & -\frac{2abc}{b+c-a} \\ -\frac{2abc}{a-b+c} & -\frac{2abc}{b+c-a} & \frac{4c^2ab}{(b+c-a)(a-b+c)} \end{pmatrix}$$

so that: $\{\alpha^2 = k \frac{4a^2bc}{(a-b+c)(b+a-c)}, \text{etc.} \}$

Since barycentrics are defined only "up to a proportionality factor", the former set of equations has only to be solved in $k, b, c$. This gives the ratios between the quantities $a, b, c$. One gets:

$$(a : b : c) \simeq (\alpha^2 (\beta^2 + \gamma^2 - \alpha^2) : \beta^2 (\gamma^2 + \alpha^2 - \beta^2) : \gamma^2 (\beta^2 + \alpha^2 - \gamma^2))$$

so that:

$$P_{[2]} = \frac{1}{\beta^2 + \gamma^2 - \alpha^2} : \frac{1}{\gamma^2 + \alpha^2 - \beta^2} : \frac{1}{\alpha^2 + \beta^2 - \gamma^2}$$

This results identifies the incenter of the reference triangle $ABC$ as the orthocenter of the excentral triangle. In the same vein, the circumcenter of $ABC$ can be identified as the nine-points center of $A'B'C'$. More details on this specific relation will be given in Subsection 22.7.

— **pldx**: Translation of the Kimberling’s Glossary into barycentrics —
5.5 Reducing up to a factor

Maple 5.5.1. Reducing "up to a factor" is a key feature in computer-aided projective geometry.

```maple
1: REDUCE := proc (qui_ :: {Matrix, Vector, list}) :: local qui
2: qui0:=convert(qui_, set) \ {0}
3: if qui0 = {} then return qui_ end if
4: qui:=FActor(convert(qui_, Vector)/qui0[1])
5: lili:=convert(qui, list)
6: nunu,dede:=map(simplify@numer,lili), map(simplify@denom,lili)
7: fac:=(lcm@op)(dede)
8: if type(nunu,list(numeric)) then
9: fac:=fac/(igcd@op)(nunu)
10: else
11: fac:=fac/(gcd@op)(nunu)
12: end if
13: qui:=qui * fac
14: qui:=qui * fac
15: if type(qui_, Vector) then
16: return qui
17: else if type(qui_, list) then
18: return convert(qui, list)
19: else
20: return LTr(Matrix(ColDim(qui_), RowDim(qui_), convert(qui,list)))
21: end if
```

Listing 5.1: The reduce procedure.

```maple
REDCOL := proc (ma :: Matrix)
Matrix([seq(reduce(Column(ma, j)), j = 1..ColDim(ma))])
```

Listing 5.2: The reducol procedure

```maple
REDUROW := proc (ma :: Matrix)
< seq(reduce(Row(ma, j)), j = 1..RowDim(ma)) >
```

Listing 5.3: The redurow procedure

```maple
WEDGE := proc (pp, pu) :: local p, q, r, u, v, w, tmp
p, q, r, u, v, w := op(convert(pp, list)), op(convert(pu, list))
tmp := [q * w - r * v, -p * w + r * u, p * v - q * u]
if type(pp, Vector[row]) then
return Vector(tmp)
else if type(pp, Vector) then
return Vector[row](tmp)
end if
```

Listing 5.4: The wedge procedure
5.6 packages

<table>
<thead>
<tr>
<th>Package</th>
<th>Lines</th>
</tr>
</thead>
<tbody>
<tr>
<td>logic.m</td>
<td>443</td>
</tr>
<tr>
<td>latex.m</td>
<td>834</td>
</tr>
<tr>
<td>geo2c.m</td>
<td>571</td>
</tr>
<tr>
<td>hyperbolic.m</td>
<td>1257</td>
</tr>
<tr>
<td>pldx.m</td>
<td>1268</td>
</tr>
<tr>
<td>homogr.m</td>
<td>1988</td>
</tr>
<tr>
<td>gone.m</td>
<td>2172</td>
</tr>
<tr>
<td>birap.m</td>
<td>3174</td>
</tr>
<tr>
<td>map2alg.m</td>
<td>3905</td>
</tr>
<tr>
<td>circles.m</td>
<td>4085</td>
</tr>
<tr>
<td>dessin.m</td>
<td>3056</td>
</tr>
<tr>
<td>quadlat2.m</td>
<td>7166</td>
</tr>
<tr>
<td>relire_sk.m</td>
<td>7223325</td>
</tr>
</tbody>
</table>

--- pldx : Translation of the Kimberling’s Glossary into barycentrics ---
Chapter 6

Maple procedures about searchkeys

Remark 6.0.1. At 2024-01-02, there were 61097 points subject to the curse of having coordinates in the Kimberling’s database. Too much for continuing to recompile the web pages and extract the formal coordinates of these points.

From 2019, I have decided to limit my "formal database" to the \( n \leq 6809 \) first points and to the 59 points having complex coordinates. In fact, only four of the later, namely the \( \{8072, 8073, 11065, 11066\} \) have been added.

On the other hand, the Kimberling (1998-2021) database can be used to obtain the "numerical barycentrics" of all the \( 6809 < n < 61098 \) remaining points (except from the 59 complex ones), allowing numerical explorations using triangle 6,9,13.

6.1 Procedure mkalgo

The result goes to \texttt{ipse/docs/maple/latex/proc.tex} and is read by \texttt{M-b d}. Import into an Algo box

\begin{verbatim}
1: CIRCLE3 := proc()
2: if ColDim(Args[1]) = 3 then
3: return Procname(Column(Args[1], 1..3))
4: end if
5: wedge3(seq((FActor@reduce@Ver)(jj), jj = Args))
6: 'if' (%[4] = 0, FActor(%), FActor(%/%[4]))
\end{verbatim}

Listing 6.1: circle3

manual: algorithmic before and after ; Ditto1()\to\% ; LColumn \to Column ; gestion des @ ;

6.2 Standardized barycentrics

Proposition 6.2.1. Standardized barycentrics are defined as follows :

\[
(x, y, z) \mapsto \begin{cases} 
(x, y, z) \times \frac{1}{x + y + z} & \text{if } x + y + z \neq 0 \\
(x, y, z) \times \left(\frac{1}{2} + \frac{1}{y} + \frac{1}{z}\right) & \text{otherwise}
\end{cases}
\]

They are defined for all points, except from the directions of the sidelines and the points \(-1 \pm i\sqrt{3} : 2\).

Remark 6.2.2. This quantity is projectively defined... but depends on a preference towards the barycentrics, rather than towards the trilinears.

Remark 6.2.3. The two exceptional points of the former proposition are not the so called umbilics, defined at Subsection 14.1.2.

Algorithm 6.2.4. \textsc{Normalize}. This procedure \textsc{Alg. 6.2} returns a normalized numerical vector (using triangle 6,9,13) from what is given on entry (a vector, a list or an expression).
Memory from the past: the former procedure *kicety* was combining this normalize procedure with the `make_a_key` instructions (which are now a part of procedure dichot).

| 1: NORMALIZE := proc (pu) ; local u, v, w, tmp, tmp1, tmp2 |
| 2: if type(pu, Vector) then tmp := subs(ency__, pu) |
| 3: else if type(pu, list) then tmp := Vector(subs(ency__, pu)) |
| 4: else tmp := Vector(subs(ency__ [rot3](pu))) |
| 5: end if |
| 6: indets(tmp) |
| 7: if %<> {} then Error("unresolved indets in normalize", %) end if |
| 8: u, v, w := tmp[1], tmp[2], tmp[3] |
| 9: if evalf(abs(u + v + w))/(abs(u) + abs(v) + abs(w))) < .1E-8 then |
| 10: return xpurge(evalf(tmp *(1/u + 1/v + 1/w))) |
| 11: else |
| 12: return xpurge(evalf(tmp/(u + v + w))) |
| 13: end if |

**Ensure:** Decides if $M \in \mathcal{L}_\infty$ and returns a normalized, numerical vector, in $\mathbb{C}^3$.

### 6.3 Numerical values

**Definition 6.3.1.** The *Kimberling’s search key* has been defined as the directed distance between $X$ and the BC sideline of the reference triangle $a = 6, b = 9, c = 13$ (for an ordinary point) and as $x' \times (1/x' + 1/y' + 1/z')$ for a point at infinity (caveat: here $x': y': z'$ are trilinears !). When using barycentrics, this quantity is given by:

$$
\begin{align*}
\text{key} = & x : y : z \notin \mathcal{L}_\infty, \quad \text{key} = \frac{x}{x + y + z} \times \frac{2S}{a} \\
\text{key} = & x : y : z \in \mathcal{L}_\infty, \quad \text{key} = \frac{x}{a} \times \left( \frac{a + b + c}{x + y + z} \right)
\end{align*}
$$

(6.1)

The actual value of factor $bc \div 2R = 2S/a$ is $4\sqrt{35}/3$. Reference values of this search key are provided by the ETC (Kimberling, 1998-2021).

**Definition 6.3.2.** The *patched search key* has been introduced to deal with points like X(5000) that would otherwise have a non real searchkey. When $x/(x + y + z) \notin \mathbb{R}$, then

$$
\text{key} = \arg \left( \frac{x}{x + y + z} \right) \in [-\pi, +\pi]
$$

If one uses Maple, arg($z$) is obtained by evalf (arctan (3z, Rz)). Using arctan (3z/Rz) would be wrong when Rz < 0. And evalf is required to avoid symbolic $\pi$.

**Example 6.3.3.** The key for the first umbilic, i.e. $S_b - 2iS : S_a + 2iS : -c^2$ (see Subsection 14.1.2 for more details about the umbilics) is obtained by:

$$
\left( \frac{1}{S_b - 2iS} + \frac{1}{S_a + 2iS} - \frac{1}{c^2} \right) \left( \begin{array}{c} S_b - 2iS \\ S_a + 2iS \\ -c^2 \end{array} \right) = \left( \begin{array}{c} 0.9541237489 \cdots -0.3042530621 \cdots i \\ 1.089086127 \cdots +1.034633282 \cdots i \\ -2.043209876 \cdots -0.7303802201 \cdots i \end{array} \right)
$$

$$
\text{key} = -0.308686862910087033702
$$

**Fact 6.3.4.** When repeating the same process with triangles 9,13,6 and 13,6,9, we obtain $\text{key}_a$, $\text{key}_b$, $\text{key}_c$. Then either $\sum a \text{key}_a = 0$ (point at infinity) or $\sum a \text{key}_a = 2S = 8\sqrt{35}$ (ordinary point)... or anything else for a may be complex point. About key conflicts:

1. At 2009-08-27, the minimal distance between two search keys was $4.8E - 7$. 

January 3, 2024 21:08 published under the GNU Free Documentation License
2. At 2017-01-11, this minimal distance was $2.95 \times 10^{-8}$. This can only become smaller as more points are added to the database. Therefore, one has to be careful when computing a search key, in order to face the possibility of huge canceling terms. Using Maple\texttt{Digits:=20} seems to be safe.

3. At 2017-12-27, point have appeared that share the same $A$–key.

<table>
<thead>
<tr>
<th>older</th>
<th>667</th>
<th>3239</th>
<th>3616</th>
<th>3617</th>
<th>3635</th>
<th>3875</th>
<th>5592</th>
</tr>
</thead>
<tbody>
<tr>
<td>newer</td>
<td>9780</td>
<td>8834</td>
<td>14078</td>
<td>15224</td>
<td>15519</td>
<td>7292</td>
<td>10896</td>
</tr>
</tbody>
</table>

Thus we can use $key_a + X \times 1E − 19$ to build the unique key required by a dichotomy, and then use $1E − 13$ as the blur limit for recognition of a key. And go back using a process that gives either $X$ or $[X_1, X_2]$.

4. At 2020-3-30, there were 24 pairs of points that share the same $A$–key. Moreover, there were now 4 pairs of points that have the same 6-9-13 barycentrics:

<table>
<thead>
<tr>
<th>nn</th>
<th>3635</th>
<th>4098</th>
<th>4691</th>
<th>22166</th>
</tr>
</thead>
<tbody>
<tr>
<td>alt</td>
<td>15519</td>
<td>24150</td>
<td>21267</td>
<td>22266</td>
</tr>
<tr>
<td>$x$</td>
<td>2</td>
<td>110</td>
<td>14</td>
<td>54</td>
</tr>
<tr>
<td>$y$</td>
<td>5</td>
<td>209</td>
<td>11</td>
<td>51</td>
</tr>
<tr>
<td>$z$</td>
<td>9</td>
<td>285</td>
<td>7</td>
<td>47</td>
</tr>
</tbody>
</table>

Therefore, a safe identification requires a special treatment for these points.

6.3.1 The new reliresk

**Algorithm 6.3.5. reliresk.** From 2019, the numerical database is pre-compiled in Maple format and is simply loaded as is (or even not loaded at all). We have fac47 = $8\sqrt{35}$, while current values (2024-01-02) are $smax = 53090; siz_{\_enc} = 53024$. As an example, we have

enc\_sort[20086] = [0.125000000000000000006623134309, (15519, 3635)]

| 1: reliresk := proc() : global fac47, smax, siz\_enc, sk, fk, enc\_sort |
| 2: read("$ipse/docs/Cherche/Geometry/maple/reliresk/pas_toujours/reliresk.m") |
| 3: "numerical database imported" |
| Ensure: sk[j] = [x,y,z]; fk[j] \in \{0, 1, 2\}; enc\_sort[J] = [key, qui] |

**Listing 6.3:** The reliresk procedure

6.4 The new buildsk

Let Ketc=https://faculty.evansville.edu/ck6/encyclopedia/ and download the sources

1. Create and go into ipse/docs/Cherche/Geometry/ETC_20xx/
2. Download Ketc/ETC.html and Ketc/ETCPartn.html as ETCPartNN.html for NN = 1..36
3. Download Ketc/Search$kki.html for $kki \in 6_9_13 9_13_6 13_6_9$ and then
4. grep "<tr align=right>" $qui | sed -e "s|<tr align=right><td>¶¶<td>¶¶<td>¶<td>¶¶<td>¶¶" > $kki.csv
5. All required Maple procedures are stored in the relire\_sk package.

**Algorithm 6.4.1. buildsk.** Start from the searchkeys given by ETC. Use them to identify the type of the point and set $fk[j]$. Then compute their normalized values and set $sj[k]$. Set the searchkey $xk[j]$ as $sk[j][1]+j \times 1E − 19$. Read the special file that deals with the few points using complex values (and overwrite what needs to be overwritten).
And then, compute the sorted version \texttt{enc\_sort} so that a key is linked to one or more values (duplicated lines are not deleted!).

\begin{verbatim}
76
6.4. The new buildsk

Ensure:

\begin{verbatim}
    global fac47, smax, sk, xk, fk, siz_enc, enc_sort
    local source, fd, tmp1, tmp2, grand, j6, j9, j13, js, jj, j, stamp, laps, tmp, lili

4: source := ipse/docs/Cheche/Geometry/ETC_2023/
5: Digits := 30 ; stamp := time() ; fac47 := evalf(8 * sqrt(35), 30)
6: grand := rand(1000..9999) * 1E - 24
7: sk := table() ; fk := table() ; xk := table()
8: try close(fd) catch : end try
9: fd := open(cat(source,"6_9_13.csv"), READ)
10: for j to qqqq do
11:     tmp := readline(fd) ;
12:     if \% = 0 then Break() end if
13:     tmp1, tmp2 := op sscanf(tmp,"%d;%a") ; sk[tmp1] := 6 * tmp2
14: end for
15: smax := -1 + j ; close(fd)
16: fd := open(cat(source,"9_13_6.csv"), READ)
17: for j to smax do
18:     tmp := readline(fd) ;
19:     if \% = 0 then Break() end if
20:     tmp1, tmp2 := op sscanf(tmp,"%d;%a") ; sk[tmp1] := sk[tmp1], 9 * tmp2
21: end for
22: close(fd) ; fd := open(cat(source,"13_6_9.csv"), READ)
23: for j to smax do
24:     tmp := readline(fd) ;
25:     if \% = 0 then Break() end if
26:     tmp1, tmp2 := op sscanf(tmp,"%d;%a") ;
27:     sk[tmp1] := sk[tmp1], 13 * tmp2
28: end for
29: close(fd)
30: for jj to smax do
32:     if abs(js) < .1e - 12 then
33:         fk[jj] := 1 ; js := 1/j6 + 1/j9 + 1/j13 ; sk[jj] := [j6 * js, j9 * js, j13 * js]
34:     else if abs(js - fac47) < .1e - 12 then
36:     else
38:     end if
40: end for
41: return smax

Ensure: tables sk, xk, fk are set ; fac47, smax, siz_enc, enc_sort
\end{verbatim}

Listing 6.4: Procedure buildsk
\end{verbatim}

Maple procedures about searchkeys

1. \textbf{buildencsort} := proc()  
   \textbf{Require:} sk\_plex requires the formal coordinates of the complex points
2. \textbf{global} fac47, smax, siz\_enc, enc\_sort, sk, zk, fk
3. \textbf{local} fd, j, jj, jjj, jk, tmp, ttmp, enc\_tmp
4. \textbf{fd} := open("ipse/public\_html/etc/sk\_plex.csv", READ)
5. for \textbf{j} to 20000 do
6. \textbf{ttmp} := readline(fd) 
7. if \% = 0 then Break() end if
8. \textbf{SubstituteRec}(ttmp, " ", "", ",", ",", ",", ",", ",") 
10. end for
11. \textbf{close(fd)} ; printf("read \%d corrections", \textbf{j} - 1) ; print()
12. \textbf{enc\_tmp} := Array(sort([seq([\textbf{zk}[\textbf{jj}], \textbf{jj} = 1..smax])]) ; \textbf{jj} := 1
13. for \textbf{j} to \textbf{smax} while \textbf{jj} <= \textbf{smax} do
14. \textbf{jk} := NULL
15. for \textbf{jjj} from \textbf{jj} to \textbf{smax} do
16. if \(1e-9 < \textbf{enc\_tmp}[\textbf{jjj},1] - \textbf{enc\_tmp}[\textbf{jj},1]\) then Break() end if
17. \textbf{jk} := \textbf{jk}, \textbf{jjj}
18. end for
20. \textbf{enc\_tmp}[\textbf{jj},2] := seq(\textbf{enc\_tmp}[\textbf{jjj},2], \textbf{jjj} = \textbf{jk}) ; \textbf{jj} := 1 + op(-1,[\textbf{jk}])
21. end for
22. \textbf{siz\_enc} := \textbf{j} - 1 ; \textbf{enc\_sort} := SubMatrix(\textbf{enc\_tmp},1..\textbf{siz\_enc},1..2)
23. return siz\_enc 
\textbf{Ensure:} create \textbf{enc\_sort} ; its size is \textbf{siz\_enc} < \textbf{smax}.

Listing 6.5: buildencsort

6.5 Complex points

\textbf{Algorithm 6.5.1. build\_sk\_plex}. The file \texttt{t6913.csv} is collected from ETC, and is not supposed to change on a daily basis. Required corrections are to be stored somewhere else, namely in the \texttt{sk\_plex} file. These corrections are computed from the barycentrics stored in \texttt{fdat}. It uses \texttt{normalize} and the key produced by \texttt{dichot}.

When writing these corrections, we have to be tricky, since Maple doesn’t have a native procedure for printing complex numbers. One can see that files \texttt{t6913.csv} and \texttt{sk\_plex} are not read/written the same way. The first one has a very strong syntax, and can be read by a customized \texttt{sscanf}. The second one is more convoluted (there is no specific format for reading complex numbers). Spaces and nothing else !

\textbf{Algorithm 6.5.2}. 

— pldx : Translation of the Kimberling’s Glossary into barycentrics —
6.5. Complex points

Listing 6.6: The writese procedure

```plaintext
1: BUILDSK := proc(qqq := 20000)
Ensure: Initializes the following global variables
2: global fac47, smax, siz_enc, sk, fkk, enc_sort
3: local j, jj, j6, j9, j13, js, ttmp, enc_tmp, tmp, xk, fd
4: Digits := 30: fac47 := evalf(8*sqrt(3)): sk, fkk := table(), table(), table()
5: fd := open("/server_root/etc/t66193.csv", "READ")
6: for smax := qqq do
7:   ttmp := readline(fd)
8:   if % = 0 then Break() end if
9:   jj, j6, j9, j13 := op( sscanf('%d', "%f", "%f")
10:  j6, j9, j13 := 6*j6, 9*j9, 13*j13 ; js := j6 + j9 + j13
11:  if abs(js) < 1E-12 then
14: else if abs(js-fac47) < 1E-12 then
16: else
18: end if
19: end for
20: smax := smax - 1 ; siz_enc := smax
21: close(fd); print("t66193 has been read: %d items", smax)
22: fd := open("/server_root/etc/sk_plex.csv", "READ")
23: for j to 10 do
24:   ttmp := readline(fd)
25:   if % = 0 then Break() end if
26:   SubstituteRec(ttmp, "", "", "", "", "", "", "", "", ""); tmp := sscanf('%d %a %a %a %a');
28: end for
29: close(fd); print("read %d corrections", j)
30: enc_tmp := [seq]([zk[jj] + 1E-18 * jj, jj], jj = 1..smax) ;
31: enc_sort := Array(sort(enc_tmp))
32: for jj to siz_enc - 1 do
33:   if 1E-9 < enc_sort[jj + 1] - enc_sort[jj, 1] then Next() end if
34:   tmp := enc_sort[jj, 2], enc_sort[jj + 1, 2]
35:   enc_sort[jj + 1, 2] := tmp ; enc_sort[jj, 2] := tmp
36: end for
```

Listing 6.7: The build_sk_plex procedure

```plaintext
BUILD_SK_PLEX := proc; local j, theplex, lefichier, fd, tmp, tmp2, tmp3, qq
theplex := seq('if'(fkk[j] = 2, j, NULL), j = 1..6802)
fd := open("/ipse/public_html/etc/sk_plex.csv", WRITE)
for j in theplex[1..5] do
  tmp := map(cat, normalize(parse(fdat[j])), 20)
  ttmp1 := op(convert(tmp, list))
  ttmp2 := (op@map)(op@[Re, Im], convert(tmp, list))
  dichot(ttmp[1]) ; ttmp3 := cat(key + j*B - 19, 20)
  try:
    fprintf(fd, "", %d", "%24.20f","%24.20f","%24.20f","%24.20f"
  catch:
    fprintf(fd, cat("", %d", ",%24.20f%+24.20f"$3, ",%24.20f"
  end try
end for
close(fd)
```
6.5.1 Procedures ency and dichot

```
ENCY := procqui ; local tm1, tm2, tmpd, norqui, rep
Require: receive a vector, or a list or an algebraic expression to be rotated
Ensure: fails with '?' or proves the existence and unicity of a specific match with the entry
Digits := max(Digits, 20) ; norqui := normalize(qui) ; tmpd := dichot(norqui[1])
if seq((j, max(map(abs, norqui - normalize(sk[j])))), j = tmpd[2])
rep := [seq('if'((j[2] < 1e-11, j[1], NULL)), j = %)]
if rep = [] then return '?' end if
return op(rep)
Listing 6.8: The ency procedure
```

```
DICHOT := procqui ; global key ; local u, o, m ensu, enso, ensm, eps
Require: receive qui ∈ cc, the normalized x-barycentric of some point
Ensure: fails with left<key<right or returns the key and either a point or a collision_list
if Im(qui) = 0 then key := qui else key := argument(qui) end if
ε := .1e - 7 ; u, o := 1, siz_enc
ensu, enso := enc_sort[u][1], enc_sort[o][1]
if enso < key then
  if |enso - key| < ε then return enc_sort[o] else return ?, [enso, key, infinity] end if
end if
if key < ensu then
  if |ensu - key| < ε then return enc_sort[u] else return '?', [-infinity, key, ensu] end if
end if
while 1 < o - u do
  m := floor(o/2 + u/2) ; ensm := enc_sort[m][1]
  if ensm < key then u := m ; ensu := ensm else o := m ; enso := ensm end if
end while
if |ensu - key| < ε then return enc_sort[u]
else if |enso - key| < ε then return enc_sort[o]
else return ?, [ensu, key, enso]
end if
Listing 6.9: The dichot procedure
```

*** descriptions of procedures like localize should be moved here ***

6.6 morley

Definition 6.6.1. The Morley's search key is the search key associated with a point defined by its Morley affix. Details and formula are provided in the corresponding chapter. See Proposition 15.3.10.
Chapter 7

Euclidian structure using barycentrics

Barycentric coordinates were intended to describe affine properties, i.e. properties that remains when points are moved freely. Therefore describing euclidian\(^1\) properties when using barycentrics is often presented as contradictory. We will show that, on the contrary, all the required properties can be described simply. The key fact is that orthogonality only depends on the directions of lines so that all what is really needed is a bijection that sends each point at infinity onto the point that characterizes the orthogonal direction.

7.1 Lengths and areas

Notation 7.1.1. In this section, \(T_0\) describes the reference triangle \(ABC\) in the affine plane, and \(T\) describes a generic triangle \(P_j\) (indices are dealt modulo 3, i.e. \(P_4 = P_1\), etc. In other words:

\[
T_0 = \begin{vmatrix}
\xi_a & \xi_b & \xi_c \\
\eta_a & \eta_b & \eta_c \\
1 & 1 & 1
\end{vmatrix},
T = \begin{vmatrix}
\xi_a & \xi_b & \xi_c \\
\eta_a & \eta_b & \eta_c \\
1 & 1 & 1
\end{vmatrix}
\]

(7.1)

We will use \(|BC| = a, |P_2P_3| = \alpha, etc together with \(S_a = (\beta^2 + \gamma^2 - \alpha^2) / 2, S_\alpha = (\beta^2 + \gamma^2 - \alpha^2) / 2, etc.\) In other words,

\[
(\xi_a - \xi_b)^2 + (\eta_a - \eta_b)^2 = c^2, etc
\]

(7.2)

Lemma 7.1.2. Let \(T\) be the matrix describing triangle \((P_j)\) wrt triangle \(ABC\), i.e. the matrix whose columns are the barycentrics \(p_i : q_i : r_i\) of the \(P_j\). Then:

\[
T = T_0^{-1} \cdot T.
\]

Proof. This is nothing but normalized \((T) = T_0^{-1} \cdot T.\)

Definition 7.1.3. Matrix \(W\). The matrix \(W\) is defined by:

\[
W = \begin{pmatrix}
-1 & 0 & 1 \\
0 & -1 & 0 \\
1 & 0 & -1
\end{pmatrix}
\]

(7.3)

Proposition 7.1.4. When a matrix \(T\), as defined in (7.1), gives the vertices \(P_j\) of a triangle, then \(T \cdot W \) gives the sideline vectors \(\overrightarrow{P_{j+1}P_{j+2}}\) of this triangle (using modulo 3 indices). On the other

\(^1\)Let us recall that computing figures, instead of figuring computations, is as far as possible of what the historical Euclide was doing in his Elements (see 7). Using the adjective Euclidean to describe the modern euclidian geometry would only be misleading.
hand, matrix $[W]$ can be used to compute the point at infinity $U \in \mathbb{P}_2(\mathbb{R}^3)$ of a line given by its barycentrics $\Delta$. In this case, we have:

$$U = \Delta \wedge \mathcal{L}_{\infty} = [W] \cdot \left(\begin{array}{c} \xi \\ \eta \\ \zeta \end{array}\right) \simeq \left(\begin{array}{c} \alpha^2 - S_\gamma \\ -S_\gamma \\ -S_\beta \end{array}\right)$$

Proof. The first part is obvious from $P_j P_{j+1} = P_{j+1} - P_j$ that holds when $P_j$ are 3-tuples in the $\zeta = 1$ plane. The second part is the very definition of the $\wedge$ operator.

Lemma 7.1.5. Matrix $K$. We have the following Al-Kashi formula:

$$[K] = [W] \cdot [T] \cdot [W] = \left(\begin{array}{ccc} \alpha^2 - S_\gamma & -S_\gamma & -S_\beta \\ -S_\gamma & \beta^2 - S_\alpha & \gamma^2 \\ -S_\beta & -S_\alpha & \gamma^2 \end{array}\right)$$

Proof. Diagonal elements are $\langle B\hat{C} | B\hat{C} \rangle$, etc and the others are $\langle B\hat{C} | C\hat{A} \rangle = - \langle \hat{C}\hat{B} | C\hat{A} \rangle$, etc.

Proposition 7.1.6. Let matrix $[T]$ define a (finite) triangle $\mathcal{T}$ with vertices $P_i = p_i : q_i : r_i$. Then area of $\mathcal{T}$ is given by:

$$S = \frac{\det[T]}{\prod (p_i + q_i + r_i)} \quad (7.4)$$

while the area $S$ of the reference triangle is given by the Heron formula:

$$S^2 = \frac{1}{16} (a + b + c)(b + c - a)(c + a - b)(a + b - c) \quad (7.5)$$

Proof. The first formula is obvious from Lemma 7.1.2 and the well-known formula

$$S = \frac{1}{2} \begin{vmatrix} \xi_a & \xi_b & \xi_c \\ \eta_a & \eta_b & \eta_c \\ 1 & 1 & 1 \end{vmatrix} \quad (7.6)$$

that gives the oriented area of a triangle. The denominator of (7.4) enforces the required invariance wrt multiplicative factors acting on barycentrics, and recognizes the fact that only triangles with finite vertices have an area.

As it should be, this formula acknowledges that area of $ABC$ is $S$. The Heron formula can be proved in many ways, one of them being:

$$4S^2 = |AB|^2 |AC|^2 \sin^2 A = b^2 e^2 - \langle \hat{A}\hat{B} | \hat{A}\hat{C} \rangle^2 = b^2 e^2 - S_a^2$$

Remark 7.1.7. A key point is that formula (7.4) is of first degree in $S$ : once the orientation of the reference triangle is chosen, all other orientations are fixed.

7.2 Embedded euclidian vector space

Definition 7.2.1. When points $P, Q$ at finite distance are given by their barycentrics $p : q : r$ and $u : v : w$, the embedded vector from $P$ to $U$ is defined as:

$$\vec{v} = \frac{U}{\mathcal{L}_{\infty} \cdot U} = \frac{P}{\mathcal{L}_{\infty} \cdot P} = \frac{1}{u + v + w} \left(\begin{array}{c} u \\ v \\ w \end{array}\right) = \frac{1}{p + q + r} \left(\begin{array}{c} p \\ q \\ r \end{array}\right) = \left(\begin{array}{c} \rho \\ \sigma \\ \tau \end{array}\right) \quad (7.7)$$

Proposition 7.2.2. Embedded vectors belong to the vector plane $\mathcal{V} : x + y + z = 0$, seen as a subspace of the Cartesian (non projective) vector space $\mathbb{R}^3$. These vectors obey to the usual Chasles rule:

$$\vec{v} = \vec{v} (P_1, P_2) = \vec{v} (P_1, P_3) + \vec{v} (P_2, P_3)$$

and space $\mathcal{V}$ is isomorphic to the usual vector space $\mathbb{R}^2$ where $P_1 P_2 = (\xi_2 - \xi_1, \eta_2 - \eta_1)$. 

January 3, 2024 21:08 published under the GNU Free Documentation License
Proposition 7.2.3. When using \((\xi, \eta, \zeta)\) coordinates, the metric of the usual euclidian plane \(V\) is described by matrix:

\[
\begin{pmatrix}
  1 & 0 & * \\
  0 & 1 & * \\
  * & * & *
\end{pmatrix}
\]

(7.8)

where * are placeholders for three arbitrary parameters. When using barycentrics (wrt triangle \(ABC\)), this matrix is replaced by \(t_0^t \cdot \text{Pyth}_3 \cdot t_0 + U \cdot \mathcal{L}_\infty + t(U \cdot \mathcal{L}_\infty)\)

(7.9)

can be used to define the metric of vector space \(V\).

Proof. This is the usual formula for a bilinear form when submitted to a change of basis. Independence from column \(U\) comes from \(L_\infty \cdot V = \{0\}\).

Theorem 7.2.4. Pythagoras theorem. Define the Pythagoras bilinear form by:

\[
\langle \overrightarrow{\text{vec}}_1, \overrightarrow{\text{vec}}_2 \rangle = t \overrightarrow{\text{vec}}_1 \cdot \text{Pyth} \cdot \overrightarrow{\text{vec}}_2
\]

where:

\[
\text{Pyth} = \frac{1}{2} \begin{pmatrix}
  0 & -c^2 & -b^2 \\
  -c^2 & 0 & -a^2 \\
  -b^2 & -a^2 & 0
\end{pmatrix}
\]

(7.10)

When restricted to \(V\), this form extends the transformation \(\mathbb{R}^2 \mapsto V : \overrightarrow{PQ} \mapsto \overrightarrow{\text{vec}}(P, Q)\) into an isomorphism of euclidian spaces. Therefore, the squared distance between two (finite) points of the triangle plane is given by:

\[
|PU|^2 = \langle P(U) \cdot \text{Pyth} \cdot \overrightarrow{\text{vec}}(P, U)
\]

(7.11)

Proof. Matrix \(\text{Pyth}\) is what is obtain when choosing column \(U\) in (7.9) in order to obtain a zero diagonal instead of using \(K\) directly. A less "stratospheric" proof is direct inspection. We have:

\[
\langle \overrightarrow{\text{vec}}(A, B) | \overrightarrow{\text{vec}}(A, B) \rangle = c^2 \quad \text{(etc)} \quad \text{and} \quad \langle \overrightarrow{\text{vec}}(A, B) | \overrightarrow{\text{vec}}(A, C) \rangle = S_a \quad \text{(etc)}
\]

Since \(S_a = bc \cos A\), property holds for a basis. And linearity extends the result to all the other vectors.

7.3 About circumcircle and infinity line

Definition 7.3.1. The power of a point \(X = x : y : z\) (at finite distance) with respect to the circle \(\Omega\) centered at \(P\) with radius \(R\) is:

\[
\text{power} (\Omega, X) = |PX|^2 - R^2
\]

Theorem 7.3.2. The power formula giving the \(\Omega\)-power of any point \(X = x : y : z\) from the power at the three vertices of the reference triangle is:

\[
\text{power} (\Omega, X) = \frac{ux + vy + wz}{x + y + z} - \frac{a^2yz + b^2xz + c^2xy}{(x + y + z)^2}
\]

(7.12)

Proof. Use (7.7) to obtain \(\overrightarrow{P^2}\) and then Theorem 7.2.4 to obtain \(\text{power} (\Omega, X)\). Substitute \(y = z = 0\) to obtain \(u\), etc. Then a simple subtraction leads to the required result.
**Definition 7.3.3.** The standard equation of the circumcircle is defined as:

$$\Gamma_{std}(x, y, z) = -\frac{a^2yz + b^2xz + c^2xy}{x + y + z}$$  \hspace{1cm} (7.13)

**Proposition 7.3.4.** The equation of any circle can be written as:

$$\Omega(x, y, z) = (ux + vy + wz) + \Gamma_{std}(x, y, r) = 0$$  \hspace{1cm} (7.14)

where $$x : y : z$$ is the arbitrary point and $$\Gamma_{std}$$ is the standard equation of the circumcircle, as defined just above.

**Proof.** Obvious from (7.12) and $$\text{power} (\Gamma, A) = 0$$, etc.

**Remark 7.3.5.** "Can be written" must be understood as "when required, multiply by $$x + y + z$$ and use polynomials". For more details, see Chapter 13.

**Corollary 7.3.6 (Heron).** Center and radius of the circumcircle are:

$$X_3 = a^2 (b^2 + c^2 - a^2) : b^2 (c^2 + a^2 - b^2) : c^2 (a^2 + b^2 - c^2)$$

$$R^2 = \frac{a^2b^2c^2}{(a + b + c)(a + b - c)(b + c - a)(c + a - b)}$$

**Computed Proof.** Direct elimination from $$\{ |XA|^2 = R^2, \text{etc} \}$$ and (7.7, 7.11).

**Proposition 7.3.7.** For a line $$\Delta = \rho : \sigma : \tau$$, not the infinity line, point $$Q = \Delta \land L_\infty$$ is on $$L_\infty$$ while $$U = \text{isogon} (Q)$$ is on the circumcircle. Therefore, rational parametrizations of the line at infinity and the circumcircle are:

$$L_\infty = \{ \sigma - \tau : \tau - \rho : \rho - \sigma \mid \rho : \sigma : \tau \neq L_\infty \}$$  \hspace{1cm} (7.15)

$$\text{circumcircle} = \{ \frac{a^2}{\sigma - \tau} : \frac{b^2}{\tau - \rho} : \frac{c^2}{\rho - \sigma} \mid \rho : \sigma : \tau \neq L_\infty \}$$  \hspace{1cm} (7.16)

**Proof.** Point $$Q$$ is the point at infinity of the line $$\rho x + \sigma y + \tau z = 0, U \in \Gamma$$ is obvious from (7.13) and bijectivity of $$Q \mapsto U$$ is clear.

**Remark 7.3.8.** Information conveyed by a 3-tuple like (7.7) is multiple. A first part is the direction of line $$PU$$, described –up to a proportionality factor– by the point $$\rho : \sigma : \tau \in L_\infty$$. Another part is the squared length $$|PU|^2$$ given by (7.11). In this formula, circumcircle appears as the conic that defines how lengths are computed in each direction.

### 7.4 Orthogonality

#### 7.4.1 Matrices and formulas

**Theorem 7.4.1.** Orthopoint. A point at infinity $$V \in L_\infty$$ defines a direction of lines. The point $$W \in L_\infty$$ that defines the orthogonal direction is called the orthopoint of $$V$$. And we have $$W \simeq \left[\text{OrtO}\right] \cdot V$$ where

$$\left[\text{OrtO}\right] = \frac{1}{2S} \begin{bmatrix} \left[Pyth\right] W \end{bmatrix} = \frac{1}{4S} \begin{bmatrix} c^2 - b^2 & -a^2 & a^2 \\ b^2 & a^2 - c^2 & -b^2 \\ -c^2 & b^2 & a^2 - c^2 \end{bmatrix}$$  \hspace{1cm} (7.17)

**Proof.** Straightforward computation: when $$V \in \mathcal{V}$$, we have indeed

$$\left[V \left[Pyth\right] \left[\text{OrtO}\right]\right] = 0$$
Remark 7.4.2. As discussed in the next section, there are many "formal rules" like:

\[
\text{orthopoint}(V) = \frac{1}{4S} \mathcal{L}_\infty \wedge \left( V \div b \right) X_4
\]

\[
= \frac{1}{4S} \left( V \wedge X_4 \right) \div b X_4
\]

\[
\text{OrtH} = \frac{1}{2S} \begin{bmatrix}
0 & -S_b & +S_c \\
+S_a & 0 & -S_c \\
-S_a & +S_b & 0
\end{bmatrix}
\]

They all provide the same result when \( V \in \mathcal{V} \).

Proposition 7.4.3. Matrix \( \text{OrtO} \) describes the \(+90^\circ\) rotation in the \( \mathcal{V} \) space, while its transpose matrix \( W \cdot \text{Pyth} \div 2S \) describes the \(-90^\circ\) rotation. Spoiler: this provides \( \Omega_y \mapsto +i\Omega_y \).

Proof. Factor \( 1/2S \) was precisely chosen to enforce the conservation of length, as it could be easily checked. We have indeed

\[
^t V \cdot \text{Pyth} \cdot V = ^t V \cdot \text{OrtO} \cdot \text{Pyth} \cdot \text{OrtO} \cdot V
\]

when \( V \). It remains to prove the orientation. Go back to the ordinary Cartesian coordinates by(7.1), substitute the squared sidelengths using (7.2), and obtain:

\[
T_0 \cdot \text{OrtO} \cdot T_0^{-1} = \begin{pmatrix}
0 & -1 & * \\
1 & 0 & * \\
0 & 0 & *
\end{pmatrix}
\]

\[\square\]

Proposition 7.4.4. The orthodir \( U \) of any line \( \Delta \) (except from the line at infinity) is defined as the orthopoint of \( \Delta \wedge \mathcal{L}_\infty \). It can be computed as \( U = \mathcal{M} \cdot ^t \Delta \) where:

\[
\mathcal{M} \equiv \text{OrtO} \cdot W = \frac{1}{2S} \left( W \cdot \text{Pyth} \cdot W \right) = \frac{1}{2S} \begin{bmatrix}
a^2 & -S_c & -S_b \\
-S_c & b^2 & -S_a \\
-S_b & -S_a & c^2
\end{bmatrix} \quad (7.18)
\]

Proof. This comes directly from the orthopoint formula. Normalization factor \( 1/2S \) is useless here, but will be required when measuring, i.e. computing angles or distances. One can check that, for example, the first column gives the direction of the first altitude (orthogonal to sideline \( BC \)).

Remark 7.4.5. Characteristic polynomial of \( \mathcal{M} \) is:

\[
\chi(\mu) = \mu^3 + \mu^2 S_a / S + 3\mu
\]

and it can be checked that its left null space is \([1, 1, 1]\), the row associated with \( \mathcal{L}_\infty \) : for any column \( X \), \( \mathcal{M} \cdot X \in \mathcal{L}_\infty \) (as it should be).

Proposition 7.4.6. Isotropic lines. A line \([f, g, h]\) that satisfies \( \Delta \cdot \mathcal{M} \cdot ^t \Delta = 0 \) is orthogonal to itself, whence the name ‘isotropic’ given to these lines. This equation can be factored:

\[
\Delta \cdot \mathcal{M} \cdot ^t \Delta = \frac{1}{2S} \left( a^2 f^2 + b^2 g^2 + c^2 h^2 - 2 S_a gh - 2 S_b hf - 2 S_c fg \right)
\]

\[
\simeq (\Delta \cdot \Omega_x) \times (\Delta \cdot \Omega_y)
\]

so that an isotropic line is a line that goes through one of the so-called umbilics of the plane, whose barycentrics are:

\[
\Omega_x \simeq \begin{pmatrix}
S_b - 2iS \\
S_a + 2iS \\
-c^2
\end{pmatrix} ; \quad \Omega_x \simeq \begin{pmatrix}
S_b - 2iS \\
S_a + 2iS \\
-c^2
\end{pmatrix}
\]

pldx : Translation of the Kimberling’s Glossary into barycentrics

— —
Proof. The possibility of this factorization comes from det $M = 0$. Let us proceed by undetermined coefficients wrt $f, g, h$. Eliminating $u : v : w$ in the equation:

$$\Delta \cdot M \cdot ^t\Delta = (fp + gq + hr) (fu + gv + hw)$$

leads to $u = a^2/p, v = b^2/q, w = c^2/r$ (i.e. these points are isogonal conjugate of each other). And then solving in $q, r$ leads to the given values. The appearance of these results is rather not symmetric... but

$$\Omega_x \simeq \begin{pmatrix}
(b^4 + c^4 - a^2b^2 - a^2c^2) a^2 + 4 i (b^2 - c^2) a^2 S
(c^4 + a^4 - a^2b^2 - b^2c^2) b^2 + 4 i (c^2 - a^2) b^2 S
(a^4 + b^4 - a^2c^2 - b^2c^2) c^2 + 4 i (a^2 - b^2) c^2 S
\end{pmatrix}$$

is rather too complex to be useful. \hfill \square

Remark 7.4.7. Spoiler: all the isotropic lines form a 'degenerate tangential conic' (see Proposition 12.4.2). Therefore adjoint of matrix $M$ is $^t\mathcal{L}_\infty \cdot \mathcal{L}_\infty$ (the "all ones" matrix) and describes the line $\Omega_x, \Omega_y$, i.e. $\mathcal{L}_\infty$. Rank is one (since $M$ has rank $n - 1$), so that product $\text{Adjoint}(M) \cdot M$ is the null matrix.

7.4.2 Discussion

As stated in Proposition 7.2.3, any matrix

$$\begin{align*}
\begin{pmatrix}
\text{Pyth}_U
\end{pmatrix} &= \begin{pmatrix}
\text{Pyth} + \frac{1}{2} \left(U \cdot \mathcal{L}_\infty + ^t(U \cdot \mathcal{L}_\infty)\right)
\end{pmatrix}
\end{align*}$$

can be used to define the euclidian metric of the $V$ vector space. Let us discuss this "degree of freedom" and note $U \simeq u : v : w$.

Proposition 7.4.8. Isotropic vectors wrt $\begin{pmatrix}\text{Pyth}_U\end{pmatrix}$ are related to the points of a visible circle $C_U$. This circle is characterized by power $(C_U, A) = u$, etc.

Proof. Obvious from the equation $^tX \cdot \begin{pmatrix}\text{Pyth}_U\end{pmatrix} \cdot X = 0$ and the power formula (7.12). \hfill \square

Remark 7.4.9. Spoiler: the Veronese representation of circle $C_U$ is therefore $u : v : w : 1$. Remember:

$$\begin{pmatrix}
(x : y : z)
\end{pmatrix} \cdot \begin{pmatrix}
u
v
w
1
\end{pmatrix} = 0 \iff (a^2yz + b^2zx + c^2xy) + (x + y + z) (ux + vy + wz) = 0$$

And therefore, any circle can do the job (when choosing the correct normalization factor). The center and the radius of $C_U$ can be chosen at will.

Example 7.4.10. The choice $U = 0 : 0 : 0$ leads to $C_0 = \Gamma = (O, R)$, which is nothing else than the circumscribed circle.

Example 7.4.11. Requiring a diagonal form for $\begin{pmatrix}\text{Pyth}_U\end{pmatrix}$ leads to

$$\begin{pmatrix}\text{Pyth}_H\end{pmatrix} = \begin{pmatrix}
S_a & 0 & 0 \\
0 & S_b & 0 \\
0 & 0 & S_c
\end{pmatrix}$$

that uses the polar circle $C_H = (H: \sqrt{-S_aS_bS_c + 2S})$ as isotropic circle (see Section 13.7).

Example 7.4.12. Spoiler: the choice given by the change of basis formula applied to the Morley formula $ZZ = 0$ leads to the $C_M = (O; 0) = (O)$, the point circle centered at $X(3)$:

$$C_M = \frac{1}{2} \begin{pmatrix}
2R^2 & 2R^2 - c^2 & 2R^2 - b^2 \\
2R^2 - c^2 & 2R^2 & 2R^2 - a^2 \\
2R^2 - b^2 & 2R^2 - a^2 & 2R^2
\end{pmatrix}$$
Proposition 7.4.13. Let $P = (p, q, r) \in \mathbb{R}^3$ be a 3-ple that does not belong to $\mathcal{V}$. Then it exists a linear transform $\psi$ such that (i) $\psi(P) = 0$, (ii) $\psi(\mathcal{V}) = \mathcal{V}$ (iii) for all $V \in \mathcal{V}$, $\langle \psi(V) | \psi(V) \rangle = \langle V | V \rangle$ while $\langle \psi(V) | V \rangle = 0$. Then its characteristic polynomial is $\chi(\mu) = \mu^3 + \mu$ and we have:

$$\text{Orth}(P) = \frac{1}{2(p + q + r)} S \begin{pmatrix} qS_b - rS_c & -(ra^2 + pS_b) & ga^2 + pS_c \\ rb^2 + qS_a & rS_a - pS_a & -(pb^2 + qS_c) \\ -(qc^2 + rS_b) & pc^2 + rS_b & pS_a - qS_b \end{pmatrix}$$

The opposite of this matrix is the only other solution to the problem.

**Computed Proof.** Assertions $M \cdot P = 0$, $\mathcal{L}_\infty \cdot M = 0$ and $\langle \psi(V) | V \rangle = 0$ when $V = x : y : -x - y$ gives nine equations. Elimination leads to the given matrix, up to a coefficient. Then $\langle \psi(V) | \psi(V) \rangle = \langle V | V \rangle$ gives the coefficient. Division by $p + q + r$ enforces the fact that $P$ is at finite distance.

Proposition 7.4.14. For any $U$, matrix $\text{Orth}(U) \doteq \frac{1}{16} \begin{pmatrix} W_1 & \text{Pyth}(U) \end{pmatrix}$ is equal to $\text{Orth}(P)$ where $P$ is the center of the isotropic circle $C_U$. Thus, matrices $\text{Orth}(P)$ and $\text{Orth}(U)$ relative to finite points $P, U$ are related by the "translation" formula:

$$\text{Orth}(P) = \text{Orth}(U) \cdot \left(1 - \frac{1}{p + q + r} \begin{pmatrix} p \\ q \\ r \end{pmatrix} \cdot \mathcal{L}_\infty \right)$$

**Proof.** Acting at infinity, both matrices induces a quarter-turn. At finite distance, it remains only to move the kernel to the right place.

**Remark 7.4.15.** Any $\text{Orth}(P)$ matrix describes a quarter-turn in the euclidian plane $\mathcal{V}$. In order to provide a better perception of this result, let $O = X(3)$, $H = X(4)$, put $W_3^2 = 16 \|OH\|^2 S^2 = \sum_4 a^6 - \sum a^4b^2 + 3a^2b^2c^2$ and consider the linear transform (collineation) whose matrix is $\phi = [X(3), X(30) / 4S, X(523)]$. When computing $|OH|^2$, vector $OH$ is involved and therefore $X(30)$ (the direction of the Euler line), while $X(523)$ is known to be the direction orthogonal to the Euler line. We have:

$$\phi = \begin{pmatrix} a^2 (b^2 + c^2 - a^2) & 2a^4 - (b^2 - c^2)^2 - a^2 (b^2 + c^2) & b^2 - c^2 \\ b^2 (c^2 + a^2 - b^2) & 2b^4 - (c^2 - a^2)^2 - b^2 (c^2 + a^2) & c^2 - a^2 \\ c^2 (a^2 + b^2 - c^2) & 2c^4 - (a^2 - b^2)^2 - c^2 (a^2 + b^2) & a^2 - b^2 \end{pmatrix}$$

$$\phi^{-1} \cdot \text{Orth} \cdot \phi = \begin{pmatrix} -16a^2b^2c^2S^2 & 0 & 0 \\ 0 & W_3^2 & 0 \\ 0 & 0 & W_3^2 \end{pmatrix}$$

$$\phi^{-1} \cdot \text{Orth} \cdot \phi = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & -1 & 0 \end{pmatrix}$$

$$\phi^{-1} \cdot \text{Orth} \cdot \phi = \begin{pmatrix} 0 & 0 & 0 \\ 0 & W_3^2 & 0 \\ 0 & 0 & W_3^2 \end{pmatrix}$$

**Remark 7.4.16.** Mind the signs in $[M]$! The following matrix

$$\begin{pmatrix} a^2 & S_c & S_b \\ S_c & b^2 & S_a \\ S_b & S_a & c^2 \end{pmatrix}$$

describes the triangle of the midpoints of the altitudes. And also the Longchamps circle $\text{13.8}$ (see $12.21.17$).
7.5 Angles between straight lines

Proposition 7.5.1. Let \( P, U_1, U_2 \) be three points at finite distance, such that \( \overrightarrow{PU_i} \neq \overrightarrow{0} \). Then :

\[
|PU_1| \cdot |PU_2| \cdot \sin \left( \overrightarrow{PU_1}, \overrightarrow{PU_2} \right) = 2S \frac{\det (P, U_1, U_2)}{(p + q + r)} \prod (u_i + v_i + w_i)
\]

\[
|PU_1| \cdot |PU_2| \cdot \cos \left( \overrightarrow{PU_1}, \overrightarrow{PU_2} \right) = 2S \frac{(P \wedge U_1) \cdot t(P \wedge U_2)}{(p + q + r)^2} \prod (u_i + v_i + w_i)
\]

Proof. The sin formula comes from (7.4), while the cos formula can be obtained by using (7.11) into \( |PU_1|^2 + |PU_2|^2 - |U_1U_2|^2 \) and rearranging.

Theorem 7.5.2. Let \( P \) be a point at finite distance, and \( U_1, U_2 \) two other points (at finite distance or not). Then the angle between straight lines \( PU_1, PU_2 \) is characterized by its tangent, according to :

\[
\tan \left( \overrightarrow{PU_1}, \overrightarrow{PU_2} \right) = (p + q + r) \frac{\det (P, U_1, U_2)}{(P \wedge U_1) \cdot [M] \cdot t(P \wedge U_2)}
\]

(7.19)

When \( U_1, U_2 \) are at infinity, the angle between all the lines having the given directions can be computed as :

\[
\tan_{\infty} \left( \overrightarrow{U_1}, \overrightarrow{U_2} \right) = \frac{2S (v_1 w_2 - w_1 v_2)}{(v_1 w_2 + w_1 v_2) S_0 + w_1 w_2 b^2 + v_1 v_2 c^2}
\]

Proof. The key point here is that formula (7.19) is square-root free. Extension to \( U_i \) at infinity is obtained by continuity after cancellation of the \( (u_i + v_i + w_i) \). Formula \( \tan_{\infty} \) is not formally symmetric (\( P = A \) has been used). But the \( u_i \) are nevertheless present since \( u_i = -v_i - w_i \).

Theorem 7.5.3. Tangent of two lines. If the triangle plane is oriented according to \( \overrightarrow{AB}, \overrightarrow{AC} = +A \), then the oriented angle from line \( \Delta_1 \) to line \( \Delta_2 \) is characterized by :

\[
\tan \left( \overrightarrow{\Delta_1}, \overrightarrow{\Delta_2} \right) = \frac{\Delta_1 \cdot [W] \cdot t \Delta_2}{\Delta_1 \cdot [M] \cdot t \Delta_2}
\]

(7.20)

where \([W]\) and \([M]\) are exactly as given in (7.3) and (7.18) (i.e. not up to a proportionality factor).

Proof. Simple use of \( \Delta_1 \wedge \Delta_2, \mathcal{L}_{\infty} \wedge \Delta_1, \mathcal{L}_{\infty} \wedge \Delta_2 \) in (7.19). Among other things, this formula tells us that \( \vartheta = 0 \) when each line contain the point at infinity of the other, while \( |\vartheta| = \pi/2 \) when each line contains the orthopoint of the other (formula is anti-symmetric).

Stratospheric proof. Start from the affine space \( \mathcal{E} \). Equations of both lines are \( a_j \xi + b_j \eta + c_j = 0 \) and their angle is given by :

\[
\tan \left( \overrightarrow{\Delta_1}, \overrightarrow{\Delta_2} \right) = \frac{a_1 b_2 - a_2 b_1}{a_1 a_2 + b_1 b_2} = \frac{\det (L_3, D_1, D_2)}{D_1 \cdot [\text{Orth}_3] \cdot t D_2}
\]

where \( L_3 = [0, 0, 1] \) and \([\text{Orth}_3]\) is the matrix of quadratic form \( \xi^2 + \eta^2 \) — precisely this one, without any of the extra terms used in (7.8). Taking now \( ABC \) for basis, we have \( \Delta_1 = D_1 \cdot T_0^{-1} \), inducing a factor \( 1/2S \) in the numerator. Let us now compare the following two expressions :

\[
T_0^{-1} = \frac{1}{2S} \begin{bmatrix} \eta_b - \eta_c & -\eta_a + \eta_c & \eta_a - \eta_b \\ -\xi_b + \xi_c & \xi_a - \xi_c & -\xi_a + \xi_b \\ \xi_b \eta_c - \xi_c \eta_b & -\xi_a \eta_c + \xi_c \eta_a & \xi_a \eta_b - \xi_b \eta_a \end{bmatrix}
\]

\[
T_0 \cdot [W] = \begin{bmatrix} -\xi_b + \xi_c & \xi_a - \xi_c & -\xi_a + \xi_b \\ -\eta_b + \eta_c & \eta_a - \eta_c & -\eta_a + \eta_b \\ 0 & 0 & 0 \end{bmatrix}
\]
### Table 7.1: All these matrices

<table>
<thead>
<tr>
<th>Name</th>
<th>#</th>
<th>Barycentrics</th>
<th>#</th>
<th>Morley</th>
<th>Det</th>
<th>Det*4iS/R^2</th>
</tr>
</thead>
<tbody>
<tr>
<td>( \text{det} )</td>
<td>2</td>
<td>([1;1;1])</td>
<td>(15.2)</td>
<td>([0;0;1])</td>
<td></td>
<td></td>
</tr>
</tbody>
</table>
| \( L_\infty \) | 3     | (7.3) \[
\begin{pmatrix}
0 & 1 & -1 \\
-1 & 0 & 1 \\
1 & -1 & 0
\end{pmatrix}
\] | (15.3)| \([0;0;0]\)  | \([1+1;0;0]\) | \(mWW\) | \(zWW*4iS/R^2\) |
| Pythagoras    | 5     | (7.10) \[
\begin{pmatrix}
0 & -c^2 & -b^2 \\
-c^2 & 0 & -a^2 \\
-b^2 & -a^2 & 0
\end{pmatrix}
\] | (15.4) | \([R^2/2;0;0;1]\) | \([0;0;1]\) | \([0;0;0]\) | \(Pyth\) | \(zPyth*R^2\) |
| Orthopoint    | 4     | (7.17) \[
\begin{pmatrix}
c^2 - b^2 & -a^2 & a^2 \\
b^2 & a^2 - c^2 & -b^2 \\
-c^2 & c^2 & b^2 - a^2
\end{pmatrix}
\] | (15.5) | \([i;0;0;0]\) | \([1;0;0;0]\) | \([0;0;0;1]\) | \(i\) | \(0;0;0;0\) | \(OrtO\) | \(zOrtO\) |
| Orthodir      | 3     | (7.18) \[
\begin{pmatrix}
a^2 - S_c & -S_b & -S_b \\
-S_c & b^2 & -S_a \\
-S_b & -S_a & -c^2
\end{pmatrix}
\] | (15.6) | \([-i;0;0;1]\) | \([0;0;0;1]\) | \([0;0;1;0]\) | \((-i)\) | \(0;0;0;0\) | \(mMM\) | \(zMM*4iS/R^2\) |

Note: The table shows the transformation of Euclidean structure using barycentrics, including the translation of Kimberling’s Glossary into barycentrics. The matrices are used to calculate the positions of various geometric entities such as orthopoles, orthodirs, and tangent of circles. The formulas for these matrices are derived from the given equations and are used to transform points and lines in the plane.
Due to the specific value of $\text{Orth}_3$, we can replace $T_0^{-1}$ by $T_0 \cdot [W]/2S$ in the change of basis formulas and obtain $\text{Orth}_3 \mapsto [K]/(2S)^2$. Using the orthodir matrix (instead of the Al-Kashi one) leads to a formula without remaining factors.

**Remark 7.5.4.** Rotations are examined at 14.2

### 7.6 Distance from a point to a line

**Definition 7.6.1.** The distance of point $P = p : q : r$ to line $\Delta = (\rho, \sigma, \tau)$ is the lower bound of the distance of $P$ to a point $U$ that belongs to $\Delta$. By continuity, this bound is attained and is equal to the distance of $P$ to its orthogonal projection $P_0$ on $\Delta$.

**Theorem 7.6.2.** Distance from point $P = p : q : r$ to line $\Delta = (\rho, \sigma, \tau)$ is given by:

$$\text{dist} (P, \Delta) = \frac{\sqrt{2S} \Delta \cdot P}{(\mathcal{L}_\infty \cdot P) \sqrt{\Delta \cdot [M] \cdot t \Delta}}$$  \hspace{1cm} (7.21)

where $\mathcal{L}_\infty = [1, 1, 1]$ and $[M]$ is as given in (7.18) (not up to a proportionality factor). **Spoiler:** when using Morley affixes, we have:

$$\text{dist} \left( \begin{pmatrix} z \\ t \\ \zeta \end{pmatrix}, [u, v, w] \right) = \frac{1}{2} \frac{uz + vt + w\zeta}{t \sqrt{uw}}$$  \hspace{1cm} (7.22)

**Proof.** One obtains easily:

$$|PP_0|^2 = \frac{4S^2 (\rho p + \sigma q + \tau r)^2}{(p + q + r)^2 (a^2 p^2 + b^2 q^2 + c^2 r^2 - 2 \sigma \tau S_a - 2 \rho \tau S_b - 2 \rho \sigma S_c)}$$

where each factor is recognizable (cf Theorem 7.5.3). **Spoiler:** same proof.

**Remark 7.6.3.** Formula (7.21) is invariant when barycentrics of $P$ or $\Delta$ are modified by a proportionality factor. Denominators are enforcing the fact that $P$ is supposed to be finite, and $\Delta$ is not supposed to be the infinity line. The square root is the operator norm of the application $P \mapsto (\rho p + \sigma q + \tau r) / \sqrt{x + y + z}$. And finally $\sqrt{2S}$ is the square root of the standard area, i.e. the standard unit of length.

**Proposition 7.6.4.** The distance between two parallel lines is defined as the distance from one of the lines to a point of the other. We have the formula:

$$\text{dist} (\Delta, \Delta + \mu \mathcal{L}_\infty) = |\mu| \frac{\sqrt{2S}}{\sqrt{\Delta \cdot [M] \cdot t \Delta}}$$

**Proof.** Obvious from (7.21).

### 7.7 Brocard points and the sequel

#### 7.7.1 Some results

**Proposition 7.7.1.** Brocard points. It exists exactly one point $\omega^+$ and one point $\omega^-$ such that:

$$\angle (A \omega^+, AC) = \angle (B \omega^+, BA) = \angle (C \omega^+, CB)$$

$$\angle (AB, A \omega^-) = \angle (BC, B \omega^-) = \angle (CA, C \omega^-)$$

They are given by $\omega^+ = a^2 b^2 : b^2 c^2 : c^2 a^2$ and $\omega^- = c^2 a^2 : a^2 b^2 : b^2 c^2$. Moreover, when defined exactly that way, both angles are equal. This quantity is called the Brocard angle and one has:

$$\cot \omega = \frac{a^2 + b^2 + c^2}{4S}$$  \hspace{1cm} (7.23)
Proof. Equating the tangents of the angles and eliminating, one obtains a third degree equation with one simple real root and the others that involves \( \sqrt{-S^2} \). As it should be Brocard points are isogonal conjugates of each other.

**Proposition 7.7.2. cot versus Conway.** We have the following equalities between Conway symbols and some cotangents:

\[
[S_a, S_b, S_c, S_\omega] = [2S \cot A, 2S \cot B, 2S \cot C, 2S \cot \omega]
\]

\[
\cot \left( \frac{A}{2} \right) = \frac{bc + S_a}{2S} ; \quad \cot \left( \frac{B}{2} \right) = \frac{ac + S_b}{2S} ; \quad \cot \left( \frac{C}{2} \right) = \frac{ab + S_c}{2S}
\]

Proof. First three are obvious, the \( \omega \) one is given just above. This proves the Volenec (2005) formula:

\[
\cot \omega = \cot A + \cot B + \cot C
\]

**Remark 7.7.3.** A subsection about the so-called Brocard triangles is located at Subsection 22.9.

**Remark 7.7.4.** The ETC points on the Brocard line are 9 (Mittenpunkt), 512 (at infinity), 881, 882, 2524, 2531.

**Lemma 7.7.5.** Since \( \tan \omega > 0 \) and \( |\omega| \leq \pi/6 \), we have:

\[
\cos (\omega) = \frac{c^2 + a^2 + b^2}{2\sqrt{a^2b^2 + a^2c^2 + b^2c^2}} , \quad \sin(\omega) = \frac{2S}{\sqrt{a^2b^2 + a^2c^2 + b^2c^2}} \quad (7.24)
\]

**Fact 7.7.6.** Each Brocard point is at the intersection of three isogonal circles, according to:

\[
\angle (\omega^+ B, \omega^+ C) = \angle (BA, BC) \\
\angle (\omega^- B, \omega^- C) = \angle (CB, CA)
\]

**Proposition 7.7.7. Neuberg circles.** The locus of \( A \) when \( B, C, \omega \) are given is a circle (Neuberg circle of vertex \( A \)). Barycentric equation, center and radius are:

\[
a^2yz + b^2xz + c^2xy - a^2(x + y + z)(y + z) = 0
\]

\[
N_a \approx \left( \begin{array}{c} a^2 \left( c^2 + a^2 + b^2 \right) \\ (a^2 + b^2) c^2 - b^4 - a^4 \\ (a^2 + c^2) b^2 - c^4 - a^4 \end{array} \right) \approx \left( \begin{array}{c} -a \cos (\omega) \\ b \cos (C + \omega) \\ c \cos (B + \omega) \end{array} \right)
\]

\[
\rho_A = \frac{a}{2} \sqrt{\cot^2 \omega - 3}
\]

Proof. Straightforward computation, replacing \( b^2 \) by \( |A'C|^2 \) etc. The form given proves the circular shape.

**Proposition 7.7.8. Tarry point.** Triangle \( N_aN_bN_c \) is perspective with \( ABC \) and perspector is the Tarry point \( X(98) \):

\[
\frac{1}{a^2b^2 + a^2c^2 - b^4 - c^4} : \frac{1}{b^2c^2 + b^2a^2 - c^4 - a^4} : \frac{1}{c^2a^2 + c^2b^2 - a^4 + b^4}
\]

Conversely, Neuberg center \( N_a \) is common point of the \( A \) cevian of \( X_{98} \) and the perpendicular bisector of side \( BC \), etc.

Proof. Direct computation.

**Proposition 7.7.9. The Steiner angles \( \omega_1 > \omega_2 \) are defined as follows. \( 2\omega_1 \) is the maximal value of \( A \) when \( \omega \) is given and \( 2\omega_2 \) is the minimal value. We have the following relations:

\[
\cot 2\omega_j + 2/ \cot \omega_j = \cot \omega \\
\cot \omega_1 = \cot \omega - \sqrt{\cot^2 \omega - 3} \\
\cot \omega_2 = \cot \omega + \sqrt{\cot^2 \omega - 3} \\
\sin (2\omega_j + \omega) = 2 \sin \omega \\
\omega + \omega_1 + \omega_2 = \pi/2
\]
Proposition 7.7.10. Any Neuberg circle is viewed from another vertex under angle $2\vartheta$ where:

$$\cos \vartheta = 2 \sin \omega = \sin (2\omega_j + \omega)$$

Proof. The polar of $B$ cuts circle $N_\alpha$ in two points $T_1, T_2$ (equation of second degree, $\Delta = a^4 + b^4 + c^4 - a^2 b^2 - a^2 c^2 - b^2 c^2$). And we have $2\vartheta = \angle (BT_1, BT_2)$. A better choice is $T_0 = \text{midpoint}(T_1, T_2)$ and $\vartheta = \angle (BT_1, BT_0)$ ... taking orientation into account! \hfill \square

### 7.7.2 Results related to the Kiepert RH

This section has moved to Proposition 13.21.1.

### 7.7.3 Spoiler: study of the Neuberg pencil

Let us consider the level curves of the Brocard angle $\omega(M, B, C)$ when vertex $M$ moves in the $ABC$ plane. Barycentric coordinates are surely not the best system here, and using the Veronese map is probably not required. Therefore, this subsection should rather be considered as a spoiler related to the chapter devoted to pencils of cycles.

1. Let us note $M \simeq x : y : z$, $|MB| = \gamma$, $|MC| = \beta$ and $u/v = \tan \omega$. Then

$$\frac{u}{v} = \frac{a^2 + \beta^2 + \gamma^2}{4 S_M}$$

where $\beta^2 = \frac{z (x + z) a^2 + x (x + z) c^2 - b^2 x z}{(x + y + z)^2}$, etc. ; $S_M = \frac{x S}{x + y + z}$

2. It can be seen that numerator of $u/v - \cdots$ collects nicely, leading to equation

$$v \times \mathcal{C}(x, y, z) = 4 u S \times (x (x + y + z)) = 0$$

3. In the second term, one recognizes that $x (x + y + z)$ describes the line $BC$ seen as a cycle, i.e. completed by the line at infinity. On the other hand, the conic $C_0(x, y, z)$ can be recognized as the circle whose equation is the column : $V \simeq S_{\omega} : a^2 : a^2 : 1$. And therefore is centered at $\frac{O_b}{V} \simeq 0 : 1 : 1$ and has radius $\sqrt{V \cdot \frac{O_b}{V}} = ia \sqrt{3}/2$.

4. Thus the set of the level curves is the pencil of cycles generated by the two special cases, i.e. the Neuberg pencil. Since $C_0$, the cycle centered on the radical axis, is virtual, the pencil is an isotomic pencil, whose limit points $P_k$ are on the real circle associated with $C_0$: these points are the vertices of the equilateral triangles whose basis is $BC$.

5. From $a^2/(u^2 + v^2) = \cos(\omega_M)^2$, we have $u = v \cot(\omega_M)$ so that the level curves are the circles:

$$V \simeq S_{\omega} - 2 S \cot(\omega) : a^2 : a^2 : 1$$

Thus $\sqrt{V \cdot \frac{O_b}{V}} = \frac{1}{2} a \sqrt{\cot^2 \omega - 3}$.

6. And therefore, $C_0$ is the (virtual) locus of vertices $M$ such that $\omega_M = 90^\circ$. 
7.7.4 Spoiler: Brocard angle of a pedal triangle

1. Consider \( M \simeq x : y : z \) and its pedal triangle. Apart from a common factor \( 1 \div (2R \ (x + y + z))^2 \), area and squared sidelengths are:

\[
S_M = S (a^2yz + b^2zx + c^2yx) ; \ a^2_M = a^2 (z (y + z) b^2 + y (y + z) c^2 - a^2yz) ; \ etc.
\]

Thus the level curves of the Brocard angle are given by

\[
\frac{u}{v} = \cot (\omega_M) = \frac{a^2_M + b^2_M + c^2_M}{4S_M}
\]

2. As before, the numerator collects nicely and the locus is given by:

\[
v \ C \ (x, y, z) - 4u \ S (a^2yz + b^2zx + c^2yx) = 0
\]

3. This locus is a circle, whose Veronese image \( V \), center \( E \) and radius \( \rho \) are:

\[
V \simeq \begin{pmatrix}
  b^2c^2v \\
  c^2a^2v \\
  a^2b^2v \\
  2uS + S_w v
\end{pmatrix}
\]

\[
E \simeq \begin{pmatrix}
  a^2 (2Sv + Sw u) \\
  b^2 (2Sv + Sw u) \\
  c^2 (2Sv + Sw u)
\end{pmatrix} \approx \begin{pmatrix}
  a \cos (A - \omega_M) \\
  b \cos (B - \omega_M) \\
  c \cos (C - \omega_M)
\end{pmatrix} ; \ \rho = \frac{R \sqrt{\cot^2(\omega_M) - 3}}{\cot (\omega_M) + \cot (\omega)}
\]

4. One sees that \( v = 0 \) describes the circumcircle. Projections of \( M \) are aligned on the Simson line, so that \( \omega_M = 0 \).

5. One sees that \( \omega_M = \pm 30^\circ \), i.e. \( v = \cot (\omega_M) = \pm \sqrt{3} \), \( u = 1 \) characterizes two points (\( \rho = 0 \)). They are \( X(15), X(16) \), the isodynamic points. No other pedal triangle is equilateral.

6. The locus characterized by \( \omega_M = \omega_{ABC} \) is centered at \( X(182) \). One identifies the 3-6-Brocard circle. The locus characterized by \( \omega_M = -\omega_{ABC} \) is a straight line (\( \rho = \infty \)). Therefore, this line is the perpendicular bisector of segment \( X(15), X(16) \): the line \( X(187), X(512) \).

7.8 Orthogonal projector onto a line

**Proposition 7.8.1.** The matrix \( \pi_\Delta \) of the orthogonals projector onto line \( \Delta \simeq [p, q, r] \) – not the line at infinity – is given by:

\[
\pi_\Delta = \text{Id} - \left[ \Delta \cdot \left[ \Delta \cdot \left[ \Delta \cdot \frac{1}{\Delta} \right] \right] \right] \left[ \Delta \cdot \left[ \Delta \cdot \left[ \Delta \cdot \frac{1}{\Delta} \right] \right] \right] \quad (7.25)
\]

where matrix \( \left[ \Delta \right] \) is defined by \( 7.18 \).

**Proof.** Let \( \vec{\delta} = [\Delta] \cdot \frac{1}{\Delta} \) be the orthodir of \( \Delta \) and \( P \) a generic point of \( \Delta \). Then \( \pi_\Delta = \text{Id} - \left( \vec{\delta} \cdot \Delta \right) \div \left( \Delta \cdot \vec{\delta} \right) \). Then \( \pi_\Delta (P) = P \) since \( \vec{\delta} \cdot \Delta \cdot P = \vec{\delta} \cdot \left( \Delta \cdot P \right) = 0 \). On the other hand, one has \( \vec{\delta} \cdot \Delta \cdot \vec{\delta} = \vec{\delta} \cdot \left( \Delta \cdot \vec{\delta} \right) \), so \( \pi_\Delta (\vec{\delta}) = 0 \). As a result, \( \pi_\Delta \left( P + \lambda \vec{\delta} \right) = P \), as it should be.

**Proposition 7.8.2.** The matrix \( \sigma_\Delta \) of the orthogonals reflection wrt line \( \Delta \simeq [p, q, r] \) – not the line at infinity – is given by:

\[
\sigma_\Delta = \text{Id} - 2 \left[ \Delta \cdot \left[ \Delta \cdot \left[ \Delta \cdot \frac{1}{\Delta} \right] \right] \right] \left[ \Delta \cdot \left[ \Delta \cdot \left[ \Delta \cdot \frac{1}{\Delta} \right] \right] \right] \quad (7.26)
\]

where matrix \( \left[ \Delta \right] \) is defined by \( 7.18 \).
Figure 7.1: Level curves of the Brocard angle of the pedal triangle of point M
Proof. Obvious from the preceding proof.

Proposition 7.8.3. Cosine of a projection. Consider line $\Delta_1 \simeq [p, q, r]$ and use $\vec{e}_1^\perp = (q - r, r - p, p - q)$ as unit vector for this direction. Consider also line $\Delta_2 \simeq [u, v, w]$ and use $\vec{e}_2^\perp = (v - w, w - u, u - v)$ as unit vector for that other direction. Then orthogonal projection $\pi$ onto $\Delta_1$ transforms $\Delta_2$-vectors into $\Delta_1$-vectors according to:

$$\pi(\vec{e}_2^\perp) = \frac{\vec{e}_1^\perp \cdot \Delta_1 \cdot \Delta_2}{\Delta_1 \cdot \Delta_1^\perp} \Delta_2^\perp$$

Proof. Formula is homogeneous, as it should be. Vectors $\vec{e}_i^\perp$ are not normalized, which is the reason why this formula is square-root free. Formula $2S[\Delta] = \begin{bmatrix} W & \text{Pyth} \end{bmatrix} \cdot \begin{bmatrix} W \end{bmatrix}$ indicates that scaling factor can be interpreted in terms of a "cosine of projection".

Proposition 7.8.4. Consider the homothety of center $P = p : q : r$ (not on the infinity line) and ratio $k \ (\text{not } 0 \iff)$. Then points $U$ are transformed as $U \mapsto h(P, k) \cdot U$ while lines $\Delta$ are transformed as $\Delta \mapsto \Delta \cdot h(P, 1/k)$ where:

$$h(P, k) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} + \frac{1 - k}{k(p + q + r)} \begin{bmatrix} p & p & p \\ q & q & q \\ r & r & r \end{bmatrix} \simeq k + (1 - k) \frac{P \cdot \mathcal{L}_\infty}{\mathcal{L}_\infty \cdot P}$$

(7.27)

Proof. Applied to column $P$, the last formula gives $P$. Applied to a vector $\vec{V}$, this gives $k \vec{V}$ since a vector is defined by $\mathcal{L}_\infty \cdot \vec{V} = 0$. Another method: we want $U \mapsto X$ such that, given $U$, we have $(X - P) = k(U - P)$. Expressed in barycentrics, this leads to:

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} \simeq \begin{bmatrix} u \\ v \\ w \end{bmatrix} + \frac{(1 - k)(u + v + w)}{k(p + q + r)} \begin{bmatrix} p \\ q \\ r \end{bmatrix}$$

Property about lines comes from the fact that action over lines and action over points are inverse of each other.

7.9 The hortocenter romance

The three perpendicular bissectors of a triangle intersect at a single point. The proof is well known. Note them $\mu(B, C)$, etc and define $P$ as $\mu(A, B) \cap \mu(B, C)$. Then $PA = PB$ together with $PB = PC$ (from Pythagoras theorem). This implies $PA = PC$ and proves $P \in \mu(A, C)$.

And now, define $A' = B + C - A$, etc. The $\mu(B'C')$, etc intersect at some point $H$, while $A = (B' + C')$, so that $\mu(B'C')$ is also the altitude issued from $A$ and perpendicular to $BC \parallel B'C'$. As a result, we have proven that the three altitudes of a triangle intersect at a single point. This enlightening proof appears to be a recent one... while the property itself is not listed in the Greek geometry books (Bogomolny, 2015).

In this section, we will explore what happens when this result is taken as an axiom.

Definition 7.9.1. Orthopoints. Let us take a triangle $ABC$ in the projective plane and choose the point $H \simeq p : q : r$ as the "rightfull center" together with choosing the line $\mathcal{L}_\infty \equiv [f, g, h]$ as the line at infinity. Points $H_a \equiv (HA) \cap (BC)$, etc are called the cevian feet of $H$. And then directions $D_a \equiv (BC) \cap \mathcal{L}_\infty$ and $T_a \equiv (AH) \cap \mathcal{L}_\infty$ are called orthopoints of each other (this relation will be extended later into an involution of $\mathcal{L}_\infty$).

Definition 7.9.2. Holy Garden and hortocenter. Let $R_a$ be the fourth-harmonic of $H$ wrt $H_a, T_a$, etc. Then $A, B, C, R_a, R_b, R_c$ are co-conic. Let us call this conic $\Gamma$ the holy garden, i.e. the most beautiful circle. Its perspector is the Lemoine point (and therefore is usually named $K$). Let us draw the polar of $T_a$. It goes through $D_a$. The three polars intersect at a single point (the pole of $\mathcal{L}_\infty$). This point is the center of the holy garden, i.e. the hortocenter. Let us call it $O$, since the orthocenter is called $H$.  

—— pldx : Translation of the Kimberling’s Glossary into barycentrics ——
Proof. One has

\[
H_a, T_a, R_a \simeq \begin{pmatrix} 0 \\ q \\ r \end{pmatrix}, \begin{pmatrix} -qg - rh \\ qf \\ fr \end{pmatrix}, \begin{pmatrix} -p(qg + rh) \\ q(2fp + qg + rh) \\ (2fp + qg + rh)r \end{pmatrix}
\]

\[
K, O \simeq \begin{pmatrix} p(qg + rh) \\ q(rh + pf) \\ (pf + qg) \end{pmatrix}, \begin{pmatrix} gh(qg + rh) \\ hf(rh + pf) \\ fg(pf + qg) \end{pmatrix}
\]

Definition 7.9.3. Gravity center. By \( O \), the hortocenter, let us draw the line \( OT_a \) having the same direction as \( (HA) \) and obtain the points \( G_a = (BC) \cap (OT_a) \), etc. The triangle \( G_aG_bG_c \) is perspective with \( ABC \), defining a point \( G \). To avoid confusions with the Garden Center \( O \), let us call \( G \) as the Gravity center. It occurs that \( O, G, H \) are colinear, defining the celebrated Euler line.

Proof. One has:

\[
G \simeq \begin{pmatrix} gh \\ fh \\ fg \end{pmatrix} = \text{tripolar}(L_\infty) ; \text{Euler} \simeq [f(qg - rh), g(rh - pf), h(pf - qg)]
\]

Definition 7.9.4. Isogonal conjugacy and orthopoint transform. It happens that \( G \ast K = O \ast H \). Let us extend this result to the whole plane and define the isogonal conjugacy as \( M^* = G \ast K \div M \). (Spoiler: this is a quadratic Cremona transform). Moreover, we can see that the pairs \( (T^*_a, D^*_a) \), etc are aligned with \( O \). Let us use this property to extend the orthopoint relation to the whole \( L_\infty \), i.e.

\[
N = \text{orthopoint}(M) \iff \det |O, M^*, N^*| = 0
\]

This leads to: \( N \simeq \text{OrtH} \cdot M \) where \( \text{OrtH} \simeq \begin{pmatrix} 0 & \frac{pr}{f} & \frac{pq}{f} \\ \frac{rq}{f} & 0 & -\frac{pq}{g} \\ -\frac{h}{r} & \frac{pr}{h} & 0 \end{pmatrix} \)

Proof. Write the system \( \det |O, M^*, N^*| = L_\infty \cdot M = L_\infty \cdot N = 0 \) and solve in \( r, q', r' \). Substitute into \( p' : q' : r' \), re-introduce \( r \) and take the gradient wrt \( M \). One can remark that \( \ker[\text{OrtH}] \) is \( H \simeq p : q : r \) while \( \ker[\text{OrtH}] = L_\infty \).

Proposition 7.9.5. The characteristic polynomial of \( \text{OrtH} \) is:

\[
\chi[\text{OrtH}](X) = X^3 + \frac{pqr(fq + qg + rh)}{fg} X
\]

Non vanishing roots can be taken as \( \pm i \) or as \( \pm 1 \) according to the signum of \( pqr(fq + qg + rh) \div fg \) (as it should be, this signum behaves projectively).
Chapter 8

Brief extension to 3D spaces

Previous chapters were dealing with 2D spaces (planes), represented as the projective of a 3D vector space. In this chapter, we are using a 4D vector space to describe a 3D geometric space.

8.1 Basic results

Definition 8.1.1. A 3D point is a projective column in \( \mathbb{P}(\mathbb{R}^4) \), most of the time noted as:

\[
X = \begin{pmatrix}
x_1 \\
x_2 \\
x_3 \\
x_4
\end{pmatrix} = \left( \frac{\vec{X}}{x} \right)
\]

Definition 8.1.2. We will say that four points \( X_j \) are coplanar when

\[
\det_{1..4} (X_j) = 0
\]

Theorem 8.1.3 (Universal factoring). Given four columns, we have:

\[
\det (X_1X_2X_3X_4) = X_1 \cdot (X_2 \wedge X_3) \cdot X_4 \quad \text{where}
\]

\[
\begin{pmatrix}
x_2 \\
y_2 \\
z_2 \\
t_2
\end{pmatrix} \wedge \begin{pmatrix}
x_3 \\
y_3 \\
z_3 \\
t_3
\end{pmatrix} = \begin{pmatrix}
0 & t_3 z_2 - t_2 z_3 & t_2 y_3 - t_3 y_2 & y_2 z_3 - y_3 z_2 \\
t_2 z_3 - t_3 z_2 & 0 & t_3 x_2 - t_2 x_3 & x_3 z_2 - x_2 z_3 \\
t_3 y_2 - t_2 y_3 & t_2 x_3 - t_3 x_2 & 0 & x_2 y_3 - x_3 y_2
\end{pmatrix}
\]

Proof. Operator \( \det \) is multilinear: this ensures the existence of the central matrix. The actual values of the coefficients are obtained by partial derivatives.

Definition 8.1.4. Matrix \( \Delta \simeq (X_2 \wedge X_3) \) is anti-symmetric, and thus depends of 6 parameters. We call it the punctual matrix. Two concurrent matrix notations are used in the litterature:

\[
\begin{pmatrix}
D_{1,2} & -D_{1,3} & D_{1,4} \\
-D_{1,2} & 0 & D_{2,4} \\
+D_{1,3} & -D_{2,3} & 0 \\
-D_{1,4} & -D_{2,4} & -D_{3,4}
\end{pmatrix} = \begin{pmatrix}
0 & B_z & -B_y & E_x \\
-B_z & 0 & B_x & E_y \\
B_y & -B_x & 0 & E_z \\
-E_x & -E_y & -E_z & 0
\end{pmatrix} = \begin{pmatrix}
B \\
E_y \\
E_x \\
E_z
\end{pmatrix}
\]

Notation 8.1.5. Beside the electromagnetic notation given above, there is another representation, called the Plucker column representation. We have:

\[
\text{col} (\Delta) = \begin{pmatrix}
D_{2,3} \\
D_{1,3} \\
D_{1,2} \\
D_{1,4} \\
D_{2,4} \\
D_{3,4}
\end{pmatrix} = \begin{pmatrix}
B_x \\
B_y \\
B_z \\
E_x \\
E_y \\
E_z
\end{pmatrix} = \begin{pmatrix}
\vec{B} \\
\vec{E}
\end{pmatrix}
\]

(8.1)
Remark 8.1.6. Caveat: (1) $D_{12}$ is $B_z$, not $B_x$; (2) a minus sign is used at place $[1,3]$ (3) $B$ is the antisymmetric matrix such that $B \cdot \vec{X} = \vec{X} \times \vec{B} = (X_x : X_y : X_z) \wedge (B_x : B_y : B_z)$... mind the order! (4) $E$ is defined by $E \cdot \vec{Y} = \vec{Y} \times \vec{E}$.

Remark 8.1.7. When $X_2 \neq X_3$, the points collinear with $X_2, X_3$ are the elements of $\ker \Delta$.

Moreover, the characteristic polynomial is:

$$\chi_\Delta (\lambda) = \lambda^4 + \lambda^2 \sum_{j=6}^2 D_{j,k}^2 + \left( \sum_{j=3}^2 D_{j,k,l} \right)^2$$

so that $\sum_3 D_{j,k,l} = E_x B_x + E_y B_y + E_z B_z$ is null.

**Proposition 8.1.8.** When $X \notin \Delta$, the row characterizing the plane containing both point $X$ and line $\Delta$ is obtained as $\Pi = \text{t} M \cdot \Delta$.

**Proposition 8.1.9.** Beside the punctual matrix, let us define the planar matrix $\Delta^*$ so that the column characterizing the point common to plane $\mathfrak{P}$ and line $\Delta$ is given by $X = \Delta^* \text{t} \mathfrak{P}$ (obviously, $\Delta \notin \mathfrak{P}$ is assumed). Then, using notations of (8.1), we have:

$$\Delta^* = \begin{pmatrix}
0 & D_{3,4} & -D_{2,4} & D_{2,3} \\
-D_{3,4} & 0 & D_{1,4} & -D_{1,3} \\
D_{2,4} & -D_{1,4} & 0 & D_{1,2} \\
-D_{2,3} & -D_{1,3} & -D_{1,2} & 0
\end{pmatrix} = \begin{pmatrix}
0 & E_z & -E_y & B_z \\
-E_z & 0 & E_x & B_y \\
E_y & -E_x & 0 & B_z \\
-B_z & -B_y & -B_z & 0
\end{pmatrix} = \begin{pmatrix}
\mathcal{E} \\
\mathcal{B} \\
0
\end{pmatrix}$$

$$\text{col} \left( \Delta^* \right) = \begin{pmatrix}
D_{1,4} \\
D_{2,4} \\
D_{3,4} \\
D_{2,3} \\
D_{1,3} \\
D_{1,2}
\end{pmatrix} = \begin{pmatrix}
E_x \\
E_y \\
E_z \\
B_x \\
B_y \\
B_z
\end{pmatrix} = \begin{pmatrix}
\mathcal{E} \\
\mathcal{B}
\end{pmatrix}$$

**Proof.** This result is obtained by solving $\mathfrak{P} \cdot X = 0$, $\Delta \cdot X = \vec{0}$ and then generically expressing $X$ as $\Delta^* \text{t} \mathfrak{P}$.

**Proposition 8.1.10.** Given two (different) planes $\mathfrak{P}_1, \mathfrak{P}_2$, the punctual matrix $\Delta$ of their diedral line can be obtained directly as:

$$\Delta \simeq \text{t} \mathfrak{P}_1 \cdot \mathfrak{P}_2 - \text{t} \mathfrak{P}_2 \cdot \mathfrak{P}_1$$

Given two (different) points $P_1, P_2$, the planar matrix $\Delta^*$ of line $P_1P_2$ can be obtained directly as:

$$\Delta^* \simeq P_1 \cdot \text{t} P_2 - \text{t} P_2 \cdot P_1$$

Remember that $\Delta$ describes a punctual line, while $\Delta^*$ describes a planar line.

**Proof.** These objects are clearly projective objects and antisymmetric matrices of rank 2. The first one satisfies $\Delta \cdot X = 0 : 0 : 0 : 0$ as soon as $\Pi_1 \cdot X = \Pi_2 \cdot X = 0$. The second one satisfies $Y \cdot \Delta^* = [0,0,0,0]$ as soon as $Y \cdot P_1 = Y \cdot P_2 = 0$.

**Proposition 8.1.11.** The so-called Klein quadratic form defined by

$$\text{Klein} \left( \Delta, \Delta^* \right) \quad = \quad \text{trace} \left( \Delta \cdot \Delta^* \right)$$

$$= \quad D_{1,2}E_{4,4} + D_{1,3}E_{2,4} + D_{2,3}E_{1,4} + D_{1,4}E_{2,3} + D_{2,4}E_{1,3} + D_{3,4}E_{1,2}$$

$$= \quad \text{t} \text{col} \left( \Delta^* \right) \cdot \text{col} \left( \Delta \right) = \mathcal{C} \cdot \mathcal{E} + \mathcal{B} \cdot \mathcal{F}$$
is indeed a quadratic form. Then Klein \( \Delta_{23}, \Delta_{4} \) is exactly equal to \( \det (X_1X_2X_3X_4) \) when \( \Delta_{4} = (X_1 \wedge X_4) \) and \( \Delta_{23} = (M_2 \wedge M_3) \). And therefore, Klein \( \Delta_{1}, \Delta_{1} \) = 0 is the condition for the two lines to be coplanar.

**Maple 8.1.12.** In order to obtain independent points on a line \( \Delta \) then take \( \Delta^* \) and call \( \text{map(reduce, ColumnSpace(\%))} \). This gives a list of two columns.

**Proposition 8.1.13.** Using the electromagnetic notation, we have the following formulas:

- Line through two points \( (X \wedge Y) = (\vec{B} \overrightarrow{E}) = \left(y \vec{X} - x \vec{Y}\right) \)
- Line by a point \( X \) and a direction \( \vec{W} \) \( (X \wedge \left(\vec{W} : 0\right)) = \left(x \vec{W} \times \vec{X}\right) \)
- Plane through a line and a point \( \vec{t}X \cdot \vec{A} = \left[ x \vec{E} + \vec{X} \times \vec{B} ; \vec{E} \cdot \vec{V}\right] \)
- Incident punctual lines \( \vec{W} \cdot \vec{V} + \vec{B} \cdot \vec{F} = 0 \) or vanishes

\[ \begin{vmatrix} 53 & -58 & 0 & 76 \\ -97 & -9 & -16 & 40 \\ -74 & 32 & -43 & 45 \\ 79 & -34 & 58 & 39 \end{vmatrix} ; \det P = 24803093 ; \det A = \begin{vmatrix} 1 & -1 & 0 & 0 \\ -1 & -1 & -1 & 0 \\ 1 & 1 & -1 & 0 \end{vmatrix} ; \det A = 4 \]

Let \( Q_1, Q_2, Q_3, Q_4 \) be the images of the \( P_3 \) by the collineation. We have:

\[ Q \equiv A \cdot P = \begin{vmatrix} 150 & -49 & 16 & 36 \\ 118 & 35 & 59 & -161 \\ -155 & 51 & -31 & -120 \\ 30 & -99 & 27 & 71 \end{vmatrix} ; \det Q = 99212372 \]

Using the \( \wedge \) operator, we compute line \( P_{23} = (P_2 \wedge P_3) \) and \( Q_{23} = (Q_2 \wedge Q_3) \):

\[ P_{23} = \begin{vmatrix} 0 & 394 & 1066 & 899 \\ -394 & 0 & -3364 & -2494 \\ -1066 & 3364 & 0 & 928 \\ -899 & 2494 & -928 & 0 \end{vmatrix} ; Q_{23} = \begin{vmatrix} 0 & -1692 & -6786 & -4094 \\ 1692 & 0 & 261 & -703 \\ 6786 & -261 & 0 & -3451 \\ 4094 & 703 & 3451 & 0 \end{vmatrix} \]

And also the planes \( P_{123} = \wedge_{3} (P_1, P_2, P_3) \) and \( Q_{123} \):

\[ P_{123} = \begin{vmatrix} 46081 & -31028 & 309494 & 220893 \end{vmatrix} ; Q_{123} = \begin{vmatrix} -729354 & -192255 & -883572 & -162149 \end{vmatrix} \]

**Proposition 8.1.15.** Action of a collineation. We have the following properties:

1. \( \vec{t}P \cdot P_2 \cdot P_4 \cdot P_1 = \det P \cdot \overrightarrow{Q} \cdot Q_2 \cdot Q_3 \cdot Q_4 = \det Q = \det P \cdot \det A \)
2. \( Q_{23} = \det A \cdot A^{-1} \cdot P_{23} \cdot A^{-1} \)
3. \( Q_{123} = \det A \cdot P_{123} \cdot A^{-1} \)

— pldx : Translation of the Kimberling’s Glossary into barycentrics ——
4. \( P_{123} \wedge P_{234} = \det P \)\quad \text{dual} \left( P_{234} \right) = d P_{123} \cdot P_{234} - d P_{123} = \det P_{234} \\
5. The same dual \( \left( \overrightarrow{B}, \overrightarrow{E} \right) = e \left( \overrightarrow{E}, \overrightarrow{B} \right) \) is used for \( P \) and \( Q \).

**Exercise 8.1.16.** Loosely speaking and using Cartesian coordinates: a line goes through \((-1,1,5)\) and is incident with line \( D_1 : \{x + y = 2, \ x - y = 2\} \) together with line \( D_2 : \{1,3,1\} + \lambda(2,-2,1) \).

We have two planes \( P_{11} \simeq [1,1,0,-2], \ P_{12} \simeq \begin{cases} 1, -1, 0, -2 \end{cases} \) and three points \( P_0 \simeq 1 : 1 : 2 : 1, \ P_2 \simeq 1 : 3 : 1 : 1, \ P_22 \simeq 2 : 2 : 1 : 0 \). Thus \( D_1 = \text{dual} \left( \left( \overrightarrow{P_{11}} \wedge \overrightarrow{P_{12}} \right) \right), \ D_2 = \left( \overrightarrow{P_{21}} \wedge \overrightarrow{P_{22}} \right) \), and we have to solve the set of equations:

\[
\Delta : P_0 = 0 : 0 : 0 : 0, \ \phi(\Delta, D_1) = \phi(\Delta, D_2) = 0
\]

After expansion, we obtain:

\[
\left\{ \begin{array}{l}
-6B_y - 3E_z; -B_y - B_z + E_z; B_z + 5B_z + E_y; B_z - 5B_y + E_z; \\
E_x - E_y = -5E_z; -8B_z + B_y + 5B_z - 2E_z + 2E_y - E_z
\end{array} \right.
\]

so that \( \overrightarrow{B}, \overrightarrow{E} = -3 : 1 : 1 : 4 : 14 : -2 \). The incidence points are: \( P_1 \simeq 2 : 0 : 4 : 1 \) (who clearly belongs to both planes) and \( P_5 \simeq 5 : -1 : 3 : 1 \) (obtained with \( \lambda = 2 \)). Finally, \( P_0, P_4, P_5 \) are aligned, since \( P_2 \) is the middle of the other two.

One can also obtain the rows that describes the plane through \( P_0, D_1 \), the plane through \( P_0, D_2 \) and obtain \( \Delta \) using Proposition 8.1.10.

### 8.2 Euclidian cartesian metric

**Notation 8.2.1.** In the "usual" cartesian space, points are noted \( \overrightarrow{M} : m \simeq M_x : M_y : M_z : m \), and the metric is described by:

\[
\mathcal{L}_{4c} \simeq [0,0,0,1] : \left[ \begin{array}{cc}
\text{Pyth}_{4c} & \mathcal{M}_{4c}
\end{array} \right] = \left( \begin{array}{ccc}
1 & 0 & 0 \\
0 & 1 & 0 \\
0 & 0 & 1 \\
0 & 0 & 0
\end{array} \right)
\]

The \( c \) in the \( 4c \) index is "cartesian", while indices 4 are spared for a later use (tetrahedron barycentric space).

**Proposition 8.2.2.** Assume that \( \mathfrak{P} \) is not the plane at infinity. Then the orthogonal projection of point \( P \simeq X : Y : Z : T \) on \( \mathfrak{P} \simeq [E,F,G,H] \) is given by:

\[
\text{pr} \left( P, \mathfrak{P} \right) = (E^2 + F^2 + G^2) \left[ \begin{array}{c}
X \\
Y \\
Z \\
T
\end{array} \right] - \left( EX + FY + GZ + HT \right) \left[ \begin{array}{c}
E \\
F \\
G \\
0
\end{array} \right] = \left( \mathfrak{P} \cdot \mathcal{M}_{4c}, \mathfrak{P} \right) (P) - \left( \mathfrak{P} \cdot P \right) \left( \mathcal{M}_{4c}, \mathfrak{P} \right)
\]

and therefore

\[
\text{dist} \left( P, \mathfrak{P} \right) = \frac{\mathfrak{P} \cdot P}{\mathcal{L}_{\infty} \cdot P} \frac{1}{\sqrt{\left( \mathfrak{P} \cdot \mathcal{M}_{4c}, \mathfrak{P} \right)}}
\]

**Proof.** Express that the generic \( M \in \mathfrak{P} \) is generated by the three points \( [H : 0 : 0 : -E], [0 : H : 0 : -F], [0 : 0 : H : -G] \). Then compute \( [PM]^t \), minimize and substitute. \( \square \)

**Proposition 8.2.3.** The orthogonal projection of point \( P \simeq X : Y : Z : T \) on \( \Delta = \left[ \overrightarrow{B}, \overrightarrow{E} \right] \) is given by:

\[
\text{pr} \left( P, \Delta \right) = T_{4c} \left( \begin{array}{c}
B_x E_y - B_y E_z \\
B_x E_z - B_z E_x \\
B_y E_x - B_x E_y \\
B_z^2 + B_y^2 + B_x^2
\end{array} \right) + (XB_z + YB_y + ZB_z) \left( \begin{array}{c}
B_x \\
B_y \\
B_z \\
0
\end{array} \right)
\]

January 3, 2024 21:08 published under the GNU Free Documentation License
Proof. Minimize
\[ \left| t \operatorname{nor} \begin{pmatrix} 0 \\ -E_z \\ +E_y \\ -B_x \end{pmatrix} + (1 - t) \operatorname{nor} \begin{pmatrix} +E_z \\ 0 \\ -E_x \\ -B_y \end{pmatrix} \right| ^2 \]
\[ \begin{cases} X \\ Y \\ Z \\ T \end{cases} \]
\( \blacksquare \)

**Proposition 8.2.4.** It exists a unique point situated at equal distance from four given points when (1) they are all at finite distance (2) they are not coplanar.

**Proof.** The computation is straightforward, and the denominator of \( R^2 = (kuvw)^2 \times \det (KUVW)^2 \). Nevertheless, the length of the formal formula is roughly equal to 470,000 !

### 8.3 Rotations in the 3D Euclidean space

In this section, the usual Cartesian coordinates \( x : y : z \) are used to describe the usual 3D Euclidean space, \( \mathcal{E}_3 \) using the usual metric \( |v| = \sqrt{x^2 + y^2 + z^2} \).

**Proposition 8.3.1.** In the Euclidean space \( \mathcal{E}_3 \), the orthogonal projector onto the line directed by the (not zero) vector \( V \equiv t[f, g, h] \) is given by
\[ \pi^\perp = \frac{1}{f^2 + g^2 + h^2} V^\top V \]

On the contrary, the matrix
\[ \omega = \frac{1}{\sqrt{f^2 + g^2 + h^2}} \begin{bmatrix} 0 & h & -g \\ -h & 0 & f \\ g & -f & 0 \end{bmatrix} \]
describes "project onto \( V^\perp \text{ and quarter-turn} \)" while the projector itself is given by \( \pi^\perp = -\omega^2 \).

Thus the matrix of the 3D-rotation of angle \( \tau \) around axis \( V \) is given by:
\[ \rho = \text{Id} + \sin \tau \omega + (1 - \cos \tau) \omega^2 \] (8.2)

**Proof.** One can see that \( \omega^2 + \omega^4 = 0 \). Thus \( -\omega^2 \) is a projector. And the rest follows. One can check that \( \chi_\omega = X (X^2 + 1) \), while \( \chi_\rho = (X - 1) (X^2 - 2X \cos \tau + 1) \).
\( \blacksquare \)

### 8.4 Euclidian metric in the tetrahedron space

**Proposition 8.4.1.** The matrix describing the metric in the tetrahedron \( A_0B_0C_0D_0 \) space is
\[ \text{Pyth}_4 = -\frac{1}{2} \begin{bmatrix} 0 & c^2 & b^2 & A^2 \\ c^2 & 0 & a^2 & B^2 \\ b^2 & a^2 & 0 & C^2 \\ A^2 & B^2 & C^2 & 0 \end{bmatrix} \]

where \( a \cong B_0C_0 \) and \( A = A_0D_0 \) (in this context, vertices are noted with index 0 while the sidelength are noted with bare letters). And the matrix giving the direction orthogonal to a given plane is symmetric matrix whose fourth column is:
\[ \text{col} \left( \begin{bmatrix} 4 \\ 4 \end{bmatrix} \right) = \delta \left( \begin{bmatrix} 2A^2a^2 + a^4 - (B^2 + C^2 + b^2 + c^2) & A^2 - (b^2 - c^2) (B^2 - C^2) \\ 2B^2a^2 + b^4 - (C^2 + A^2 + a^2 + c^2) & B^2 - (c^2 - a^2) (C^2 - A^2) \\ 2C^2a^2 + c^4 - (A^2 + B^2 + a^2 + b^2) & C^2 - (a^2 - b^2) (A^2 - B^2) \\ (b + a + c)(a + b - c)(a + c - b)(b + c - a) \end{bmatrix} \right) \] (8.3)

— pldx : Translation of the Kimberling’s Glossary into barycentrics —
Proposition 8.4.2. The squared volume of the reference tetrahedron is

\[ V^2 = \frac{1}{144} \left( -a^2b^2c^2 + \sum_{i=1}^{3} a^2 \left[ \frac{a^2}{2} - a^2 + (A^2 - B^2) \right] \right) \]

\[ = \frac{1}{36} \mathcal{L}_\infty \cdot \text{adjoint} \left( \begin{bmatrix} \text{Pyth}_4 \end{bmatrix} \right) \cdot \mathcal{L}_\infty = \frac{1}{2304} |\delta_D|^2 \div S_D^2 \]

Proof. Cf. the Heron’s formula: \( \mathcal{L}_\infty \cdot \text{adjoint} \left( \begin{bmatrix} \text{Pyth} \end{bmatrix} \right) \cdot \mathcal{L}_\infty = 4S^2 \).

Proposition 8.4.3. Let \( \text{medAB} \) be the plane perpendicular to line \( AB \) at \( (A + B)/2 \), i.e., the perpendicular bisector of the edge. Let \( \text{cutAB} \) be parallel to \( \text{medAB} \) through \( (C + D)/2 \). Then the six \( \text{med} \) planes intersect at a point \( \Omega \) called the circumcenter of the tetrahedron. Its barycentrics \( \text{wrt} AB \cdots\) are:

\[ \Omega \simeq \frac{1}{6} \left[ a^2A^2(B^2 + C^2 - a^2) + b^2B^2(a^2 + C^2 - B^2) + c^2C^2(a^2 + B^2 - C^2) - 2a^2B^2C^2 \right] \]

\[ \approx \frac{1}{6} \left[ a^2(2c^2 - a^2) + b^2B^2(a^2 + c^2 - B^2) + \frac{1}{2}c^2C^2(a^2 + B^2 - c^2) - 2a^2B^2c^2 \right] \]

The circumscribed sphere is described by the \( \left( \begin{bmatrix} \text{Pyth}_4 \end{bmatrix} \right) \) quadratic form while its radius \( R \) is given by:

\[ R^2 = (Aa + Bb + Cc)(Bb + Cc - Aa)(Cc + Aa - Bb)(Aa + Bb - Cc) \div (24V)^2 \]

Moreover, the six cut planes intersect at a point \( M \) called the Monge point of the tetrahedron. And we have \( 2(\Omega + M) = A + B + C + D \).

Proof. Obtain \( \text{medAB} \) as the \( X \)-gradient of \( XA^2 - XB^2 \), etc, and take the wedge of the three planes \( \text{medAB}, \text{medBC}, \text{medCD} \). The result is symmetric, proving the existence of \( \Omega \). Then compute \( R \) from the Pythagorean formula (the obtained value is symmetric, as it should be). Finally, compute \( \text{cutAB} \), etc, by solving \( (\text{medAB} + x\mathcal{L}_4) \cdot (C + D) = 0 \), etc and obtain \( M \) using \( \Lambda_\Delta \). \( \square \)

Proposition 8.4.4. The distance between a point \( P \) and a plane \( \Psi \) is given by:

\[ \text{dist}(P, \Psi) = \frac{\Psi \cdot P}{\mathcal{L}_\infty \cdot P} \sqrt{\frac{12V}{\Lambda_4 \cdot \mathcal{L}_\infty}} \]

Proof. This formula is homogeneous in \( P \), in \( \Psi \) and in \( a,b,c,A,B,C \). Moreover, it obeys to the general model \( \text{dist}(x, \ker \phi) = \phi(x)/\|\phi\| \).

\( \square \)

Proposition 8.4.5. There are 8 points that are at the same distance from the faces of the standard tetrahedron. They are the centers of eight spheres tangents to these faces. The coordinates of the centers and the radii are:

\[ I_j = \pm S_A : \pm S_B : \pm S_C : \pm S_D : \frac{1}{\rho_j} = \frac{1}{h_A} + \frac{1}{h_B} + \frac{1}{h_C} + \frac{1}{h_D} \]

where \( h_j = 3V/S_j \) are the altitudes of the tetrahedron.

Proof. The distances from point \( x : y : z : t \) to the four faces of the reference tetrahedron are:

\[ \frac{x}{S_A} \cdot \frac{y}{S_B} \cdot \frac{z}{S_C} \cdot \frac{t}{S_D} \times \frac{3V}{x + y + z + t} \]

\( \square \)

Proposition 8.4.6. The distance \( d_j \) from the in/excenter \( I_j \) to the circumcenter is given by:

\[ d_j^2 - R^2 = -\left( \sum_{i=1}^{6} |AB|S_iS_j \right) \div \left( \sum_{i=1}^{4} S_i \right)^2 \]
Proof. Compute \( \text{nor}(\Omega) \cdot [\text{Pyth}_1] \cdot \text{nor}(\Omega) \) and obtain \(-R^2\), proving that \([\text{Pyth}_1]\) gives the power of a point wrt the sphere (and not some multiple). And then use the barycentrics of \( L_j \). Thus \( \sum \) sums the product of each sidelength by the areas of the two adjacent faces (taken with the relevant sign).

Example 8.4.7. The tetrahedron used in Hecquet (1980) is characterized by
\[
a^2 = 19, \quad b^2 = 13, \quad c^2 = 7, \quad A^2 = 21, \quad B^2 = 28, \quad C^2 = 37
\]

It’s existence granted by the reality of the areas:
\[
\frac{S_A}{26} = \frac{S_B}{19} = \frac{S_C}{14} = \frac{S_D}{11} = \frac{\sqrt{3}}{4}
\]

One has \( V^2 = 105/2, \quad R^2 = 973/90, \quad r^2 = 18/35, \quad d^2 = 2851/630. \)

8.5 HH: hyperbolic hyperboloids

Remark 8.5.1. Caveat 1: the \( \mathcal{H} \) used here means "hyperbolic hyperboloid" and therefore is not the \( \mathcal{H} \) used at Section 26.11, in the Sister Marie Cordia Karl section.

Caveat 2: HH are also called one-sheet hyperboloids.

Definition 8.5.2. Let \( L_1, L_2, L_3 \) be three lines in the 3D space, no two of them being incident. For each point \( M \) on \( L_3 \), it exists a line \( \Delta \) going through \( M \) and incident to \( L_1 \) and \( L_2 \). The union of these lines \( \Delta \) is called the hyperbolic hyperboloid defined by the three \( L_j \).

Proposition 8.5.3. It exist a \( 4 \times 4 \) matrix \([\mathcal{H}]\) such that \( [\Delta^*] \cdot [\mathcal{H}] \cdot [\Delta^*] = [0_4] \) is verified by the (planar) matrix of \( \Delta \) when \( M_3 \) moves on \( L_3 \). Moreover, any point \( M \) on such a \( \Delta \) verifies: \( t^M \cdot [\mathcal{H}] \cdot M = 0 \).

Proof. Use the four points \( A, B \in L_1 \) and \( C, D \in L_2 \) as barycentric basis, and note \( L_3 \) as \( (\overrightarrow{B}, \overrightarrow{E}) \).

Let \( \Delta \) be a line incident to \( L_1 \) at \( M_1 = tA + (1-t)B \) and to \( L_2 \) at \( M_2 = sC + (1-s)D \). This gives:
\[
\Delta \simeq \begin{bmatrix}
0 & 0 & (1-s)(1-t) & s(t-1) \\
0 & 0 & (s-1)t & st \\
(s-1)(1-t) & (1-s)t & 0 & 0 \\
s(1-t) & -st & 0 & 0
\end{bmatrix}
\]

and now write that \( \Delta \) and \( L_3 \) are incident. This leads to an homomorphic relation between \( s \) and \( t \). Eliminate \( t, k \) in \( M = kM_1 + (1-k)M_2 \) and obtain \( t^M \cdot [\mathcal{H}] \cdot M = 0 \) where
\[
M \simeq \begin{bmatrix}
((E_x - E_y + B_z + B_y)k + (E_y - B_z)k) t \\
((E_x - E_y + B_z + B_y)k + (E_y - B_z)k) (1-t) \\
(E_x - E_y) t + E_y (1-k) \\
(B_z + B_y) t - B_z (1-k)
\end{bmatrix}; \quad [\mathcal{H}] \simeq \begin{bmatrix}
0 & 0 & -B_y & E_x \\
0 & 0 & B_z & E_y \\
-B_y & B_z & 0 & 0 \\
E_x & 0 & 0 & 0
\end{bmatrix}
\]

Proposition 8.5.4. Suppose that \( M, N \) and \( M + N \) belong all three to \( \mathcal{H} \). Then any point of line \( \Delta = MN \) belongs to \( \mathcal{H} \), while \( t^\Delta \cdot [\mathcal{H}] \cdot t^\Delta = [0_4] \) is satisfied.

Proof. Use \( \mathcal{H}_0 \) and consider \( M \simeq \overrightarrow{V} : v \); \( N \simeq \overrightarrow{W} : w \), write the three equations and eliminate. This leads to
\[
v = \frac{(V_y B_y - V_y B_z) V_z}{E_x V_z + E_y V_y}, \quad w = \frac{(B_y W_x - B_z W_y) V_z}{E_x V_z + E_y V_y}, \quad w_z = \frac{(E_x W_x + E_y W_y) V_z}{E_x V_z + E_y V_y}
\]

and the conclusion follows.

Proposition 8.5.5. By any point \( M \) of \( \mathcal{H} \), two straight lines can be drawn which belong to \( \mathcal{H} \). One of them, say \( \Delta_M \), is incident to \( L_1, L_2, L_3 \) while the other one, say \( \delta_M \), is incident to all the lines \( \Delta_N \) where \( N \in \mathcal{H} \).
Proposition 8.5.7. Direct computation.

barycentrics wrt the null polynomial in $u,v,w_x$. Then compute the lines $MN$, check that parameters are cancelling and get:

$$
\Delta \simeq \begin{bmatrix}
0 & 0 & +uU_y & -uU_z \\
0 & 0 & -uU_x & +uU_z \\
-uU_y & +uU_x & 0 & 0 \\
U_yU_z & -uU_z & 0 & 0 \\
\end{bmatrix};
\delta \simeq \begin{bmatrix}
0 & -E_xB_xU_x^2 & -B_yU_z & E_xU_z \\
* & 0 & +B_yU_z & E_yU_z \\
* & * & 0 & -E_xU_x - E_yU_y \\
* & * & * & 0 \\
\end{bmatrix}
$$

And then Klein $(\Delta_M, \delta_N) = 0$ proves the incidences. One can check that $L_1 = \delta (A), L_2 = \delta (C)$ together with $L_3 = \delta (-E_yU_y - E_zU_z; E_zU_y; E_zU_z; E_zu)$.

Remark 8.5.6. When dealing with a circumscribed QH, i.e. when $\delta = \begin{bmatrix}
0 & H_z & H_y & K_y \\
H_z & 0 & H_x & K_y \\
H_y & H_x & 0 & K_y \\
K_x & K_y & K_z & 0 \\
\end{bmatrix}$, both series of lines $\Delta, \delta$ are involving the same radical $W$, where

$$W^2 = H_2^2K_x^2 + K_y^2H_y^2 + H_2^2K_y^2 - 2H_xK_xH_yK_y - 2K_yH_yH_zK_z - 2H_zK_zH_xK_x$$

Proposition 8.5.7. The four altitudes of a tetrahedron $ABCD$ belong to a same HH. When using barycentrics wrt $ABCD$, we have

$$
\delta \simeq \begin{bmatrix}
0 & u(a,b,c,A,B,C) & u(c,a,b,C,A,B) & u(b,c,A,B,c,a) \\
* & 0 & u(b,c,a,B,C,A) & u(c,a,B,C,a,b) \\
* & * & 0 & u(a,B,C,A,b,c) \\
* & * & * & 0 \\
\end{bmatrix}
$$

where $u(a,b,c,A,B,C) = (B^2 - A^2 + b^2 - a^2) \big((A^2 - B^2)(a^2 - b^2) + (a^2 + b^2 + A^2 + B^2 - 2C^2)c^2 - c^4\big)$

so that : $\det \delta = 82944V^2 \times \prod (B^2 - C^2 + b^2 - c^2)^2$.

Proof. Read the direction $\delta_D$ of the fourth altitude $\Delta_D$ at (8.3). Compute $\Delta_D \simeq \begin{bmatrix} D & \delta_D \end{bmatrix}$, etc. And check that the four sets of equations: $\begin{bmatrix} \Delta_D^+ & \delta \end{bmatrix} = 0$, etc are compatible.

Proposition 8.5.8. Orthocentric tetrahedron. When $A^2 + a^2 = B^2 + b^2$, then $A_0B_0 \perp C_0D_0$ and conversely. When two sets of opposite edges are orthogonal, so is the third set and the HH is totally degenerate: the four altitudes are concurrent, defining an orthocenter $H$. One has:

$$H \simeq \begin{bmatrix}
S_bS_c(A^2 - S_a) \\
S_cS_a(A^2 - S_a) \\
S_aS_b(A^2 - S_a) \\
S_aS_bS_c \\
\end{bmatrix};
V_H^2 = \frac{1}{9}S^2A^2 - \frac{1}{36}a^2S_a^2$$

Proof. Direct computation.

Exercise 8.5.9. What happens when there is only one pair of orthogonal opposite edges ?
8.6 HH: some examples

Proposition 8.6.1. Consider a tetrahedron \( A, B, C, D \). Affect a coefficient to each edge, e.g. \( BC = a \), etc and \( AD = d \), etc. On each face, take the barycenter of the three vertices using the weight of the opposite side, and obtain

\[
D^+ = (a A + b B + c C) / (a + b + c) \\
A^+ = (f B + e C + a D) / (f + e + a) \\
B^+ = (f A + d C + b D) / (f + d + b) \\
C^+ = (e A + d B + c D) / (e + d + c)
\]

Then lines \( AA^+ \), \( BB^+ \), \( CC^+ \), \( DD^+ \) are imbedded in a same HH.

Proof. A simple computation gives:

\[
\begin{bmatrix}
0 & (ad - be) & (cf - ad) & (be - cf) \\
(ad - be) & 0 & (be - cf) & (cf - ad) \\
(cf - ad) & (be - cf) & 0 & (ad - be) \\
(be - cf) & (cf - ad) & (ad - be) & 0
\end{bmatrix}
\]

with \( W = (ad - be)(be - cf)(cf - ad) \).

Corollary 8.6.2. If we take \( a = |BC| \) and so on, we obtain a property relative to the four lines joining a vertex to the incenter of the opposite face in a tetrahedron. And the same occurs for the Lemoine centers. And for their isotomic images \( X(75) \) and \( X(76) \). When all coefficients are equal, the four lines are simply concurrent at the gravity center.

Exercise 8.6.3. Consider the four points:

\[
A, B, C, D \simeq \begin{pmatrix}
-13 \\
-43 \\
43 \\
1
\end{pmatrix}, \begin{pmatrix}
-21 \\
-11 \\
41 \\
1
\end{pmatrix}, \begin{pmatrix}
-13 \\
21 \\
-91 \\
1
\end{pmatrix}, \begin{pmatrix}
-1 \\
22 \\
53 \\
1
\end{pmatrix}
\]

and compute the orthocenters, by minimizing \( \delta^2(A, xB + yC + zD) \) or otherwise. Consider the lines \( AH_a \), etc. Show they aren’t independent. Compute the QH containing these four lines. One obtains

\[
\begin{bmatrix}
-6048 & 40004 & -96004 & 3786528 \\
40004 & 94944 & 64701 & -288685 \\
-96004 & 64701 & -88896 & -4296565 \\
3786528 & -288685 & -4296565 & 520174122
\end{bmatrix}
\]

Exercise 8.6.4. (Follow-up). Take two points \( P, Q \) on \( AH_a \) and two points \( R, S \) on \( BH_b \). Consider the line \( \Delta \) joining \( eP + fQ \) with \( gR + hS \). Solve in \( h \) so that \( \Delta \) becomes incident to \( CH_c \). And now check that the generic point of \( \Delta \) belongs to the HH.
Chapter 9

Pedal stuff

In a previous life, this Section was intended as foreword to Chapter 7 (orthogonality). Now, this Section is rather the symmetric aisle of Chapter 3 (cevian stuff).

9.1 Pedal triangle

**Definition 9.1.1.** The pedal triangle of point \( P \) is the triangle whose vertices are the orthogonal projections of \( P \) on the sides of the triangle.

**Remark 9.1.2.** Crossover the Channel, the pedal triangle is called "triangle podaire", while "pédal triangle" is used to denote the Cevian triangle. *Plaisante vérité qu’une rivière borne* (Pascal, 1852, p. 41). For the anti-pedal triangle, see Proposition 24.4.8.

**Proposition 9.1.3.** The pedal triangle of point \( P \) has the following barycentrics (each point is a column):

\[
\begin{pmatrix}
 p \\
 q \\
 r 
\end{pmatrix}
\begin{bmatrix}
 0 & S_c q + b^2 p & S_b r + c^2 p \\
 S_c p + a^2 q & 0 & S_a r + c^2 q \\
 S_b p + a^2 r & S_a q + b^2 r & 0 
\end{bmatrix}
\begin{pmatrix}
 S_c b^2 p + p \\
 S_b c^2 q + q \\
 S_a r^2 + r 
\end{pmatrix}
\]

(9.1)

**Proof.** As usual, \( S_a = (b^2 + c^2 - a^2)/2 \), etc. Use (7.25) and obtain directly the result.

**Proposition 9.1.4.** Condition for an inscribed triangle \( P_1 P_2 P_3 \) to be the pedal triangle of some \( P \) is:

\[
\frac{q_1 - r_1}{q_1 + r_1} a^2 + \frac{r_2 - p_2}{p_2 + r_2} b^2 + \frac{p_3 - q_3}{p_3 + q_3} c^2
\]

(9.2)

In such a case, point \( P \) is the perspector between \( P_1 P_2 P_3 \) and triangle \( \Delta \) and is given by either following expressions:

\[
\begin{bmatrix}
 b^2 c^2 p_2 p_3 - (S_c r_2 - S_a p_2) (S_b q_3 - S_a p_3) \\
 b^2 c^2 p_2 q_3 + (S_b q_3 - S_a p_3) b^2 r_2 \\
 b^2 c^2 p_3 r_2 + (S_c r_2 - S_a p_2) c^2 q_3 \\
 a^2 c^2 p_3 q_1 + (S_a p_3 - S_b q_3) a^2 r_1 \\
 a^2 c^2 q_1 q_3 - (S_a p_3 - S_b q_3) (S_c r_1 - S_b q_1) \\
 a^2 c^2 q_3 r_1 + (S_c r_1 - S_b q_1) c^2 p_3 \\
 a^2 b^2 p_2 r_1 + (S_a p_2 - S_c r_2) a^2 q_1 \\
 a^2 b^2 q_1 r_2 + (S_b q_1 - S_c r_1) b^2 p_2 \\
 a^2 b^2 r_1 r_2 - (S_b q_1 - S_c r_1) (S_a p_2 - S_c r_2) 
\end{bmatrix}
\]

107
Proof. P is on the line through \( P_1 \) and orthopoint of \( BC \) etc. The required condition is the determinant of these three lines. The various ways of writing \( P \) are the wedge product of two rows at a time. A more symmetric formula would be great...

**Proposition 9.1.5.** It exists exactly one pedal triangle of a given shape. When the shape is given by the tangents \( t_A = \tan \left( \frac{AP_B}{BP_P} \right) \), etc (bound by \( t_A + t_B + t_C = t_A t_B t_C \), the central point is given by

\[
P \simeq \begin{bmatrix}
a^2 t_B t_C (S_a t_A + 2 S) \\
b^2 t_C t_A (S_b t_B + 2 S) \\
c^2 t_A t_B (S_c t_C + 2 S)
\end{bmatrix}
\]

**Proof.** Substitute 9.1 in the tan formula (7.20) and obtain three equations. And then eliminate.

**Exercise 9.1.6.** Spoiler: using Morley affixes, the pedal triangle of can be written

\[
\text{pedal} \begin{pmatrix} z \\ t \\ \zeta \end{pmatrix} = \begin{pmatrix} \beta + \gamma + \frac{z - \beta \gamma \zeta}{t} \\ \alpha + \gamma + \frac{z - \alpha \gamma \zeta}{t} \\ \alpha + \beta + \frac{z - \alpha \beta \zeta}{t} \end{pmatrix}
\]

**Exercise 9.1.7.** The area of the \( P \)-pedal triangle is \( \frac{S}{4} \left( 1 - \frac{1}{R^2} \right) \). Spoiler: a proof is given at Proposition 26.10.11.

**Exercise 9.1.8.** From Proposition 9.1.5, there is exactly one pedal triangle of each shape. What to say about the only pedal triangle who is skew similar to a given one?

**Construction 9.1.9.** Construct the \( ABC \)-pedal triangle similar to a given triangle \( T_sT_bT_c \). Through point \( A \), draw the parallel \( B_sC_p \) to \( T_bT_c \), etc and obtain triangle \( A_sB_sC_p \). Draw circles \( BCA_p \), etc. Call \( gE \) their common (Miquel) point. Then \( E \simeq \text{isogon}(gE) \) is the center of the sought pedal triangle.

## 9.2 Isogonal conjugacy and Steiner triangle

**Definition 9.2.1.** Use barycentrics wrt \( ABC \) and suppose that the point \( P = p : q : r \) is not on a sideline. Then the point \( Q \) defined by:

\[
Q \simeq \text{isogon}(p : q : r) \simeq \frac{a^2}{p} : \frac{b^2}{q} : \frac{c^2}{r}
\]

is not a vertex and is called the isogonal conjugate of \( P \).

**Remark 9.2.2.** The isogonal conjugate of point \( P \) can be introduced as the circumcenter of the \( P \)-Steiner triangle. This will be done at Section 10.1. Another introduction comes from the following eponymous property.

**Proposition 9.2.3.** The isogonal conjugate \( Q \) of the point \( P \) (not on a sideline) is characterized by the three relations:

\[
(AB, AP) + (AC, AQ) = 0, \text{ etc}
\]

In other words, lines \( AP \) and \( AQ \) are equally inclined over lines \( AB, AC \). This property is obviously symmetric between \( P, Q \) and both points are said to form a pair of isogonal conjugates.

**Proof.** One has \( \tan (A, B, P) + \tan (A, C, Q) = 0 \), etc. Mind the order! When line \( AP \) cuts \( BC \) between \( B \) and \( C \), then line \( AQ \) does the same.

**Remark 9.2.4.** The isogonal conjugate of a point \( P \) is often noted \( P^{-1} \) in ETC since its trilinears are \( 1/p : 1/q : 1/r \) when those of \( P \) are \( p : q : r \).
9.3 Cyclopedal conjugate

Definition 9.3.1. The pedal circle of a given point is the circle circumscribed to the pedal triangle of this point. (spoiler) It’s representative is:

\[ V_P \simeq \begin{bmatrix} p (c^2q + S_a r) (b^2r + S_a q) \\ q (a^2r + S_b p) (c^2p + S_b r) \\ r (b^2p + S_c q) (a^2q + S_c p) \\ (a^2q + b^2p + c^2p) (p + q + r) \end{bmatrix} \]

Proposition 9.3.2 (Matthieu). When \( P \) and \( Q \) are isogonal conjugates, they share the same pedal circle. The center of this circle is the middle of \( P \) and \( Q \) (cf Figure 9.1).

Proof. Straightforward computation (using \( V \) or not !).

Figure 9.1: Cyclopedal congrugates are isogonal conjugates

Remark 9.3.3. The definition of "cyclopedal conjugacy" has been coined to enforce symmetry with the cyclocevian conjugacy, cf. Section 13.22. Some examples are:

<table>
<thead>
<tr>
<th>point</th>
<th>code</th>
<th>bary</th>
<th>cycp</th>
<th>circumcenter</th>
</tr>
</thead>
<tbody>
<tr>
<td>incenter</td>
<td>X (1)</td>
<td>a</td>
<td>X (1)</td>
<td>X (1)</td>
</tr>
<tr>
<td>centroid</td>
<td>X (2)</td>
<td>1</td>
<td>X (6)</td>
<td>X (597)</td>
</tr>
<tr>
<td>Lemoine</td>
<td>X (6)</td>
<td>( a^2 )</td>
<td>X (2)</td>
<td>X (597)</td>
</tr>
<tr>
<td>circumcenter</td>
<td>X (3)</td>
<td>( a^2 (\sqrt{a^2 + b^2 + c^2}) )</td>
<td>X (4)</td>
<td>X (5)</td>
</tr>
<tr>
<td>orthocenter</td>
<td>X (4)</td>
<td>( 1/ (\sqrt{a^2 + b^2 + c^2}) )</td>
<td>X (3)</td>
<td>X (5)</td>
</tr>
</tbody>
</table>

Proposition 9.3.4. (Spoiler). The center \( K_P \) of the RH through \( A, B, C, H, P \) belongs to the pedal circle of \( P \). As a result, \( K_P \) and \( K_Q \) are the common points of the \( (P, Q) \) pedal circle and the Euler circle.

Proof. This can be checked by using:

\[ K_P \simeq \begin{bmatrix} (r (p + q) b^2 - q (r + p) c^2) (S_b q - S_c r) p \\ (p (q + r) c^2 - r (p + q) a^2) (S_c r - S_a p) q \\ (q (r + p) a^2 - p (q + r) b^2) (S_a p - S_b q) r \end{bmatrix} \]
Chapter 10
Orthogonal stuff

10.1 Steiner triangle

Definition 10.1.1. The Steiner triangle of point $P$ is the triangle whose vertices are the orthogonal reflections of $P$ on the sides of the triangle.

Proposition 10.1.2. The Steiner triangle of point $P \simeq p:q:r$ has the following barycentrics (each point is a column):

$$\text{Steiner} \begin{pmatrix} p \\ q \\ r \end{pmatrix}_\beta \simeq \begin{pmatrix} -p & 2q S_c + p & 2r S_b + p \\ 2p S_c + q & -q & 2r S_a + q \\ \frac{2p^2 S_b}{a^2} + r & \frac{2q^2 S_a}{b^2} + r & -q \end{pmatrix}$$ (10.1)

Proof. Use (7.26) and obtain directly the result. □

Proposition 10.1.3. The circumcenter of the Steiner triangle has the following center and radius:

$$Q \simeq \begin{pmatrix} a^2 qr \\ b^2 rp \\ c^2 pq \end{pmatrix}; \quad \rho = \frac{\prod \sqrt{b^2 r^2 + 2q r S_a + c^2 q^2}}{(a^2 qr + b^2 rp + c^2 qb)(p + q + r)}$$

Proof. Determine $Q$ using the perpendicular bissectors $|QM_a|^2 - |QM_b|^2 = 0$ and then compute $|Q M_a|$. (Spoiler) Another method will be described at Section 14.3: compute $V^b = A_3 V_{\text{Ver}}(M_j)$, and then use (14.14) together with (14.15) □

Fact 10.1.4. One has the result:

$$\text{Steiner} (M) \cdot \text{isogon} (M) \simeq M$$

10.2 Steiner line

Proposition 10.2.1 (Steiner line). Let $P$ be a point in the barycentric plane. When its Steiner triangle $A'B'C'$ is flat, the corresponding line is called the Steiner line of point $P$. This occurs when either:

1. $P$ is on the line at infinity. Then $A' = B' = C' = P$ and $\text{Steiner} (P) = L_\infty$.
2. $P$ is on the circumcircle. Then $Q = \text{isogon} (P) \in L_\infty$ while $\text{Steiner} (U)$ goes through the orthocenter $X_4$ and has the following equation:

$$\text{Steiner} (P) \simeq \text{isogon} \begin{pmatrix} P \ast X_4 \\ b \end{pmatrix} \simeq \begin{pmatrix} Q \ast X_4 \\ b \end{pmatrix} \simeq [S_a (\tau - \sigma), S_b (\rho - \tau), S_c (\sigma - \rho)]$$

$$\simeq \begin{pmatrix} a^2 S_a & b^2 S_b & c^2 S_c \\ p & q & r \end{pmatrix}$$ (10.2)
\[
S_2 = \tau(Q_2 \div b \ X_4) \quad U_2 \in \Gamma \quad \text{reflection in} \ X_3 \quad U_1 \in \Gamma \quad \rightarrow \quad S_1 = \tau(Q_1 \div b \ X_4) \quad Q_1 = L_\infty \cap S_2
\]

**orthopoint** \(Q) = L_\infty \land \tau.Q \div X_4 \)  \hspace{1cm} (10.3)

**Steiner (P) = X_4 \land \text{orthopoint (isogon (P))} \)  \hspace{1cm} (10.4)

Figure 10.1: The orthopoint transform

**Proof.** The Steiner triangle is depicted at (10.1). Its determinant factors into:

\[
(p + q + r) \left( a^2qr + b^2rp + c^2pq \right) / R^2
\]

The case \(p + q + r = 0\) is the line at infinity and is not to be discarded, since the orthogonal projection of a point at infinity onto a line at finite distance is the point at infinity of this line. When \(P \in \Gamma\), computing (10.2) is straightforward.

**Remark 10.2.2.** The study of the properties of the so called Simson lines is delayed until Section 26.8. Indeed, their envelope is a third degree curve, and not a simple point as for the Steiner lines.

**Proposition 10.2.3.** When \(U_1\) and \(U_2\) are on the circumcircle, then:

\[
\left( \text{Steiner} (U_1), \text{Steiner} (U_2) \right) = -\frac{1}{2} \left( X_3U_1, X_3U_2 \right)
\]

where the "overbrace" denotes the oriented angle between two straight lines. Therefore, it exists a one-to-one correspondence between lines through \(X_4\) and points \(U\) on the circumcircle. Moreover, Steiner lines relative to diametrically opposed points on the circumcircle are orthogonal to each other.

**Proof.** From elementary euclidian geometry... or using \(\tan (2\vartheta) = 2 \tan \vartheta / (1 - \tan^2 \vartheta)\) and (7.20).

**Corollary 10.2.4.** For each point on the circumcircle, the isogonal conjugate is the orthopoint of the Steiner line (cf. Figure 10.1, and also far below– Figure 20.7).

**Proof.** Let \(Q_1\) be a point at infinity. Take the isogonal conjugate of \(Q_1\) and obtain \(U_1 \in \Gamma\). Take the Steiner line of \(U_1\) and obtain \(S_1\). Take the point at infinity of \(S_1\) and obtain \(Q_2\). Take the isogonal conjugate of \(Q_2\) and obtain \(U_2 \in \Gamma\). Take the Steiner line of \(U_2\) and obtain \(S_2\). Now \(Q_1\) is the point at infinity of \(S_2\) while \(Q_1, Q_2\) are orthopoints of each other and \(U_1, U_2\) are antipodes of each other on the circumcircle.

**Proposition 10.2.5.** When \(Q_1, Q_2\) are orthopoints, their barycentric product lies on the orthic axis, i.e. the tripolar of \(X_4\).

**Proof.** Use parametrization (7.15), eliminate \(k, \sigma\) in \(k X = Q_1 \ast Q_2\) and obtain an equation of first degree.

**Example 10.2.6.** The following list gives the triples \((I, J, K)\) where \(X(I)\) and \(X(J)\) are named orthopoints of each other and \(X(K)\) is their (named) barycentric product:
10.3 Parallelogy
This section has moved to Section 24.3

10.4 Orthology
This section has moved to Section 24.4

10.5 Orthopole
The section has moved to "Pedal LFTT and orthopole" (Section 26.10), while "orthojoin" has disappeared.

10.6 Spoiler: Moebius-Steiner-Cremona transform
This section should be skipped by a reader not familiar with Morley spaces (Chapter 15) and Cremona transforms (Chapter 17).

Definition 10.6.1. Let $\alpha : 1 : 1/\alpha$, etc be the reference triangle $ABC$, and $z : t : \zeta$ the Morley affix of a point $P$.

Proposition 10.6.2. The affine transform $\psi$ characterized by $\begin{bmatrix} ABC \end{bmatrix} \mapsto \begin{bmatrix} \mathcal{T} \end{bmatrix}$ sends $O$ on $H$ and satisfies:

$$z(\psi(M)) = +s_1 - z_M - \frac{\zeta s_3}{t} \zeta_M$$

Proof. The Steiner triangle of $P$ is described by:

$$\begin{bmatrix} \mathcal{T} \end{bmatrix} \simeq \frac{1}{t} \begin{bmatrix} t (\beta + \gamma) - \beta \gamma \zeta & t (\gamma + \alpha) - \gamma \alpha \zeta & t (\alpha + \beta) - \alpha \beta \zeta \\ t \gamma & t (\gamma + \alpha) - z & t (\alpha + \beta) - z \\ \beta \gamma & \gamma \alpha & \alpha \beta \end{bmatrix}$$

and, therefore, the matrix of $\psi$ is:

$$\begin{bmatrix} \mathcal{T} \end{bmatrix} L u^{-1} \simeq \begin{bmatrix} -1 & s_1 & -\frac{\zeta s_3}{t} \\ 0 & 1 & 0 \\ -\frac{z}{s_3} & -\frac{s_2}{s_3} & -1 \end{bmatrix}$$

□

Corollary 10.6.3. Applied to $Q \equiv \text{isogon}(P)$ and using Fact 10.1.4, this gives (17.5).
Proposition 10.6.4. Let us consider two points and their Steiner triangles. Define the collineation \( \phi \) by \( T_1 \mapsto T_2; \mathcal{L}_z \mapsto \mathcal{L}_z \). Its matrix is:

\[
\begin{pmatrix}
\phi \\
\psi_2 \psi_1^{-1} \simeq T_2 \quad T_1^{-1}
\end{pmatrix}
\]

\[
\begin{pmatrix}
t_1 t_2 - \zeta_2 z_1 & \frac{(\zeta_2 t_1 - \zeta_1 t_2) (z_1 \sigma_1 - \sigma_2 t_1)}{t_1} & \sigma_3 (\zeta_2 t_1 - \zeta_1 t_2) \\
0 & \frac{(t_2^2 - z_1 \zeta_1)}{t_2 t_1} & 0 \\
\frac{1}{\sigma_3} (z_2 t_1 - z_1 t_2) & \frac{(z_2 t_1 - z_1 t_2) (\zeta_1 \sigma_2 - \sigma_1 t_1)}{t_1 \sigma_3} & t_1 t_2 - z_2 \zeta_1
\end{pmatrix}
\]

its characteristic polynomial is

\[
\chi(\mu) = (\mu - 1) \left( \frac{t_1 (2 t_1 t_2 - \zeta_2 z_1 - z_2 \zeta_1)}{t_2 (t_1^2 - z_1 \zeta_1)} \mu + \frac{(t_2^2 - z_2 \zeta_1) t_1^2}{(t_1^2 - z_1 \zeta_1) t_2^2} \right)
\]

and we have \( \phi(H) = H \).

Remark 10.6.5. Considering some special cases:

1. For \( \phi \) to be parallelologic, condition is \( \phi_{11} + \phi_{33} = 0 \), i.e. \( z_2 \zeta_1 + \zeta_2 z_1 - 2 t_1 t_2 = 0 \). Each point is on the circle-polar of the other.

2. For \( \phi \) to be orthologic, condition is \( \phi_{11} - \phi_{33} = 0 \), i.e. \( z_2 \zeta_1 - \zeta_2 z_1 = 0 \). Both points are aligned with the circumcenter \( X(3) \).

3. For \( \phi \) to be an skew similarity, condition is \( \phi_{11} = \phi_{33} = 0 \). Both points are inverse in the circumcircle.

4. For triangles \( T_1, T_2 \) to be in perspective (homologic), condition is

\[
\sigma_4 (z_2 t_1 - t_2 z_1) (\zeta_2 t_1 - \zeta_1 t_2) ((z_1 t_2 - t_1 z_2) \sigma_2 + (z_1 z_2 - z_1 \zeta_2) \sigma_3 + (t_1 \zeta_2 - \zeta_1 t_2) \sigma_3 \sigma_1) = 0
\]

In other words, either the points are equal in one of the spherical maps, or they are aligned with \( X(4) \). In this case, the perspector is a point \( S \) on the circumcircle, and line \( U_1 HU_2 \) is nothing but the Steiner line of \( S \).

Proof. Compute the perspector of \( T(M_1) \) and \( T(k M_1 + (1 - k) H) \).

Proposition 10.6.6. In the map \( Z : T \), the homography \( \mu \) defined by \( A \mapsto A', B \mapsto B', C \mapsto C' \) is called the Moebius-Steiner transform related to \( U \). This transform is involutive and we have:

\[
\mu(Z : T) = \frac{t^2 Z - (t^2 \sigma_1 - t \zeta \sigma_2 + \zeta^2 \sigma_3) T}{\zeta t Z - t^2 T}
\]

Image \( \mu(U) \) is isogon(\( U \), i.e. the other focus of the inscribed conic whose \( U \) is the first focus. Moreover, \( \mu(\infty) = t/\zeta \), i.e. the symmetric of \( U \) wrt the circumcircle. Finally, \( A', B', C', H \) are concyclic when either \( U \) is on the circumcircle (the Steiner property) or \( U \) is on the orthoptic of the polar circle (centered at \( H \)).

Proof. Write the reality of the cross-ratio and obtain \( (t^2 - z \zeta) (s_1 s_3 t \zeta + s_2 t z - s_3 z \zeta - s_4 t^2) = 0 \).

10.7 Orthocorrespondents

Definition 10.7.1. Suppose \( P \) is a point in the plane of triangle \( ABC \). The perpendiculars through \( P \) to the lines \( AP, BP, CP \) meet the lines \( BC, CA, AB \), respectively, in collinear points. Let \( L \) denote their line. The trilinear pole of \( L \) is \( P^\perp \), the orthocorrespondent of \( P \). This definition is introduced in Gibert (2003). If \( P = p : q : r \) is given in barycentrics, then \( P^\perp = u : v : w \) is given by:

\[
-(b^2 + c^2 - a^2) p^2 + (a^2 - b^2 + c^2) pq + (a^2 + b^2 - c^2) pr + 2qr a^2
\]
Remark 10.7.2. If follows that if \( P = x : y : z \) is given in trilinears, then \( P^\perp \) has trilinears given cyclically by:

\[
yz + (\text{−} x \cos A + y \cos B + z \cos C)x
\]

Example 10.7.3. Pairs \((I,J)\) for which the orthocorrespondent of \( X(I) \) is \( X(J) \) include the following:

1. 57 11 651 62 2005 109 1813 125 648 156 677
2. 1992 13 13 80 2006 111 895 132 287 178 57
4. 1994 16 61 100 1332 113 2986 242 1999 187 8
7. 1997 33 2002 106 1797 118 2989 917 2989 0
8. 1998 36 2003 107 648 119 2990 130 8
9. 1999 61 2004 108 651 120 2991 156 8

Proposition 10.7.4. The orthocorrespondent of every point on the line at infinity is the centroid. Conversely, given a finite point \( U \), different from the centroid, it exists exactly two orthoassociate points \( P_1 \) and \( P_2 \) (real or not, distinct or not) that share the same orthocorrespondent \( U \). When \( P_1 \) is given, then:

\[
p_2 \simeq (q_1 + r_1)p_1 + \frac{a^2 - b^2 + c^2}{a^2 - b^2 - c^2} q_1^2 + \frac{a^2 + b^2 - c^2}{a^2 - b^2 - c^2} r_1^2
\]

When \( U \) is given, the condition of reality is \( \Delta \geq 0 \) where:

\[
\Delta \equiv S^2(u + v + w)^2 - u(w + v)S_a S_b - v(u + w)S_a S_c - w(u + v)S_b S_c
\]

\( S = \text{area and } S_a = \frac{(b^2 + c^2 - a^2)}{2} \). Then, cyclically, we have:

\[
p_1, p_2 \simeq \left( S ((u - w)(u + v - w)S_b + (u - v)(u - v + w)S_c) \pm ((u - w)S_b + (u - v)S_c) \sqrt{\Delta} \right)
\]

Proof. Write orthocorr \( P = kU \) and eliminate (rationally) \( k, p \). Obtain a second degree equation, whose discriminant is \( S^2 (v - w)^2 \Delta \). In order to obtain a symmetric form for the barycentrics \( p : q : r \), all these expressions must be simplified using \( (\sqrt{\Delta})^2 = \Delta \), then rationalized and simplified again, and finally normalized using \( p + q + r = 1 \).

10.8 Isoscelizer

An isoscelizer is a line perpendicular to an angle bisector. If \( P \) is a point, then the \( A \)-isoscelizer of \( P \) is the line \( L(P,A) \) through \( P \) perpendicular to the line that bisects vertex angle \( A \); the \( B \)- and \( C \)-isoscelizers are defined cyclically. Let \( D \) and \( E \) be the points where \( L(P,A) \) meets sidelines \( AB \) and \( AC \). Unless \( D = E = A \), the triangle \( ADE \) is isosceles.

In ETC, there are several triangle centers defined in terms of isoscelizers. These were discovered or invented by Peter Yff, in whose notebooks the word isoscelizer dates back to 1963.
Chapter 11

Circumcevian stuff

11.1 Circum-cevians, circum-anticevians

Definition 11.1.1. Circumcevian. Let $P$ be a point, not on $\Gamma$ (the circumcircle of $ABC$). Let $A'$ be the other intersection of line $AP$ with $\Gamma$ and define $B'$, $C'$ cyclically. Then $A'B'C'$ is the circumcevian triangle of $P$. Using barycentric columns,

$$
circumcevian(P) = \begin{pmatrix}
-\frac{qra^2}{c^2q + b^2p} & p & p \\
q & -\frac{rbp^2}{a^2r + c^2p} & q \\
-r & r & -\frac{pcq^2}{b^2p + a^2q}
\end{pmatrix}
$$

(11.1)

Proposition 11.1.2. A circumcevian triangle is a central triangle. When $P$ is on $\Gamma$, the corresponding triangle is totally degenerate. Three points on the circumcircle form a circumcevian triangle when the triangle obtained by killing the diagonal of the matrix is a cevian triangle.

Proof. Centrality follows directly from (11.1) (that’s a reason to keep denominators), while killing the diagonal gives the intersections with sidelines. Determinant is $\Gamma(P)^3$ over $\prod a^2q + b^2p$. □

Definition 11.1.3. Circum-anticevian. Consider the anticevian triangle $P_A P_B P_C$ of a point $P$ that is not a vertex of triangle $ABC$. Line $P_B P_C$ cuts circumcircle $\Gamma$ at $A$. Let $A'$ be the other intersection and define $B'$, $C'$ cyclically. Then $A'B'C'$ is the circum-anticevian triangle of $P$. Using barycentric columns,

$$
circumanticevian(P) = \begin{pmatrix}
\frac{qra^2}{c^2q - b^2r} & -p & p \\
q & -\frac{rbp^2}{a^2r - c^2p} & -q \\
-r & r & \frac{pcq^2}{b^2p - a^2q}
\end{pmatrix}
$$

(11.2)

Proposition 11.1.4. This triangle should be a central triangle (does the definition allows that ?). When one of the three other points $\pm p : \pm q : \pm r$ is on $\Gamma$, the circum-anticevian triangle degenerates.

11.2 Steinbart transform

Definition 11.2.1. Exceter point. The circumcevian triangle of the centroid, $X_2$, is perspective to the tangential triangle $A_6$. The perspector, $X_{22}$, is named Exceter point, for Phillips Exceter Academy in Exceter, New Hampshire, USA, where $X_{22}$ was detected in 1986 using a computer.

Definition 11.2.2. Steinbart transform. The circumcevian triangle of a point $P$ is ever perspective to the tangential triangle $A_6$. The corresponding perspector has been called Steinbart point by Funck (2003) and was called in TCCT (p. 201). This transformation carries triangle centers to triangle centers. Using barycentrics :
\[ Steinbart(P) = a^2 \left( \frac{b^4}{q^2} + \frac{c^4}{r^2} - \frac{a^4}{p^2} \right) : b^2 \left( \frac{a^4}{p^2} - \frac{b^4}{q^2} + \frac{c^4}{r^2} \right) : c^2 \left( \frac{a^4}{p^2} + \frac{b^4}{q^2} - \frac{c^4}{r^2} \right) \]

**Example 11.2.3.** On the circumcircle, Steinbart transform is the identity. Here is a list of other \((I,J)\) such that \(Steinbart(X(I)) = X(J)\):

\[
\begin{align*}
1 &: 3 & 14 & 1606 & 56 & 1616 & 162 & 1624 & 365 & 55 \\
2 &: 22 & 17 & 1607 & 57 & 1617 & 163 & 1625 & 366 & 1631 \\
3 &: 1498 & 18 & 1608 & 58 & 595 & 174 & 1627 & 651 & 1633 \\
4 &: 24 & 19 & 1609 & 59 & 1618 & 251 & 1628 & 651 & 1633 \\
5 &: 1601 & 21 & 1610 & 63 & 1619 & 254 & 1628 & 662 & 1634 \\
6 &: 6 & 25 & 1611 & 64 & 1620 & 259 & 1629 & 662 & 1634 \\
7 &: 1602 & 28 & 1612 & 81 & 1621 & 259 & 1629 & 662 & 1634 \\
8 &: 1603 & 31 & 1613 & 83 & 1078 & 266 & 56 & 1634 & 662 \\
9 &: 1604 & 54 & 1614 & 84 & 1622 & 275 & 1629 & 662 & 1634 \\
13 &: 1605 & 55 & 1626 & 509 & 1486 & 1627 & 651 & 1633 & 662 \\
19 &: 1606 & 56 & 1634 & 73 & 2369 & 184 & 2367 & 1084 & 689 \\
31 &: 767 & 58 & 2372 & 213 & 2368 & 1634 & 689 & 662 & 1634 \\
32 &: 2367 & 67 & 2373 & 219 & 2376 & 662 & 1634 & 662 & 1634 \\
37 &: 741 & 69 & 2374 & 220 & 2377 & 662 & 1634 & 662 & 1634 \\
\end{align*}
\]

Points \(X(1601)-X(1634)\) have been contributed in ETC by Jean-Pierre Ehrmann (August 2003).

**Remark 11.2.4.** See Grinberg (2003d) and his Extended Steinbart Theorem in Hyacinthos \#7984, 2003/09/23.

### 11.3 Circum-eigentransform

**Definition 11.3.1.** The circum-eigentransform of point \(U = u : v : w\), different from \(X_6\), is the eigencenter of the circumcevian triangle of point \(U\) and is denoted by \(CET(U)\). In trilinears, we have:

\[
\begin{align*}
&avw : bwu : cwv \\
&\frac{avw}{av^2 + aw^2 - bw - cw} : \frac{bwu}{bw^2 + bu^2 - cw - av} : \frac{cwv}{ca^2 + cv^2 - aw - bw}
\end{align*}
\]

and in barycentrics (cyclically):

\[
\frac{a^2vw}{a^2c^2v^2 + a^2b^2w^2 - b^2c^2uv - bc^2uw}
\]

**Proposition 11.3.2.** Point \(CET(U)\) lies on the circumcircle, and we have \(CET(U) = isog(U)\) if and only if \(U \in \mathcal{L}_\infty\).

**Remark 11.3.3.** My own computations are leading to:

\[
\frac{a^2vw}{-a^2c^2v^2 + a^2b^2w^2 - b^2c^2uv + bc^2uw}
\]

This point is also on the circumcircle, but property \(CET(U) = isog(U)\) if and only if \(U \in \mathcal{L}_\infty\) is lost. Some signs have changed: why?

**Example 11.3.4.** Apart from points on the infinity line, pairs \((I,J)\) such that \(X(J) = CET(X(I))\) include:

\[
\begin{align*}
1 &: 106 & 41 & 767 & 74 & 1294 & 238 & 741 \\
2 &: 729 & 42 & 2368 & 75 & 701 & 265 & 1300 \\
3 &: 1300 & 43 & 106 & 81 & 2375 & 670 & 3222 \\
4 &: 1294 & 44 & 106 & 110 & 99 & 694 & 98 \\
9 &: 1477 & 55 & 2369 & 125 & 827 & 895 & 2374 \\
19 &: 2365 & 56 & 2370 & 184 & 2367 & 1084 & 689 \\
25 &: 2366 & 57 & 2371 & 194 & 729 & 1279 & 1477 \\
31 &: 767 & 58 & 2372 & 213 & 2368 & 1634 & 689 \\
32 &: 2367 & 67 & 2373 & 219 & 2376 & 662 & 1634 \\
37 &: 741 & 69 & 2374 & 220 & 2377 & 662 & 1634 \\
\end{align*}
\]
Exercise 11.3.5. For a given point \( P \) on the circumcircle, which points \( U \) satisfy \( CET(U) = P \)? For example, \( CET \) carries each of the points \( X(1), X(43), X(44), X(519) \) to \( X(106) \).

11.4 Dual triangles, DC and CD Points

Definition 11.4.1. Dual triangle. Suppose \( DEF \) is a triangle (at finite distance) in the plane of triangle \( ABC \). Let \( D' \) be the isogonal conjugate of the point at infinity of line \( EF \). Define \( E' \) and \( F' \) cyclically. The triangle \( D'E'F' \) is called the dual of \( DEF \). Its vertices lie on the circumcircle.

Proposition 11.4.2. The dual triangle \( D'E'F' \) characterizes the class of all triangles homothetic to \( DEF \). Moreover, this triangle is similar to the original one.

Proof. Vertices of \( D'E'F' \) depends only on direction of sidelines \( DE, EF, FD \) and conversely. Similarity between \( DEF \) and \( D'E'F' \) can be proved in many ways. Brute force method: Pythagoras Theorem 7.2.4 applied to both triangles leads to proportional sidelengths.

Remark 11.4.3. (Proof of similarity follows from Theorem 6E in TCCT, as the "gamma triangle" there is the dual of a triangle whose sidelines are respectively perpendicular to those of \( DEF \).)

Definition 11.4.4. DC point. Suppose \( U = u : v : w \) is a point having cevian triangle \( DEF \) and dual triangle \( D'E'F' \). It happens that the later triangle is also the circum-anticevian triangle of some point. This point will be described as \( DC(U) \). Using barycentrics:

\[
DC(U) = \frac{a^2}{u(v+w)} : \frac{b^2}{v(w+u)} : \frac{c^2}{w(u+v)}
\]

Remark 11.4.5. The barycentrics of triangle \( D'E'F' \) are:

\[
\frac{a^2}{wu-vu} : \frac{-a^2}{wu+vu} : \frac{a^2}{wv+wv}
\]

\[
\frac{wu+vu}{b^2} : \frac{wu+vu}{b^2} : \frac{-b^2}{c^2}
\]

\[
\frac{vu-wv}{c^2} : \frac{vu-wv}{c^2} : \frac{wv-wv}{wv+wv}
\]

Proposition 11.4.6. To construct \( DC(U) \) from \( U \) and \( D'E'F' \), let \( A' = AD' \cap BC \) and let \( A'' \) be the harmonic conjugate of \( A' \) with respect to \( B \) and \( C \). Define \( B'' \) and \( C'' \) cyclically. The lines \( AA'', BB'' \) and \( CC'' \) concur in \( DC(U) \). We have also the formula:

\[
DC(U) = \text{cevamul (isog}(U) \cdot X(6))
\]

Example 11.4.7. There are 115 pairs \((I,J)\) such that \( I<2980 \) and \( X(J) = \text{DC}(X(I)) \). Among them \((1,81), (2,6), (3,275), (4,2), (5,288), (6,83), (7,1), (8,57), (9,1170), (10,1171)\). The longest chain for this relation is: \( 69 \rightarrow 4 \rightarrow 2 \rightarrow 6 \rightarrow 83 \rightarrow 3108 \).

Proposition 11.4.8. Inversely, the circum-anticevian triangle of a point \( P \) is the dual of the cevian triangle of a point \( CD(P) \), given for \( P = p : q : r \) by the inverse of the DC-mapping; that is:

\[
\frac{1}{a^2 qr + b^2 pr + c^2 pq} : \frac{1}{a^2 qr - b^2 pr + c^2 pq} : \frac{1}{a^2 qr + b^2 pr - c^2 pq}
\]

In other words, we have:

\[
CD(P) = \text{isog}(\text{cevadiv}(P), X(6))
\]

11.5 Saragossa points

Definition 11.5.1. Saragossa points. Let \( P \) be a point not on the circumcircle of \( ABC \). Let \( T' = A'B'C' \) be the cevian triangle of \( P \) and \( T'' = A'', B'', C'' \) the circumcevian triangle of \( P \). Consider triangle \( T \) that is the crosstriangle of \( T' \) and \( T'' \), i.e. the triangle whose vertices are \( U = B''C'' \cap B'C', V = C'A' \cap C''A'' \) and \( W = A'B'' \cap A'B' \). Then (Figure 11.1) triangles \( ABC, T', T'' \) and \( T \) are pairwise perspective ((Grinberg, 2003c). The first, second and third Saragossa points of \( P \) are the perspectors of \( T \) with, respectively, \( ABC, T' \) and \( T'' \). The name Saragossa refers to the king who proved Ceva’s theorem before Ceva did (Hogendijk, 1995).
Proposition 11.5.2. The barycentrics of the Saragossa points of $P = p : q : r$ are (cyclically):

\[
g_1(a, b, c) = \frac{a^2}{a^2qr - (a^2qr + b^2pr + c^2pq)}
\]

\[
g_2(a, b, c) = p - \left(\frac{1}{c^2q} + \frac{1}{b^2r}\right) (a^2qr + b^2pr + c^2pq)
\]

\[
g_3(a, b, c) = 2p - \left(\frac{1}{c^2q} + \frac{1}{b^2r}\right) (a^2qr + b^2pr + c^2pq)
\]

Points $P, Q_2, Q_3$ are clearly collinear. When one of the Saragossa points is equal to $P$ then $P$ is $X(6)$ or lies on the circumcircle.

*Proof.* Computations are straightforward.

Example 11.5.3. The following table give the Saragossa points of the $X(I)$ whose number is given in the first line.

<table>
<thead>
<tr>
<th></th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
<th>6</th>
<th>19</th>
<th>21</th>
<th>24</th>
<th>25</th>
<th>28</th>
<th>31</th>
</tr>
</thead>
<tbody>
<tr>
<td>58</td>
<td>251</td>
<td>4</td>
<td>54</td>
<td>1166</td>
<td>6</td>
<td>284</td>
<td>961</td>
<td>847</td>
<td>2</td>
<td>943</td>
<td>81</td>
<td></td>
</tr>
</tbody>
</table>

11.6 Vertex associates

**Definition 11.6.1.** Vertex associate. Consider the circumcevian triangles $A_pB_pC_p, A_uB_uC_u$ of points $P, U$ (not both on the circumcircle) and draw their may be degenerate vertex triangle $T$ i.e. the triangle whose sidelines are $A_pA_u, B_pB_u, C_pC_u$. It happens that $T$ is perspective to $ABC$ : the corresponding perspector $X$ is called the vertex associate of $P$ and $U$.

**Proposition 11.6.2.** When $P \in \Gamma$ but $U \notin \Gamma$, $A_pB_pC_p$ and $T$ are totally degenerate at $P$, so that $X = P$ (regardless of $U$). Otherwise, the barycentrics of the vertex associate of $p : q : r$ and $u:v:w$ are (cyclically):

\[
\text{vertex third } (P, U) = \frac{a^2}{wrq^2u - p(wb^2 + vc^2)u(rb^2 + c^2q)}
\]

*Proof.* When both $P, U$ are on $\Gamma$, both circumcevians are totally degenerate and $X$ is not defined.
Remark 11.6.3. The definition of vertex conjugate allows $X = U$. To extend the geometric interpretation to the case that $X = U$, as $X$ approaches $U$, the vertex triangle approaches a limiting triangle which we call the tangential triangle of $U$, a triangle perspective to $ABC$ with perspector $U$-vertex conjugate of $U$.

**Proposition 11.6.4.** When $P$ is not on $\Gamma$, but $U$ is on $P_\Gamma$ (the $\Gamma$-polar of $P$), then $\mathcal{T}$ is totally degenerate to a point $X$ that is the $\Gamma$-pole of line $PU$. Finally, triangle $PUX$ is autopolar wrt $\Gamma$.

**Proof.** Concurrence of $A_pA_u$, $B_pB_u$, $C_pC_u$ in a point $X$ is straightforward, and $X \in P_\Gamma$ too. \hfill $\Box$

**Proposition 11.6.5.** Operation $\text{vertexthird}$ is commutative and "formally involutory" i.e. :

\[
\text{vertexthird}(P, \text{vertexthird}(P, U)) \simeq U
\]

unless $P$ lies on the circumcircle (where $\text{vertexthird}(P, U) = P$, regardless of $U$).

**Proof.** Commutativity is from the very definition, and the formal involutory property is from straightforward computations. It remains only to track degeneracies. Determinant of the vertex triangle $\mathcal{T}$ is the square of determinant of the corresponding trigone ($\mathcal{T}$ is either a genuine triangle, or totally degenerate). The factors are the conditions for $U \in \Gamma$, $P \in \Gamma$ and the condition for one point to be on the $\Gamma$-polar of the other. \hfill $\Box$

**Exercise 11.6.6.** In the general case, what can be said about the way the three circumcevian triangles are lying on $\Gamma$ ?

**Example 11.6.7.** Here are some vertex conjugates $X(I), X(J), X(K)$ :

\[
\begin{array}{ccccccccccc}
1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 \\
1 & 56 & 3415 & 84 & 3417 & 2163 & 3418 & 3420 & \\
2 & 3415 & 25 & 3424 & 3425 & 1383 & \\
3 & 84 & 3424 & 64 & 4 & 3426 & 3427 & \\
4 & 3417 & 3425 & 4 & 3 & 3431 & \\
5 & & & & & 3432 & \\
6 & 2163 & 1383 & 3426 & 3431 & 6 & \\
7 & 3418 & 3427 & 3433 & \\
8 & & & & & 3435 & \\
9 & 3420 & & & & & 1436 & \\
\end{array}
\]

**Proposition 11.6.8.** For a given $U$, not on the circumcircle, the associated first Saragossa point (11.3) is the sole and only point $X$ such that $\text{vertexthird}(U, X) = X$.

**Proof.** We want $U = \text{VT}(X, \text{VT}(X, U)) = \text{VT}(X, X)$ while $\text{sarag1} (\text{VT}(P, P)) = P$ is straightforward. \hfill $\Box$

**Proposition 11.6.9.** Vertex association wrt $X_3$ maps the Darboux cubic to the Darboux cubic ($X_3$ is the reflection center of this cubic, whose pole is $X_6$ and pivot $X_20$). The appearance of $(I,J)$ in the following list means that $X(I), X(J)$ are on the Darboux cubic and that $X(3), X(I), X(J)$ are vertex associates :

\[
\begin{array}{ccccccccccc}
1 & 3 & 4 & 20 & 40 & 1490 & 1498 & 2131 & 3182 & \\
84 & 64 & 4 & 3346 & 3345 & 3347 & 3348 & 3183 & 3354 & \\
\end{array}
\]
Chapter 12

About conics

Notation 12.0.1. \( C \) is a conic, \( \mathbf{C} \) is the matrix of the punctual equation while \( \mathbf{C}^* \) is the matrix of the tangential equation. Point \( P \) is (often) the perspector, \( U \simeq u : v : w \) is (often) the center, \( Q \simeq f : g : h \) the auxiliary point (of an inconic), \( F_j \simeq f_j : g_j : h_j \) the focuses i.e. \( F_0 \) for a parabola and \( F_1, F_2 \) otherwise.

12.1 Tangent to a curve

Definition 12.1.1. An algebraic curve \( C \) is the set of all the points \( x : y : z \) that satisfy a polynomial equation \( C(x,y,z) = 0 \). In order to be a projective property, the polynomial \( C(x,y,z) \) is required to be homogeneous (this is ever assumed in what follows).

Proposition 12.1.2. Consider an algebraic curve \( C \) (not necessarily a conic). The line tangent to \( C \) at point \( P = p : q : r \) is given by :

\[
\overrightarrow{\text{grad}}(C)_{p,q,r} = \left[ \left( \frac{\partial C}{\partial x} \right)_{X=P}, \left( \frac{\partial C}{\partial y} \right)_{X=P}, \left( \frac{\partial C}{\partial z} \right)_{X=P} \right] \tag{12.1}
\]

Proof. Let \( P \in C \) be the contact point and \( P + kQ \) be a point in the vicinity. If we require \( P + kQ \in C \), we must have \( C(P + kQ) - C(P) = O(k^2) \) and this is \( \overrightarrow{\text{grad}}(C)_{p,q,r} \cdot U = 0 \). But polynomial \( C \) is homogeneous and we have \( \overrightarrow{\text{grad}}(C)_{p,q,r} \cdot P = dq(C) \cdot C \). The result follows.

Exercise 12.1.3. Use parametrization (7.16) to describe the points \( P \) of the circumcircle. Obtain the tangent at \( P \). Take the orthodir and obtain the normal. Differentiate and wedge to catch the contact point of the envelope of all the normals... and obtain X(3).

Definition 12.1.4. Pole and polar. The polar line of point \( X \) with respect to an algebraic (homogeneous) curve \( C \) is the line whose affix is the gradient of \( C \) evaluated at point \( X \). Point \( X \) is called a pole of its polar.

Remark 12.1.5. When point \( X \) is a simple point on an algebraic curve, its polar is nothing but the line tangent at \( X \) to the curve. Finding all the points whose polar is a given line is not an easy task in the general case.

Definition 12.1.6. To avoid misunderstandings, it is often useful to specify the curve used to polarize. So we will use circumpolar to describe polarity wrt the circumcircle, and conipolar to describe polarity wrt a given specified conic.

Remark 12.1.7. Triangle \( ABC \) can be seen as the curve \( xyz \). Gradient evaluated at point \( P \simeq p : q : r \) is \( [qr,rp,pq] \) i.e. the already defined tripolar. And we can see why the tripolar transform is not "as nice as" the usual conipolar transform: the degree of the underlying curve is not 2: there is no more "reciprocity".

123
12.2 Folium of Descartes

The curve known as the "folium of Descartes" is surely not a conic! But, in our opinion, it could be useful to see how some general methods are working in the general case, before using them in the rather specific situation of the algebraic curves of degree two.

**Definition 12.2.1.** The folium $F$ is the curve which Descartes used to check his methods regarding the coordinate system. The Cartesian equation of this curve is $x^3 + y^3 - 6xy = 0$, and its homogeneous equation is

$$X^3 + Y^3 - 6XYT = 0$$

**Proposition 12.2.2.** The folium presents a double point at $0 : 0 : 1$. If we cut by the line $Y = pX$, we obtain the parametrization: $M \simeq 6p : 6p^2 : 1 + p^3$. A better parametrization is:

$$M \simeq 3(1 + q)(1 - q)^2 : 3(1 + q)^2(1 - q) : 3q^2 + 1$$

Then the tangent $\Delta_M$ at $M \in F$ is given by

$$N = x : y : z \in \Delta_M \text{ when } \begin{bmatrix} 3X^2 - 6T, 3Y^2 - 6T, -6XY \end{bmatrix} \cdot \begin{bmatrix} x \\ y \\ z \end{bmatrix} = 0$$

**Proof.** Homography $q = (p - 1)/(p + 1)$ has been used to move point $p = -1$ at $q = \infty$ in order to have a "one piece" curve. Tangency condition is $\text{grad}\phi \cdot M\hat{N} = 0$, while, due to the Darboux property, we already have $\text{grad}\phi \cdot M = 0$.

**Example 12.2.3.** When $q_1 = 1/3$ and $q_2 = -3$, we obtain points $M_1 = 4 : -5 : 8$ and $M_2 = 24 : -12 : 7$. Tangents are $\begin{bmatrix} 4 & -5 & 8 \end{bmatrix}$, $\begin{bmatrix} 17 & 20 & 24 \end{bmatrix}$ while their common point is (see Figure 12.1) given by:

$$\begin{bmatrix} 4 & -5 & 8 \end{bmatrix} \land \begin{bmatrix} 17 & 20 & 24 \end{bmatrix} = 56 : -8 : -33 \approx -1.70 : +0.24 : 1$$

Visible asymptote is the tangent at the visible $T = 0$ point, i.e. at $+1 : -1 : 0$. Using the gradient at that point, we see that asymptote is $[3, 3, -6] \simeq [1, 1, -2]$.

![Figure 12.1: Folium of Descartes](image)

**Proposition 12.2.4.** Tangential equation of the folium (i.e. the condition for a line $\Delta \simeq [u, v, w]$ to be tangent to the curve is:

$$F^* (u, v, w) = 48u^2v^2 - 32w(u^3 + v^3) + 24uvw^2 - w^4 = 0$$

January 3, 2024 21:08 published under the GNU Free Documentation License
Proof. Tangency requires a contact point, so that:
\[
\{(q - 1) (3q^3 + 9q + 3q^2 + 1) = Kv, (q + 1) (3q^3 + 9q - 3q^2 - 1) = Ku, 6 (q + 1)^2 (q - 1)^2 = Kw\}
\]
is required. Apart from \(w = 0\) or \(q = \pm 1\), last equation gives a \(K\) value, that can be substituted into the other equations. Writing that the remaining two polynomials have the same roots, we obtain the resolvant:
\[
\begin{pmatrix}
-w - 6v & -9w - 6v & -3w + 6v & -3w + 6v & 0 & 0 \\
0 & -w - 6v & -9w - 6v & -3w + 6v & -3w + 6v & 0 \\
0 & 0 & -w - 6v & -9w - 6v & -3w + 6v & -3w + 6v \\
w + 6u & -9w - 6u & 3w - 6u & -3w + 6u & 0 & 0 \\
w + 6u & -9w - 6u & 3w - 6u & -3w + 6u & 0 & 0 \\
0 & 0 & w + 6u & -9w - 6u & 3w - 6u & -3w + 6u
\end{pmatrix}
\]
Suppressing the non vanishing factors leads to the given result.

Example 12.2.5. Start from a point \(N \simeq x : y : z\) and search the \(u, v, w\) such that:
\[
\{ux + vy + wz = 0, \Psi = 0\}
\]
We have three different possibilities, that are exemplified by:
\[
\begin{pmatrix}
u \\
v \\
v \\
u \\
u
\end{pmatrix}
\begin{pmatrix}
v \\
v \\
v \\
v \\
v
\end{pmatrix}
\begin{pmatrix}
w \\
w \\
w \\
w \\
w
\end{pmatrix}
\]

\[
N \simeq 1 : 1 : 1,
\begin{pmatrix}
1.0 & -0.51903 - 1.16372i & -0.48097 + 1.16372i \\
1.0 & -0.31967 + 0.71764i & -0.68033 - 0.71764i \\
1.0 & -0.51903 + 1.16372i & -0.48097 - 1.16372i \\
1.0 & -0.31967 - 0.71764i & -0.68033 + 0.71764i
\end{pmatrix}
\]

\[
N \simeq 4 : 2 : 1,
\begin{pmatrix}
1.0 & -1.52334 & -0.95333 \\
1.0 & -0.73833 - 1.09791i & -2.52334 + 2.19582i \\
1.0 & -0.73833 + 1.09791i & -2.52334 - 2.19582i
\end{pmatrix}
\]

\[
N = -\frac{1}{2} : -\frac{1}{2} : 1,
\begin{pmatrix}
1.0 & 1.35307 & 1.17653 \\
1.0 & 0.73906 & 0.86953 \\
1.0 & -0.43584 & 0.28208 \\
1.0 & -2.29442 & -0.64721
\end{pmatrix}
\]

In other words, the curve and its asymptote are dividing the plane into three zones. From a magenta point (see Figure 12.1b) no tangents can be drawn to the curve, but two from a green point and four from a cyan point.

Proposition 12.2.6 (Plucker formulas). Let \(d, \delta, \kappa\) be respectively the degree of a curve \(C\), its number of nodes (double points with two tangents), its number of cups (double point with a single tangent) and \(d', \delta', \kappa'\) be the corresponding numbers for the dual curve \(C'\) then we have the following relations:
\[
\begin{align*}
d' &= d (d - 1) - 2\delta - 3\kappa \\
\kappa' &= 3d (d - 2) - 6\delta - 8\kappa \\
d &= d' (d' - 1) - 2\delta' - 3\kappa' \\
\kappa &= 3d' (d' - 2) - 6\delta' - 8\kappa' \\
g &\geq \frac{1}{2} (d - 1) (d - 2) - \delta - \kappa = \frac{1}{2} (d' - 1) (d' - 2) - \delta' - \kappa'
\end{align*}
\]

Proof. Proof is not obvious since the formula turns weird when points with multiplicity greater than two are occurring. There is no simpler formula giving \(d\) than \(g = g'\). A remark : we have \(3d - \kappa = 3d' - \kappa'\). Application : see Figure 12.2. Cups are occurring at \(x = y = 1/2\) and at \(x = j/2, y = j^2/2\) and conjugate.
12.3 General facts about conics

**Definition 12.3.1.** A conic $C$ is a curve whose barycentric equation is an homogeneous polynomial of second degree. This can be written using the usual matrix apparatus:

$$X \in C \iff {}^tX \cdot C \cdot X = (x, y, z) \begin{pmatrix} m_{11} & m_{12} & m_{13} \\ m_{12} & m_{22} & m_{23} \\ m_{13} & m_{23} & m_{33} \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = 0$$

**Lemma 12.3.2.** Comatrix. Let $M$ be an $n \times n$ matrix and $M^*$ the matrix of the cofactors (at the right place, such that cofactors of a row form a column). Then:

$$M \cdot M^* = M^* \cdot M = \det(M) 1_n$$

Moreover, rank $M^* = n$ when rank $M = n$, rank $M^* = 1$ when rank $M = n - 1$, and otherwise $M^*$ is the 0 matrix.

**Proof.** When $\det(M) \neq 0$, both matrices can be inverted. In the second case a row of $M^*$ describes the hyperplane obtained by any non-zero wedge of two columns of $M$. In the last one, all minors are 0 since rank is less than $n - 1$.

**Remark 12.3.3.** Comatrix $M^*$ is also called the adjoint matrix of $M$. Frenchies are usually proud to dispose wrongly the cofactors... and then are reduced to use the transpose of what they are wrongly calling "comatrices".

**Definition 12.3.4.** Proper conic. A quadratic form can be written as the sum of as many squares of independent linear forms as its rank, leading to the following classification:

1. When rank is one, $C$ is a straight line, whose points are each counted twice (strange object).

2. When rank is 2, $C$ is the union of two intersecting lines. When these lines are complex conjugate of each other (not real), only their intersection is real and this point appears as an isolated point. When one of these lines is the line at infinity, $C$ is considered as some kind of extended circle (see Chapter 13).

3. When $\det(C) \neq 0$, intersection of $C$ and any straight line contains exactly two points (real or not, may be a double point). A such conic is called a proper conic.
Proposition 12.3.5. Conic by five points. Let $P_j \simeq q_j : r_j$ be five fixed points and $P = P_6 \simeq x : y : z$ be a generic sixth point. In this section we define the Veronese map as

$$V_{\operatorname{Ver}} : [p : q : r] \mapsto [p^2 : q^2 : r^2 : pq : qr : rp]$$

Let $\hat{Q}$ be the $6 \times 6$ matrix $\left[ V_{\operatorname{Ver}} (P_j) \right]$. Then the six points $P_1 \ldots P_6$ are coconic when the six Verones are co-hyperplanar, i.e. when $Q(x, y, z) = \det \hat{Q} = 0$. Let $Q$ be the matrix of $Q$ so that $Q = \operatorname{Ver} (P) \cdot P$. Then:

1. When none of the 10 $P_j$-triples are collinear, then $Q$ defines a proper conic.
2. When four of the five points $P_j$ are collinear, then $Q \equiv 0$ and no conic is defined.
3. Otherwise, the conic degenerates into the reunion of two lines.

Proof. In all of the cases, a simple computation leads to the "ten determinants formula"

$$\det \hat{Q} = \prod_{j,k,l} \det (P_j P_k P_l)$$

while the relation $P_1 = a P_1 + b P_2$ leads to the factorization:

$$\det \hat{Q} = ab \det (P_1 P_2 P_4) \det (P_1 P_2 P_3) \times \det (P_1 P_2 P) \det (P_2 P_3 P) \quad \square$$

Proposition 12.3.6. Proper parametrization. Given a proper conic $C$, a basis can be found where the equation of $C$ becomes $xz - y^2 = 0$. And then, a parametrization is $u^2 : uv : v^2$.

Proof. Rewrite the equation as a sum of three squares. And then use $x^2 + z^2 = (x - iz)(x + iz)$. More precisely, defining

$$K^2 \doteq m_{22} m_{33} - m_{23}^2 ; \chi \doteq \begin{bmatrix} m_{12} & m_{22} & m_{23} \\ 1 & 0 & 0 \\ m_{22} m_{13} - m_{23} m_{12} & 0 & K^2 \end{bmatrix} \cdot \begin{bmatrix} 1 & 0 & 1 \\ 0 & 2 & 0 \end{bmatrix}$$

leads to $^{t} \chi \cdot \left[ C \right] \chi \simeq \begin{bmatrix} 0 & 0 & K^2 \\ 0 & 2 m_{22} \det C & 0 \\ K^2 & 0 & 0 \end{bmatrix} \quad \square$

Definition 12.3.7. Pole and polar. According to the general definitions, the pole of a point $X$ wrt a proper conic $C$ is the line $^{t}X \left[ C \right]$, i.e. the locus of the points $Y$ such that $^{t}X \left[ C \right] Y = 0$, while the pole of line $\{Y \mid \Delta Y = 0\}$ is the point $X = \text{Adjoint} \left( \left[ C \right] \right) ^{t} \Delta$.

Remark 12.3.8. This polarity is not to be confused with polarity wrt the main triangle. Therefore, it can be useful to describe line $^{t}X \left[ C \right]$ as the conipolar of $X$ and line $^{t}\text{isotom} (X)$ as the tripolar of $X$.

Remark 12.3.9. The relation "point $Q$ belongs to the polar of $P$" is symmetric. When point $P$ belongs to $C$, its polar wrt the conic is nothing but the line tangent at $X$ to the conic.

Proposition 12.3.10. For two distinct points $A, B$, and for two distinct lines $\Delta_1, \Delta_2$, we have

$$\text{polar} (A \lor B) = (\text{polar} A) \lor (\text{polar} B)$$
$$\text{polar} (\Delta_1 \lor \Delta_2) = (\text{polar} \Delta_1) \lor (\text{polar} \Delta_2)$$

Proof. When $M$ is a $3 \times 3$ invertible matrix, then, for any columns $A, B$, we have:

$$(^{t}A \cdot M) \lor (^{t}B \cdot M) = M^{*} \cdot (^{t}(A \lor B))$$
This can be seen by taking coordinates, and expanding \((tA \cdot M) \wedge (tB \cdot M) - M^* \cdot (A \wedge B)\) to 0. Another proof is using a generic row, called \(X\) in what follows, and check that:

\[
X \cdot (tA \cdot M) \wedge (tB \cdot M) = \det \left[ X \cdot M^{-1} \cdot M, tA, tB \right] = \det \left[ X \cdot M^{-1}, (tA, tB) \right] \times \det M = \det \left[ X \cdot M^* \cdot (tA \wedge tB) \right] = X \cdot M^* \cdot (tA \wedge tB)
\]

\[\square\]

**Definition 12.3.11.** Taking the polar lines of the vertices of a triangle \(T\) gives a trigone. The associate triangle is called the polar triangle of \(T\) and noted polar \(T\). When both triangles are equal, we say that \(T\) is autopolar wrt \(C\).

**Proposition 12.3.12.** When quadrangle \(A, B, C, D\) is inscribed in a proper conic \(C\), its diagonal triangle, i.e. \(AB \cap CD\), \(AC \cap BD\), \(AD \cap BC\) is autopolar wrt \(C\).

**Proof.** From 12.3.6, we can describe the problem by:

\[
[C] \simeq \begin{bmatrix} 0 & 0 & -1 \\ 0 & 2 & 0 \\ -1 & 0 & 0 \end{bmatrix}; A, B, C, D \simeq \begin{pmatrix} u_1^2 \\ u_1 v_1 \\ v_1^2 \end{pmatrix}, \begin{pmatrix} u_2^2 \\ u_2 v_2 \\ v_2^2 \end{pmatrix}, \begin{pmatrix} u_3^2 \\ u_3 v_3 \\ v_3^2 \end{pmatrix}, \begin{pmatrix} u_4^2 \\ u_4 v_4 \\ v_4^2 \end{pmatrix}
\]

and \((AB \cap CD) \cdot [C] \cdot (AC \cap BD) = 0\) is easy to verify. \[\square\]

**Construction 12.3.13.** The polar line of a point \(P\) wrt a conic \(C\) is the locus of the points \(AC \cap BD\) where \(AB\) and \(CD\) are chords of \(C\) that meet in \(P\). As a result, the required conipolar is the line \(AC \cap BD; AD \cap BC\).

**Proof.** Obvious from the previous proposition. \[\square\]

**Proposition 12.3.14.** Triangle \(T\) is autopolar wrt a proper conic \(C\) if and only if the matrix of \(C\) wrt triangle \(T\) is diagonal.

**Proof.** Use \(T\) as barycentric basis. Then eliminate and see that either \(C\) is diagonal, or contains a null column. \[\square\]

**Proposition 12.3.15.** Perspector of a conic wrt a triangle. When the polar triangle of \(T\) is not \(T\) itself, then both triangles are in perspective. This defines a perspector and a perspectrix (related to the triangle). When \(T\) is the reference triangle itself, the polar triangle is \([C]^{-1}\) so that:

\[
P \simeq \begin{pmatrix} \langle m_{22} m_{13} - m_{23} m_{12} \rangle \\ \langle m_{33} m_{12} - m_{32} m_{13} \rangle \\ \langle m_{11} m_{23} - m_{12} m_{23} \rangle \end{pmatrix} \simeq \Delta \simeq \begin{pmatrix} m_{11} m_{12} m_{23} m_{12} m_{13} \end{pmatrix}
\]

When nothing vanishes, the following shorter formulas can be used:

- isotom \(P \simeq m_{11} m_{23} m_{13} m_{12} : m_{22} m_{13} m_{23} m_{12} : m_{33} m_{12} m_{13} m_{23}\)
- tripolar \(\Delta \simeq m_{23} : m_{13} : m_{12}\)

**Proof.** The \(T = ABC\) case is easy to compute. And suffices to prove the general case. \[\square\]

**Definition 12.3.16.** Dual of a conic. The dual of a given conic \(C_1\) is the conic \(C_2\) such that point \(x : y : z\) belongs to \(C_2\) when line \((x, y, z)\) is tangent to \(C_1\). When dealing with proper conics, we have \([C_2] = \text{Adj}([C_1])\) and conversely. When rank is 2, the dual is rank 1: all tangents have to pass through the common point.

**Definition 12.3.17.** The center \(U\) of a conic \(C\) is the pole of the line at infinity \(\mathcal{L}_\infty\) with respect to the conic. Its barycentrics are:

- \(-m_{23}^2 + (m_{13} + m_{12}) m_{23} + m_{22} m_{33} - m_{13} m_{22} - m_{12} m_{33}\)
- \(-m_{13}^2 + (m_{12} + m_{23}) m_{13} + m_{33} m_{11} - m_{12} m_{33} - m_{23} m_{11}\)
- \(-m_{12}^2 + (m_{23} + m_{13}) m_{12} + m_{11} m_{22} - m_{23} m_{11} - m_{13} m_{22}\)
**Definition 12.3.18.** A parabola is a conic whose center is at infinity (more about parabolas in Section 12.20). Two parallel lines make a non proper parabola. The union of the line at infinity and another line is ... some kind of circle rather than a "special special" parabola.

**Fact 12.3.19.** When a conic goes through its center and this center is not at infinity, the conic is the union of two different straight lines. When a conic is a single line whose points are counted twice, det \( C \) vanishes and center has no meaning.

**Proposition 12.3.20.** When \( C \) is not a parabola, its center is the symmetry center of the conic.

*Computed Proof.* Substitute (3.1) into the equation and obtain \( C (x, y, z) \) times the square of the condition to be a parabola.

**Proposition 12.3.21.** Let \( P \) be a point not on the sidelines of \( ABC \). Six points are obtained by intersecting a sideline with a parallel through \( P \) to another sideline. These points are on the same conic \( C \). Equation, perspector \( T \) and center \( U \) are :

\[
C = \sum_{p} (q + r)qrst - \sum (p^2 + pr + 2qr)pyz
\]

\[
T = \frac{2pr + 2pq + qr}{2pq + pr + 2qr} : \frac{2pq + pr + 2qr}{pq + 2pr + 2qr}
\]

\[
U = \frac{p(2qr + pr + pq - p^2)}{q(2pr + pq + qr - q^2)} : r(2pq + pr + qr - r^2)
\]

Center \( U \) is at infinity (and \( C \) is a parabola) when \( P \) is at infinity or on the Steiner inconic. Points \( P \) and \( Q = p( q + r - p) : q(r + p - q), r(p + q - r) \) are leading to the same center \( U \). Point \( Q \) is at infinity when \( P \) is on the Steiner inconic.

*Proof.* Equation in \( Q \) is of third degree. The discriminant factors into minus a product of squares. Other computations are straightforward. Examples are \([P, U], [115, 523], [1015, 513], [1084, 512], [6, 182], [3, 182], [9, 1001], [1, 1001], [190, 1016], [664, 1275]\) for other points.

### 12.4 Tangential conics

**Definition 12.4.1.** A point-conic is a 'conic as usual', i.e. a locus of points, whose equation is \( ^t M \cdot C \cdot M = 0 \). In the general case, a line is tangent to \( C \) when \( \Delta \cdot C \cdot \Delta = 0 \). A line-conic or tangential conic is what is obtained when seeing \( C \) in equation \( \Delta \cdot C \cdot \Delta = 0 \) as the primitive object, and seeing the punctual conic \( ^t M \cdot C \cdot M = 0 \) as a derivative object.

**Proposition 12.4.2.** Degenerate line-conic. When \( \det (C) = 0 \), the equation splits in two first degree factors, and a line belongs to the conic when it goes through either of the fixed points (the centers) defined by each factor (supposed distinct).

**Example 12.4.3.** The degenerate line-conic of the isotropic lines (see Proposition 7.4.6).

**Proposition 12.4.4.** The degenerate conic formed by the two tangents drawn from a point \( X_0 \) to a given conic \( C \) is given by

\[
^t X \cdot \mathcal{D}_0 \cdot X \simeq \left( ^t X_0 \cdot C \cdot X_0 \right) * \left( ^t X \cdot C \cdot X \right) + (-1) * \left( ^t X_0 \cdot C \cdot X \right)^2
\]

\[
\mathcal{D}_0 \simeq \left( ^t X_0 \cdot C \cdot X_0 \right) \mathcal{C} - \mathcal{C} \cdot X_0 \cdot ^t X_0 \cdot \mathcal{C}
\]

*bfp.* From 12.3.6, we can describe the problem by:

\[
\mathcal{C} \simeq \begin{bmatrix} 0 & 0 & -1 \\ 0 & 2 & 0 \\ -1 & 0 & 0 \end{bmatrix} ; X_1, X_2 \simeq \begin{pmatrix} u_1^2 \\ u_1 v_1 \\ v_1^2 \end{pmatrix}, \begin{pmatrix} u_2^2 \\ u_2 v_2 \\ v_2^2 \end{pmatrix}
\]

\[
tan_1, tan_2 \simeq [v_1^2, -2u_1 v_1, u_1^2], [v_2^2, -2u_2 v_2, u_2^2] ; X_0 \doteq tan_1 \wedge tan_2 \simeq 2u_1 v_2 : u_1 v_2 + u_2 v_1 : 2 v_1 v_2
\]

Then compute \((tan_1 \cdot X) * (tan_2 \cdot X)\) and check the formula. 

---

* pldx : Translation of the Kimberling’s Glossary into barycentrics
more geometrico. Consider the pencil of all the conics that are bi-tangent to conic \( C \). Among them, we have \( C \) itself, the required \( D_0 \) and the polar line of \( X_0 \), when counted twice. Thus:

\[
D_0 \simeq \alpha (C) + \beta (C \cdot X_0 \cdot X_0 \cdot C)
\]

It remains to choose \( \alpha, \beta \) so that \( D_0 \) goes through \( X_0 \).

12.5 Locusconi

Theorem 12.5.1. locusconi. Suppose that the projective coordinates of a point \( P(s) \) are second degree polynomials in a given parameter \( s \). Then the locus of \( P(s) \) is a conic. Moreover, we have the following algorithm.

Require: \( \text{tmp} \) is the column of the coordinates, and \( s \) is the parameter to eliminate

\[
\text{Locusconi} := \text{proc} \ tmp, \ s, \ \text{conitan} :: \ \text{uneval} ; \ \text{local} \ tmp1, \ tmp0, \ coni2
\]

\[
\begin{align*}
\text{tmp1} & := \text{Matrix}([\text{seq} (\text{map}(\text{coeff}, \text{reduce}(\text{tmp}, s, j), j = [2, 1, 0]))]) \\
\text{tmp0} & := \text{Matrix}([[0, 0, 2], [0, -1, 0], [2, 0, 0]]) \\
\text{coni2} & := \text{reduce} (\text{tmp1} . \text{tmp0} . \text{Tr}(\text{tmp1}) )
\end{align*}
\]

if nargs = 3 then assign (conitan, coni2) end if

\[
\text{reduce} (\text{Adjoint}(\text{coni2}))
\]

Listing 12.1: The locusconi procedure

Proof. When using \( Z : T : Z \) as the algebraic basis, the equation of the fundamental circle is \( Z^2 - T^2 \). We are doing the same here, using \( 1 : t : t^2 \) as algebraic basis. This leads to the tangential equation of the conic... and we save it immediately. Don’t compute it afterwards by taking the adjoint of the main result!

Exercise 12.5.2. Use the example \( P(s) = \begin{pmatrix} 7 s^2 - 12 \\ 3 s^2 + 2 s \\ 3 s^2 - 2 s - 6 \end{pmatrix} \). Show that the brute force identification method, i.e. solve \( ^t P \cdot [m_{jk}] \cdot P = 0, m_{jk} \) amounts to write

\[
\begin{bmatrix}
49 & 9 & 9 & 42 & 18 & 42 \\
0 & 12 & -12 & 28 & 0 & -28 \\
-168 & 4 & -32 & -72 & -44 & -156 \\
0 & 0 & 24 & -48 & -24 & 48 \\
144 & 0 & 36 & 0 & 0 & 144
\end{bmatrix}
\]

and then to compute the 6 cofactors of this \( 5 \times 6 \) matrix, while locusconi amounts to write

\[
\begin{bmatrix}
7 & 0 & -12 \\
3 & 2 & 0 \\
3 & -2 & -6
\end{bmatrix}
\]

and then to solve this system as: \( \text{Adjoint}( [m_{jk}] ) = ^t Q \cdot \text{Adjoint}( \Gamma ) \cdot Q \).

12.6 Founding configuration

Definition 12.6.1. Let us start from triangle \( ABC \) and perspector \( p : q : r \). Its tripolar \( \Delta \simeq [qr : rp : pq] \) goes through points \( Q_a \simeq 0 : +q : -r \), \( Q_b \simeq -p : 0 : +r \), \( Q_c \simeq +p : -q : 0 \). From
Proposition 3.8.12, all the triangles that can be written as $T_t \simeq \begin{pmatrix} tp & p & p \\ q & tq & q \\ r & r & tr \end{pmatrix}$ share $P$ and $\Delta$ as perspector and perspectrix. And their vertices are on the $P$-cevian lines (dotted cyan).

**Proposition 12.6.2.** We have $\text{cross\_ratio} (A, A_P, P, A_t) = \text{cross\_ratio} (\infty, 0, 1, t) = t$, etc.

**Proof.** Obvious since the cevian vertex $A_p$ is nothing but $A_0$.

**Proposition 12.6.3.** Vertices of $T_t$ and $T_s$ are on the same conic (noted $C(s,t)$) if and only if

$$2ts + t + s - 4 = 0 ; \quad s = \sigma(t) = \frac{-t + 4}{2t + 1}.$$  

Fixed points of $\sigma$ are $t = 1$, $t = -2$. And we have the decomposition:

$$C_t = (t^2 + 2) \begin{bmatrix} 2q^2r^2 & -pqr^2 & -pq^2r \\ -pqr^2 & 2p^2r^2 & -p^2qr \\ -pq^2r & -p^2qr & 2p^2q^2 \end{bmatrix} - 2 (t - 1)^2 \begin{bmatrix} q^2r^2 & 0 & 0 \\ 0 & p^2r^2 & 0 \\ 0 & 0 & p^2q^2 \end{bmatrix}$$

**Proof.** The corresponding $6 \times 6$ determinant factors into $(t - 1)^2 (s - 1)^2 (t - s)^3 q^4 r^4 p^4 (2st + s + t - 4)$.

**Proposition 12.6.4.** The center $K_t$ of conic $C_t$ belongs to line $PP^2$. More precisely, we have

$$(t^2 + 2) (p + q + r) \begin{pmatrix} p \\ q \\ r \end{pmatrix} - 2 (t - 1)^2 \begin{pmatrix} p^2 \\ q^2 \\ r^2 \end{pmatrix}$$

And we have the formula:

$$\text{cross\_ratio} (K(t_j)) = \text{cross\_ratio} (t_1, t_2, t_3, t_4) \times \text{cross\_ratio} (\sigma(t_1), \sigma(t_2), t_3, t_4)$$

$$= \text{cross\_ratio} (t_1, t_2, t_3, t_4) \times \text{cross\_ratio} (t_1, t_2, \sigma(t_3), \sigma(t_4))$$

**Proof.** Mind the fact that cross\_ratio $(\sigma(t_1), \sigma(t_2), \sigma(t_3), \sigma(t_4)) = \text{cross\_ratio} (t_1, t_2, t_3, t_4)$ !
Exercise 12.6.5. Prove that cross ratio $(K_{in}, P^2 : P, K_{out}) = -1$. that involves the conic $t = 1 = \sigma (1)$ which is centered at $P$ and degenerates into $3 (jq r + j^2 r p q + p q z) (j^2 qr x + j r p y + p q z)$ and the conic $t^2 + 2 = 0$ which is centered at $P^2$.

Example 12.6.6. We have the following cases of interest:

$C (−2, −2)$ degenerates into $−6 (qr x + rp y + pq z)^2$, i.e. the tripolar line $Q_a Q_b Q_c$.

$C_1 = C (1, 1)$ is centered at $P$, and degenerates into

$3 (jq r + j^2 r p q + p q z) (j^2 qr x + j r p y + p q z) = 0$

$C_{diag} = C (± i \sqrt{2})$ is centered at $P^2$ and is diagonal.

$C_{out} = C (∞, −1/2)$ is the so called $P$-circumconic (see Section 12.7). It goes through the vertices $(t = ∞)$ and the extra points $K_a = −p : 2q : 2r$, etc $(t = −1/2)$.

$C_{in} = C (0, 4)$ is the so called $P$-inconic (see Section 12.8). It goes through the cevian points $A_P B_P C_P (t = 0)$ and the extra points $R_a = 4p : q : r$, etc $(t = +4)$.

$T (−1)$ is the anticevian triangle $P_A P_B P_C$, obtained from the orange trigone $AQ_a, BQ_b, CQ_c$.

Proposition 12.6.7. The six sidelines of the following triangles:

$T (κ (t)), T (σ (κ (t)))$ where $κ (t) = \frac{2 − t}{t}$

are tangent to the conic $C (t, σ (t))$ at the vertices of the defining triangles. Note that $(σ κ) = κ$, enforcing the symmetry, while $Q_a, X_b, X_c$ are aligned for any triangle $T$.

Proof. Obvious computations

Example 12.6.8. Three tangents to $C_{in} = C (0, 4)$ are provided by $T (−1/2)$, see the green lines $Q_a R_a K_b K_c$, etc. As it should be, the other triangle is $T (∞)$, i.e. the $ABC$ triangle itself.

Six tangents to $C_{out} = C (∞, −1/2)$ are provided by $T (−5)$ and, as it should be, by the anticevian triangle $T (−1)$, see the orange lines $Q_a A_P B_P C_P$, etc.

Moreover, we have the alignments $Q_a R_b R_c$, not drawn), $Q_a B_P C_P$ (black dotted).

12.7 Circumconics

Definition 12.7.1. A circumconic is a conic that contains the vertices $A, B, C$ of the reference triangle. Its equation can be written as :

$CC (P) = ^t X \begin{bmatrix} \mathbb{C}_c \end{bmatrix} X = 0 \quad \text{where} \quad \begin{bmatrix} \mathbb{C}_c \end{bmatrix} = \begin{bmatrix} 0 & r & q \\ r & 0 & p \\ q & p & 0 \end{bmatrix}$ (12.4)

Construction 12.7.2. Graphical tools can construct any conic from five points. Given the perspector $P \simeq p : q : r$, the other points on the cevian lines are $−p : 2q : 2r$, etc i.e points $2q B + 2r C − p A$.

Proposition 12.7.3. When $p : q : r$ is the perspector, a parametrization of $CC(P)$ is

$M (t) \simeq \frac{p}{1} : \frac{q}{t} : \frac{−r}{1 + t}$ (12.5)

Proof. Direct inspection.

Theorem 12.7.4 (circumconics). We have the following four properties :

(i) Point $P$ is the perspector of the conic, and the polar triangle of $ABC \ wrt \ CC (P)$ is the anticevian triangle of $P \ wrt \ ABC$ (in other words, $CC (P)$ is tangent at $A$ to $P_B P_C$ etc.

(ii) When $U$ is the center of $CC (P)$ then $P$ is the center of $CC (U)$. Both are related by :

$U = \text{cevadiv} (X_2, P) = P * \text{anticomplem} (P)$
(iii) Circumconic $CC(P)$ is the $P$ isoconjugate of $\mathcal{L}_\infty$. Inter alia,

$$X \in \text{circumcircle} \iff \text{isog}(X) \in \mathcal{L}_\infty$$

(iv) The polar line of $M \simeq x : y : z$ wrt $CC(P)$ is $\simeq [qz + ry, rx + pz, py + qx]$ ... aka the polarmul of $(P, M)$.

**Proposition 12.7.5. Points at infinity.** A circumscribed conic is an ellipse, a parabola or an hyperbola when its perspector is inside, on or outside the Steiner in-ellipse. Moreover, its points at infinity, expressed from the perspector $P = p : q : r$ have the following barycentrics:

$$M_\infty \simeq \begin{pmatrix}
-2p \\
p + q - r - W \\
p + r - q + W
\end{pmatrix} \quad \text{where } W^2 = p^2 + q^2 + r^2 - 2pq - 2qr - 2rp$$

When the point at infinity of a circumparabola is $u : v : w$, its perspector is $P = u^2 : v^2 : w^2$.

**Proof.** Immediate computation. Mind the fact that $W^2 = -3$ when $P$ is at $X(2)$. For the second part, start from $T = u : v : -u - v$, and compute the circumconic relative to $u^2 : v^2 : (u + v)^2$. □

**Remark 12.7.6.** Properties of the CircumRH, aka the circumscribed rectangular hyperbola, are collected at Subsection 12.21.2

**Proposition 12.7.7.** The four common points of two (non equal) circumconics $CC(P)$ and $CC(Q)$ are the three vertices and the tripole of line $PQ$.

**Proof.** Conics that share five distinct points are equal. The value of $X$ follows by direct inspection. □

**Exercise 12.7.8.** A stupid person would rewrite this property as "the fourth point is isomot $(P \land Q)$.

But the clever reader wouldn’t. Explain why!

**Proposition 12.7.9.** The perspector $P$ of the circumconic through point $Q$ lies on the tripolar of $Q$. In other words,

$$P = Q \ast T \quad \text{where } T \in \mathcal{L}_\infty$$

Therefore, the perspector of the circumconic which goes through additional points $Q_1, Q_2$ is the intersection of the tripolars of $Q_1$ and $Q_2$ (caveat: this is not the tripole of line $Q_1 \land Q_2$). In other words, :

$$P = Q_1 \ast Q_2 \ast t(Q_1 \land Q_2)$$

**Proof.** Direct inspection. □

**Remark 12.7.10.** Collineations can be used to transform any circumconic into the circumcircle or the Steiner out-ellipse, so that many proofs can be done assuming such special cases. More details in Proposition 16.4.3.

**Proposition 12.7.11.** Let be given $\Delta$ and $K$ where $\Delta$ is a line (tripole $G$) that cuts the sidelines $BC, CA, AB$ in $A', B', C'$ and $K$ is a circumconic (perspector $P$) . Let $M = x : y : z$ be a point on $\Delta$, and $A''$ the other intersection of $MA$ with $K$ and cyclically $B'' \in MB \cap K, C'' \in MC \cap K$.

Then lines $A'A'', B'B'', C'C''$ are concurrent on a point $Q \in K$. Moreover, the conjugacy that exchanges $P$ and $G$ exchanges also $M$ and $Q(M)$.

**Proof.** Use $\Delta \simeq [\rho, \sigma, \tau], P = p : q : r, M = 1/\rho : t/\sigma : -(1 + t)/\tau$ (where $t$ is a parameter) and obtain

$$Q = \frac{p}{\rho x} : \frac{q}{\sigma y} : \frac{r}{\tau z}$$

□

— pldx : Translation of the Kimberling’s Glossary into barycentrics —
12.8 Inconics

Definition 12.8.1. An inconic is a conic that is tangent to the three sides of the reference triangle.

Theorem 12.8.2 (inconics). The punctual and tangential equations of an inconic $C_i$ can be written as:

$$t \cdot C_i \cdot X, \quad \Delta \cdot C_i^t \cdot \Delta$$

where $C_i \simeq \begin{bmatrix} f^2 & -fg & -fh \\ -fg & g^2 & -hg \\ -fh & -hg & h^2 \end{bmatrix}$, $C_i^t \simeq \begin{bmatrix} 0 & h & g \\ h & 0 & f \\ g & f & 0 \end{bmatrix}$ (12.6)

where $\Delta = [f, g, h]$ is the so-called auxiliary line of the conic. Let us note $P = \text{tripolar}(\Delta)$ and $Q = \text{isotom}(P)$, etc (mind the order!). They are the cevians of point $P$. This point $P$ is the perspector between triangle $ABC$ and its polar triangle with respect to the conic, while $\Delta$ is its perspectrix. Finally, the center of $C_i$ is the complement of $\text{isotom}(P)$.

Proof. By definition, $C_i^t$ must have a zero diagonal. Then $C_i$ is its adjoint. Perspectivity is obvious, while center is the pole of the line at infinity.

Corollary 12.8.3. Direct relations between center and perspector are as follows:

$$U = \text{complem}(\text{isot}(P)) = \text{crossmul}(X_2, P) = P \cdot \text{complem}(P)$$
$$P = \text{isot}(\text{anticomplem}(U)) = \text{crossdiv}(U, X_2)$$

Construction 12.8.4. Graphical tools can construct any conic from five points. Given the perspector $P \simeq p : q : r$, the points on the cevian lines are $A_1 = qB + rC$, etc and $A_2 = qB + rC + 4pA$, etc. And therefore,

$$\text{cross_ratio}(A, A_1, P, A_2) = 4$$

Proposition 12.8.5. When $p : q : r$ is the perspector, a parametrization of $IC(P)$ is

$$M(t) \simeq p : q t^2 : r (1 + t)^2$$

(12.7)

Proof. Direct inspection.

Proposition 12.8.6. The 'complem formula' is as follows:

$$\frac{M_{cc}(t)}{\text{tripolar}(P) \cdot M_{cc}(t)} + 2 \frac{M_{ic}(t)}{\text{tripolar}(P) \cdot M_{ic}(t)} = 3 \frac{P}{\text{tripolar}(P) \cdot P} = P$$

where $M_{cc}(t)$ is the standard parametrization of $CC(P)$ and $M_{ic}(t)$ the standard parametrization of $IC(P)$. This amounts to use $\text{tripolar}(P)$ as 'line at infinity' to normalize the columns in the assertion $M_{cc} + 2M_{ic} \simeq P$.
Proof. From (12.5), (12.7) and the obvious:

\[
1 \times \left( \frac{1}{l^2 + l + 1} \begin{bmatrix} t (1 + t) p \\ (1 + t) q \\ -b r \end{bmatrix} \right) + 2 \times \left( \frac{1/2}{l^2 + l + 1} \begin{bmatrix} p \\ q t^2 \\ r (1 + t)^2 \end{bmatrix} \right) = 3 \times \left( \frac{1}{3} \begin{bmatrix} p \\ q \\ r \end{bmatrix} \right)
\]

\[\Box\]

Construction 12.8.7. An inscribed conic can be generated as follows. Given the perspector \(P \simeq p : q : r\), draw the cevians, obtain \(A_P B_P C_P\) and draw the cevian triangle. Draw an arbitrary line \(\Delta\) through \(B\). Define \(B_a = (BP_A \cap \Delta)\) and \(B_c = (BP_C \cap \Delta)\). Then \(M = B_a C_P \cap B_c A_P\) is on the conic. Thereafter, you can define \(\Delta\) by a point \(X\) on the circumcircle, and generate \(C_i\) as the locus of \(M\) (parametrization by a turn).

Proof. Use \(X = x : y : z\) so that \(\Delta \simeq [-z, 0, x]\). Thereafter:

\[
B_a \simeq \left( \frac{p r x}{p q z - q r x} \right), \quad B_c \simeq \left( \frac{p r x}{q r x - p q z} \right), \quad M \simeq \left( \frac{p r^2 x^2}{q (p z - r x)^2} \right)
\]

One can check that \(M\) is on the conic and that \(t = \frac{px}{qx} - 1\). \(\Box\)

![Figure 12.5: How to generate an inscribed conic from its perspector.](image)

Proposition 12.8.8. Points at infinity. An inscribed conic, with auxiliary point \(Q \simeq f : g : h\), perspector \(P \simeq p : q : r\) and center \(U \simeq u : v : w\) is an ellipse, a parabola or an hyperbola when quantity

\[
W^2 = f g h (f + g + h) = \frac{p q + p r + r q}{p^2 q^2 r^2} = (u + v + w) (v + w - u) (u + v - w) (w + u - v)
\]

is, respectively, positive, null or negative. The boundaries are the line at infinity and the sidelines of \(ABC\) for \(Q\), the Steiner circum-ellipse for \(P\) and the line at infinity and the sidelines of the medial triangle for \(U\). Moreover, its points at infinity have the following barycentrics:

\[
M_\infty \simeq \left( \frac{(h + g)^2}{f g - (f + g + h) h \pm 2 \sqrt{-W^2}} \right)
\]

Proof. Immediate computation for the \(M_\infty\), followed by \(P = \text{isotom}(Q), U = \text{anticomplem}(Q)\). \(\Box\)
Proposition 12.8.9. When center $U$ is given, the perspectors $P_1, P_c$ of the corresponding in- and circum-conics are related by:

$$P_1 * P_c = U \quad ; \quad P_i = \text{anticomplem}(P_i)$$

When perspector $P$ is given, the centers $U_i, U_c$ are aligned with $P = p : q : r$ together with $p^2 : q^2 : r^2$.

Proof. Direct inspection.

Proposition 12.8.10. Let $P$ be a fixed point, not on the sidelines, and $Q$ be a point moving point on tripolar $(P)$. The envelope of all the lines $\Delta = \text{tripolar}(Q)$ is the inconic $I_C(P)$. Moreover, the contact point of $\Delta$ is $T = Q * Q \div P$.

Proof. This result has already be given. But now, this is the right place to prove it. Write:

$$P = p : q : r$$

$$Q = p(\sigma - \tau) : q(\tau - \rho) : r(\rho - \sigma)$$

$$\Delta \simeq (1 \div p(\sigma - \tau) : 1 \div q(\tau - \rho) : 1 \div r(\rho - \sigma))$$

$$T = p(\sigma - \tau)^2 : q(\tau - \rho)^2 : r(\rho - \sigma)^2$$

and check that: $\Delta \cdot \text{Adjoint}(\bar{C}) = 0$, $\Delta \cdot T = 0$ and $\langle T \cdot \bar{C}, T \rangle = 0$ where $\bar{C}$ is given in (12.6)

12.9 Poncelet porism

Definition 12.9.1. We will say that two conics $C_{\text{in}}, C_{\text{out}}$ form a Poncelet configuration when it exists a triangle $A, B, C$ inscribed in $C_{\text{out}}$ and circumscribed to $C_{\text{in}}$

Proposition 12.9.2. Poncelet porism. Let $A, B, C$ be the existing triangle, taken as reference, and $P \simeq p : q : r$, $U \simeq u : v : w$ the perspectors of the circumconic $C_{\text{out}}$ and inconic $C_{\text{in}}$. Then any point $M \in C_{\text{out}}$ can be taken as the initial vertex of a poristic triangle.

The vertices and the contact points are given by

$$M_t \in C_{\text{out}} \simeq p : q : t$$

$$N_t \in C_{\text{in}} \simeq u : vt^2 : w (1 + t)^2$$

where parameters $t = t_1, t_2, t_3$ are bound by the relations

$$t_1 + t_2 + t_3 = \mu$$

$$t_1 t_2 + t_2 t_3 + t_3 t_1 = -\mu - 1 - (qu \div pv) - (ru \div pv)$$

$$t_1 t_2 t_3 = (qu \div pv)$$

In other words, parameter $\mu$ describes the poristic triangle as a whole, while the $t_j$ describe the individual vertices and contact points.

Proof. Start from $M_t, M_s \in C_{\text{out}}$ and write that $M_t M_s$ is tangent to $C_{\text{in}}$. This gives an equation in $s$ where

$$s_1 + s_2 = -\frac{pv w (1 + t) + q w(1 + t) + r w (1 + t) + q w t}{t(1 + t) pw} \div s_1 s_2 = \frac{qu}{t pv}$$

And now, compute the line

$$M_{s_1} \wedge M_{s_2} \simeq [qr, pr s_1 s_2, pq(1 + s_1 + s_2 + s_1 s_2)]$$

This gives $M_2 M_3 \simeq \begin{bmatrix} 1 & 1 & -1 \\ u & v t & w (1 + t) \end{bmatrix}$, which is indeed tangent to $C_{\text{in}}$, while the contact point is $N_t \simeq u : vt^2 : w (1 + t)^2$.

From now on, using $t = t_1$ as main parameter would be an error: due to the symmetry, each object would be described three times, increasing the degree of the expressions by a factor 3.

January 3, 2024 21:08 published under the GNU Free Documentation License
Proposition 12.9.3. Synchronization: $M' \cong M \div b$ and $N \div U$ belong, respectively, to the out-Steiner and in-Steiner conics. And then $N' \ast M' \ast M' \cong G = X(2)$.

Proposition 12.9.4. When both conics are circles, we have $R(R-2r) = d^2$ where $R, r$ are both radiuses and $d$ the distance between the centers. When they are the circum- and the in-circle of $ABC$ then we have the so called Brisse Transform (2001):

$$N = \text{isogon} (M) \ast \text{isogon} (M) \ast X(7)$$

Proof. First part is the Euler’s formula, second part is the previous proposition.

12.10 Conic cross-ratios

Proposition 12.10.1. All the lines $\lambda$ through a given point $P$ form a linear projective family $F$. Consider a transversal line $D$ (i.e. a line that doesn’t go through $P$). Then cross ratio remains unchanged by application $F \mapsto D$, $\lambda \mapsto \lambda \cap D$.

Proof. Obvious since the wedge operator is a linear transform $\lambda \mapsto \lambda \wedge D$: the parametrization is preserved.

Proposition 12.10.2. Consider four fixed points $A, B, C, U$ with no alignments and define the moving cross-ratio of a point $M$ in the plane as the cross-ratio of lines $MA, MB, MC, MU$. The level lines of this function are the conics passing through $A,B,C,U$. Using barycentrics with respect to $ABC$, we have the more precise statement: the level line of a given $\mu$ is the circumscribed conic whose perspector is $P = -u:w:\mu r$ on the tripolar of $U = u:w:\mu v$.

Definition 12.10.3. Conic-cross-ratio. Consider a fixed proper conic $C$, and four points $A,B,C,U$ lying on this conic. Then the cross-ratio of lines $MA, MB, MC, MU$ does not depend upon the choice of the auxiliary point $M$ as long as this point $M$ remains on the conic. If we consider $C$ as a circumconic with perspector $P = p:q:r$ with respect to triangle $ABC$, we have :

$$\text{cross_ratio} (A,B,C,U) = \mu \text{ when } U \cong \begin{pmatrix} (\mu^2 - \mu) p \\ (1 - \mu) q \\ \mu r \end{pmatrix}$$

Remark 12.10.4. In the complex plane, the quantity defined in Theorem 3.2.7 is nothing but the usual cross-ratio, as computed from complex affixes. In order to be sure that, given four points on a circle, the conic cross-ratio and the usual C cross-ratio are the same quantity, let us consider the stereographic projection.

Definition 12.10.5. Stereographic projection. Start from $P(c,s)$. Define $M(C,S)$ by doubling the rotation and $T(0,t)$ by intersection of $SM$ with the y-axis. Apply Thales to similar triangles $SOT$ and $SKM$, and Pythagoras to $OHP$. This gives :

$$c^2 + s^2 = 1, \ (C, S) = (c^2 - s^2, 2cs), \ t = \frac{S}{1 + C}$$

and leads to $t = s/c$, proving that $(SO, ST) = (ON, OP)$ and therefore that

$$\cos 2\theta = \frac{1 - t^2}{1 + t^2}, \ \sin 2\theta = \frac{2t}{1 + t^2}$$

Moreover, circular cross-ratio between $M_j$ points is equal to linear cross-ratio between $T_j$ points (this is the definition), while complex cross-ratio between $M_j$ points is equal to complex cross-ratio between $T_j$ points due to the homography :

$$z = (1 + it)/(1 - it) ; \ t = i(1 - z)/(1 + z)$$

— pldx : Translation of the Kimberling’s Glossary into barycentrics —
12.11 Some in- and circum-conics

A list of specific in- and circum-conics is given in Table 12.1. Kiepert RH is studied at Brocard Section.

**Example 12.11.1.** The Steiner ellipses (centers=perspector=\(X_2\)) are what happen to both the circum- and in-circle of an equilateral triangle when this triangle is transformed into an ordinary triangle by an affinity. The Steiner circumellipse (S) is the isotomic conjugate of \(L_\infty\) and the isogonal conjugate of the Lemoine axis. Since \(isog(L_\infty)\) is the circumcircle (C), the mapping
In this case, the remaining three intersections are the cevians of some point \(A\). Steiner in-ellipse is the envelope of the line whose tripole is at infinity (more about Steiner in-ellipse in Section 12.18).

**Example 12.11.2.** The MacBeath-inconic was introduced as follows:

Lemma 1: Let \(O, H\) be the common points of a coaxal system of circles. Let a variable circle of the system cut the line of centers at \(C\). Let \(T\) be a point on the circumference such that \(TC = k \ast OC\), where \(k\) is a fixed ratio. Then the locus of \(T\) is a conic with foci at \(O, H\) (Macbeath, 1949)

Its perspector is \(X(264)\), center \(X(5)\) and foci \(X(3)\) and \(X(4)\). Its barycentric equation is:

\[
\sum \frac{a^4}{(c^2 + a^2 - b^2)(a^2 + b^2 - c^2)} x^2 - 2 \sum \frac{b^2 c^2 yz}{b^2 + c^2 - a^2} \]

and this conic goes through \(X(I)\) for \(I = 339, 1312, 1313, 2967, 2968, 2969, 2970, 2971, 2972, 2973, 2974\)

**Example 12.11.3.** The MacBeath-circumconic, is the dual to the MacBeath-inconic. Its perspector is \(X(22)\), center \(X(5)\) and foci \(X(3)\) and \(X(4)\). Its barycentric equation is:

\[
a^2(b^2 + c^2 - a^2)yz + b^2(c^2 + a^2 - b^2)zx + c^2(a^2 + b^2 - c^2)xy = 0
\]

and it goes through \(X(I)\) for \(I = 110, 287, 648, 651, 677, 895, 1331, 1332, 1797, 1813, 1814, 1815\)

### 12.12 Cevian conics

**Proposition 12.12.1.** Cevian conic. For any points \(P = p : q : r\) and \(Q = u : v : w\), not on a sideline of \(ABC\), the cevians of \(P\) and \(Q\) are on a same conic, whose equation in \(x : y : z\) is \(\text{conicev}(P, Q)\) given by:

\[
\frac{1}{w} x^2 + \frac{1}{q} y^2 + \frac{1}{r} z^2 - \left(\frac{1}{qw} + \frac{1}{vp}\right) yz - \left(\frac{1}{ru} + \frac{1}{wp}\right) zy = 0
\]

\[
\text{conicev}(P, Q) \approx \begin{bmatrix} 2 q r w & -(p v + u q) r w & -(p w + r u) q v \\ -(p w + u r) r v & 2 r p w & -(q w + r v) p u \\ -(q w + r v) q v & -(q w + r v) p u & 2 q p u \end{bmatrix}
\]

**Proof.** Apply \((x, y, z) \rightarrow [x^2, x y, y^2, y z, z^2, z x]\) to the six points and check that rank isn’t 6. \(\Box\)

**Example 12.12.2.** Any inconic is a cevian conic: \(IC(P) = \text{conicev}(P, P)\). For example, the incircle is \(IC(X_t) = \text{conicev}(X_t, X_t)\).

A non trivial example is the nine-points circle, aka \(\text{conicev}(X_2, X_4)\).

**Proposition 12.12.3.** Assume that a conic \(C\) encounters the sidelines of \(ABC\) in six (real) points, none of them being a vertex \(A, B, C\). Three of these points are the cevians of some point \(P\) if and only if:

\[
m_{33} m_{22} m_{11} - m_{11} m_{23}^2 - m_{22} m_{13}^2 - m_{33} m_{12}^2 - 2 m_{13} m_{23} m_{12} = \det M - 4 m_{13} m_{23} m_{12} = 0
\]

In this case, the remaining three intersections are the cevians of some point \(Q\) and both \(P, Q\) are given by:

\[
P, Q \approx \begin{bmatrix} + m_{13} m_{12} + m_{11} m_{23} + m_{13} \sqrt{m_{12}^2 - m_{23} m_{11}} / m_{11} \\ - m_{22} m_{13} - m_{23} m_{12} + m_{23} \sqrt{m_{12}^2 - m_{23} m_{11}} / m_{22} \\ - \sqrt{m_{12}^2 - m_{23} m_{11}} \end{bmatrix}
\]

---

plet : Translation of the Kimberling’s Glossary into barycentrics
Proof. Hypotheses are implying \( m_{11} \neq 0 \) (A \( \notin \mathcal{C} \)) and \( m_{12}^2 - m_{22}m_{11} \geq 0 \) (existence of intersections).

**Exercise 12.12.4.** The fourth common point \( F \) between cevian conics \( \text{conic}_P(P, Q_1) \) and \( \text{conic}_P(P, Q_2) \) can be obtained from the tripolars of \( Q_1, Q_2 \). We have:

\[
F \simeq \text{anticomlem} \left( \frac{X + P}{b} \right) \ast X \quad \text{where} \\
X \simeq \text{tripolar}(Q_1) \cap \text{tripolar}(Q_2) = (Q_1 \wedge Q_2) \ast Q_1 \ast Q_2
\]

**Exercise 12.12.5.** When \( Q = X(2) \), then \( \text{conic}_P(P, Q) \) contains also points \((P + A)/2, \) etc (de Villiers, 2006).

**Proposition 12.12.6.** For any points \( P = p : q : r \) and \( Q = u : v : w, \) not on a sideline of ABC, the eight points \( \pm p : \pm q : \pm r \) and \( \pm u : \pm v : \pm w \) (i.e. \( P, Q \) and their anticevians) are on a same conic, whose equation in \( x : y : z \) is \( \text{conicev}(P, Q) \) given by:

\[
\text{conicev}(P, Q) \simeq \begin{bmatrix}
g^2 r^2 - h^2 q^2 & 0 & 0 \\
0 & h^2 p^2 - f^2 r^2 & 0 \\
0 & 0 & f^2 q^2 - g^2 p^2
\end{bmatrix}
\]

Proof. Straightforward computation. Formally, the coefficients are \([f^2 : g^2 : h^2] \wedge [p^2 : q^2 : r^2].\)

### 12.13 Direction of axes

**Proposition 12.13.1.** Let \( \mathcal{C} \) be a conic, but not a circle, and \( \gamma \) an auxiliary circle. Consider, in any order, the common points \( X_1, X_2, X_3, X_4 \) of \( \mathcal{C} \) and \( \gamma. \) Then axes of \( \mathcal{C} \) have the same directions as bisectors of angle \((X_1X_2, X_3X_4).\)

Proof. Use rectangular Cartesian coordinates. Then \( \mathcal{C} \) is \( y^2 = 2px + qx^2 \) and \( \gamma \) is \( (x - a)^2 + (y - b)^2 = r^2. \) Substitution \( y^2 = Y \) gives:

\[
2by = (1 + q)x^2 + (2p - 2a)x + b^2 + a^2 - r^2 \quad ; \quad Y = 2px + qx^2
\]

By substitution and reorganization:

\[
2b \left( \frac{y_2 - y_1}{x_2 - x_1} + \frac{y_4 - y_3}{x_4 - x_3} \right) = 4(p - a) + (1 + q)(x_1 + x_2 + x_3 + x_4)
\]

But the fourth degree equation \( 0 = Y - y^2 = (1 + q)^2 x^4 - 4(1 + q)(a - p)x^3 \cdots \) leads to \( \sum x_i = 4(a - p)/(1 + q). \) This proves that lines \( X_1X_2 \) and \( X_3X_4 \) are symmetric wrt the axes and the conclusion follows. By the way, it has been proven that points \( X_i \) can be sorted in any order without changing the result.

**Definition 12.13.2. Gudulic point.** The gudulic point of a circumconic is defined as its fourth intersection with the circumcircle. Its barycentrics are rational wrt the barycentrics of the perspector.

\[
Gu = \text{isotom} \left( b^2 r - c^2 q : c^2 p - a^2 r : a^2 q - b^2 p \right)
\]

\[
= \text{tripolar} \left( X(6), P \right)
\]

(12.8)

**Definition 12.13.3. Gudulic point** (general method). The former proposition shows that any pair of orthogonal directions can be specified by giving a point \( M \) on the circumcircle \( \Gamma \) of \( ABC \) such that the bisectors of \((BC, AM)\) have the required directions. This method was firstly used by Lemoine (1900) who called it "the point \( M \) method". In order to have a more specific name, the expression "gudulic point" was coined in a discussion at www.les-mathematiques.net. May be in honor of St Gudula of Brussels.
12. About conics

Proposition 12.13.4. When $C$ is a circumconic, but not the circle itself, directions of axes are given by the bisectors of $(BC, AG_u)$ where $G_u$ is the fourth common point of $C$ and $\Gamma$. When $C$ is not a circumconic, it exists nevertheless an unique point $G_u \in \Gamma$ so that axes of $C$ have the same directions as the bisectors of $(BC, AG_u)$. We have the formulas:

\[
C \cong \begin{bmatrix}
m_{11} & m_{12} & m_{13} \\
m_{12} & m_{22} & m_{23} \\
m_{13} & m_{23} & m_{33}
\end{bmatrix} \Rightarrow G_u \cong \begin{bmatrix}
1/(m_{11} - 2m_{12} + m_{22})b^2 - (m_{11} - 2m_{13} + m_{33})c^2 \\
1/(m_{22} - 2m_{23} + m_{33})c^2 - (m_{11} - 2m_{12} + m_{22})a^2 \\
1/(m_{11} - 2m_{13} + m_{33})a^2 - (m_{22} - 2m_{23} + m_{33})b^2
\end{bmatrix}
\]

\[
CC(P) \mapsto \begin{pmatrix}
1/(rb^2 - qc^2) \\
1/(pc^2 - ra^2) \\
1/(qa^2 - pb^2)
\end{pmatrix} ; \quad IC(U) \mapsto \begin{pmatrix}
1/(w^2b^2 - v^2c^2) \\
1/(w^2c^2 - v^2a^2) \\
1/(v^2a^2 - u^2b^2)
\end{pmatrix}
\]

Proof. First part is the preceding proposition. For the second part, we have:

\[
\tan(BC, BU) + \tan(AG_u, BU) = 0
\]
\[
\tan(BU_1, BU) + \tan(BU_2, BU) = 0
\]

where $U_1, U_2$ are the points at infinity of the conic, $U$ is the unknown direction of either axis and $G_u$ is the required gudulic point. We use the usual parametrization of points at infinity (using $t, t_1, t_2, s$) to describe $U, U_1, U_2$ isogon($G_u$). We extract $t_1 + t_2$ and $t_1t_2$ from the very equation of the conic, and substitute. This gives a quadratic equation in $t$ alone, and a linear equation in $s$, with $t$ as parameter. Eliminating leads to $s$ and thus to $G_u$. $\Box$

Proposition 12.13.5. (Spoiler) Let $\alpha, \beta, \gamma, \delta$ be four turns on the unit circle. Then bisectors of the two lines obtained by pairing these four points have clinant $\tau^2$ where $\tau^4 = \alpha\beta\gamma\delta$.

Proof. Write that $\tan(AB, OT) + \tan(CD, OT) = 0$, and factor the numerator. $\Box$

12.14 Focuses of a conic

Definition 12.14.1. A point $F$ is a focus for a curve when the isotropic lines from this point are tangent to the curve.

Proposition 12.14.2. The geometric focuses of a conic are examples of the preceding definition. But a conic has, in the general case, four analytical focuses: the two geometrical ones, and two extra, not visible, focuses that stay on the other axis.

Proof. Let us consider ellipse $x^2/a^2 + y^2/b^2 = 1$ and isotropic line $(x - x_0) + iy(y - y_0) = 0$. Their intersection is given by a second degree equation whose discriminant is $(x_0 + iy_0)^2 - (a^2 - b^2) = 0$. It vanishes when $y_0 = 0, x_0 = \pm f$ (as usual) but also when $x_0 = 0, y_0 = \pm i f$. $\Box$

Proposition 12.14.3. (spoiler) When using the Morley space to compute the focuses, one equation applies to map $Z : T$ and the other one applies to map $\overline{Z} : T$. When going back to the Morley space, there are four ways of pairing the two focuses of the first map with the two focuses of the second map.

Proof. When, for a visible conic, the first map says $z = u \pm v\sqrt{w}$, the second map says $\zeta = \overline{u} \pm \overline{v}\sqrt{w}$ (with $w \in \mathbb{R}$). But the conjugate of $\sqrt{w}$ is either $\sqrt{w}$ or $-\sqrt{w}$, depending of the sign of the real number $w$. $\Box$
Proposition 12.14.4. Let be given two points \( F_1 \approx f_1 : g_1 : h_1 \) and \( F_2 \approx f_2 : g_2 : h_2 \). All the conics that admit \( F_1, F_2 \) as foci form a tangential pencil. This pencil is generated by (1) the tangential conic \( \{ F_1, F_2 \} \) of all the lines \( \Delta \simeq [u, v, w] \) through \( F_1 \) or \( F_2 \), whose equation is:

\[
(f_1 u + g_1 v + h_1 w)(f_2 u + g_2 v + h_2 w)
\]

and (2) the tangential conic \( \{ \Omega^+, \Omega^- \} \) of all the isotropic lines (through one or another umbilic), whose equation is:

\[
\Delta \begin{bmatrix} M \end{bmatrix}^t \Delta.
\]

Proof. Giving the focuses gives four lines that are tangent to the conic. \( \square \)

Remark 12.14.5. The "Joint Orthoptic Circle" of two confocal conics is now described at 12.24.

Remark 12.14.6. Non parabolic inconics are considered at 12.17, while inscribed parabola and circumscribed parabola are considered at Section 12.20.

12.14.1 Soddy Conic

Definition 12.14.7. The Soddy conic is the circumconic whose perspector is \( X(7) \).

Proposition 12.14.8. Center is \( X(3160) \). Gudulic point is \( X(927) \). Direction of axes are \( X(516) \) and \( X(514) \). The foci are \( X(175) \) and \( X(176) \). This curve is bi-tangent with the polar circle (along the \([a, b, c]\) line).

Proof. Foci can be obtained by the Plucker method. This leads to a fourth degree equation, that factors easily. For the contact, one can check that:

\[
\text{Soddy} = \text{polar}_- \text{circle} + (ax + by + cz)^2
\]

\( \square \)

12.15 PPPP, the four points pencil

Definition 12.15.1. The conics that are going through four fixed points \( M_1 = A, M_2 = B, M_3 = C, M_0 = D \) form a linear pencil, called a PPPP, i.e. a "four points pencil".

Notation 12.15.2. Using triangle \( ABC \) as barycentric basis, the fourth point is described as \( 1/p : 1/q : 1/r \) while the pencil will be parametrized by \( E^* \simeq 1 : t : -1 - t \in \mathcal{L}_\infty \), i.e. by \( E \simeq a^2 : b^2 / t : -c^2 / (1 + t) \).

Proposition 12.15.3. The matrix of the \( ABCDE \) conic is

\[
\begin{bmatrix}
0 & c^2 (b^2 q - a^2 pt) & -b^2 (a^2 p(1 + t) + c^2 r) \\
0 & c^2 (b^2 q - a^2 pt) & a^2 (b^2 q (1 + t) + c^2 rt) \\
-b^2 (a^2 p(1 + t) + c^2 r) & a^2 (b^2 q (1 + t) + c^2 rt) & 0
\end{bmatrix}
\]

The tangents at \( ABC \) determine a trigone. Let \( T_P \simeq P_a P_b P_c \) be the corresponding triangle. Then

\[
P_a \simeq a^2 (b^2 q (1 + t) + c^2 r) : b^2 (a^2 p(1 + t) + c^2 r) : c^2 (a^2 pt - b^2 q)
\]

Triangle \( T_P \) is perspective with \( T_0 = ABC \), perspector \( P \), and with \( T_G \) (the midpoints triangle), perspector \( U \). Point \( P \) is the usual perspector of \( C \) when seen as an \( ABC \) circumconic, while \( U \) is the center of \( C \). The locus of \( P \) when \( E \in \Gamma \) is the tripolar of \( D \), while the locus of \( U \) is the conic \( ABCD \) through the six midpoints of quadrangle \( ABCD \).

Proof. One has \( \text{loc} (P) = [p, q, r] \) and \( \text{loc} (U) = \begin{bmatrix} -2p & p + q & p + r \\
p + q & -2q & q + r \\
p + r & q + r & -2r \end{bmatrix} \). \( \square \)
Moreover, triangles $T_0$ and $T_N$ are perspective (at $D$), while $T_0$ and $T_R$ are perspective at:

$$K_R \simeq \begin{bmatrix} qr (2 S_a a^2 - p^2 r^2 + q^2 b^2 + c^2 r^2) \\ rp (2 S_b r p + p^2 a^2 - q^2 b^2 + c^2 r^2) \\ pq (2 S_c q p + p^2 a^2 + q^2 b^2 - c^2 r^2) \end{bmatrix}$$

Moreover, $R_a$ belongs to line $N_b N_c$, so that triangles $T_N$ and $T_R$ are perspective at:

$$K \simeq \text{crossmul} (P, K_R) \simeq \begin{bmatrix} (a^2 p + S_b r + S_c q) (2 S_a a^2 - p^2 r^2 + q^2 b^2 + c^2 r^2) \\ (b^2 q + S_c p + S_a r) (2 S_b r p + p^2 a^2 - q^2 b^2 + c^2 r^2) \\ (c^2 r + S_a q + S_b p) (2 S_c q p + p^2 a^2 + q^2 b^2 - c^2 r^2) \end{bmatrix}$$

Proof. Direct computation. Moreover, one has $R_a * R_b * R_c \simeq K * K_R * D$.

\section*{12.16 FF, the focal tangential pencil}

**Definition 12.16.1.** All of the tangential conics that share the same foci (homofocal conics) form a linear pencil. We will use $B,C$ to note the foci, and $A$ for a specific point (on the punctual conic).

**Proposition 12.16.2.** The FF pencil is generated by \( \text{hyp} A^* \) and \( \text{lip} A^* \) which are, respectively, the hyperbola and the ellipse through $A$. One has:

$$\begin{align*}
\text{hyp} A & \simeq \begin{bmatrix} 0 & 2 c (b - c) & 2 b (c - b) \\ 2 c (b - c) & a^2 - (b - c)^2 & -a^2 - (b - c)^2 \\ 2 b (c - b) & -a^2 - (b - c)^2 & a^2 - (b - c)^2 \end{bmatrix} \\
\text{lip} A & \simeq \begin{bmatrix} 0 & 2 c (b + c) & 2 b (b + c) \\ 2 c (b + c) & (b + c)^2 - a^2 & a^2 + (b + c)^2 \\ 2 b (b + c) & a^2 + (b + c)^2 & (b + c)^2 - a^2 \end{bmatrix}
\end{align*}$$

(12.9)

Proof. The four isotropic tangents provide four equations implying the matrix $C^*$. Taking the adjoint matrix of the result and saying that $A \in \mathcal{C}$ leads to a second degree equation, which provides two solutions (as a reminder of the fact that the set of the conics is not a linear pencil).

**Proposition 12.16.3.** Tangent at $A$ to \( \text{hyp} A \) is the internal bisector (through $I_0, I_a$), while the tangent to \( \text{lip} A \) is the external one (through $I_b, I_c$). Moreover (spoiler!) the (12.16) formulas are giving:

$$\begin{align*}
\sigma, \pi, f^4 &= \frac{(b - c)^2}{2} - \frac{a^2}{4}, \frac{-S^2 (b - c)^2}{(b + c - a) (b + c + a)}, \frac{a^4}{16} \\
\sigma, \pi, f^4 &= \frac{(b + c)^2}{2} - \frac{a^2}{4}, \frac{+S^2 (b + c)^2}{(a + b - c) (a + c - b)}, \frac{a^4}{16}
\end{align*}$$

Proof. Obvious computations. Moreover, the value of $f^4$ shouldn’t be a surprise, while the signum of $\pi$ characterizes beyond any doubt which one is the hyperbola and which one is the ellipse among this pair of conics.
12.17 Focuses of an inconic

**Proposition 12.17.1.** Let \( F = p : q : r \) be a point not lying on the sidelines. As in Figure 12.7, we note \( F_a, F_b, F_c \) and \( F'_a, F'_b, F'_c \) the projections and the reflections of \( F \) about the sidelines; \( G \) the isogonal conjugate of \( F \), \( \omega = (F + G) / 2 \) and \( P_a = F G_a' \cap F'_a G \), etc. Then points \( P_a, P_b, P_c \) are the cevians of \( P = (\text{isotom} \circ \text{anticomplem}) (\omega) \), and are the contact points of \( C \equiv \text{inconic} (P) \). This conic admits \( \omega \) as center and \( F, G \) as geometrical focuses. Moreover circle \( F'_a F'_b F'_c \), centered at \( G \) is the circular directrix of this conic wrt focus \( F \) while circle \( F_a F_b F_c \) (the pedal circle of \( F \) and \( G \)) is the principal circle of \( C \) (tangent at major axis).

When either \( F \) or \( G \) is at infinity, the other is on the circumcircle, \( P \) is on the Steiner circumconic, and \( C \) is a parabola.

**Proof.** One obtains easily:

\[
G'_a = -\frac{a^4}{p} : \frac{S_a a^2}{p} + \frac{a^2 b^2}{q} : \frac{S_b a^2}{p} + \frac{a^2 c^2}{r}
\]

leading to the symmetric expression:

\[
|F'_a G|^2 = \frac{(c^2 q^2 + 2S_a qr + b^2 r^2) (a^2 r^2 + 2S_b rp + c^2 p^2) (b^2 p^2 + 2S_c pq + a^2 q^2)}{(r + q + p)^2 (a^2 qr + b^2 pr + c^2 pq)^2}
\]

proving that \( \gamma \equiv F'_a F'_b F'_c \) is centered at \( G \). When \( P \) is inside \( ABC \), we can define an ellipse \( C \) by focus \( F \) and circular directrix \( \gamma \). We have \( P_a \in C \) since \( |FP_a| = |F'_a P_a| \). Moreover \( BC \) is...
The bisector of \((P_aF, P_aF')\), again by symmetry. Therefore \(C\) is the inscribed conic tangent at \(P, P_b, P_c\) and we have:

\[
\omega \simeq \left( \frac{rb^2 + qc^2}{q^2c^2 + 2 S_a r q + r^2 b^2} \right) ; P \simeq \left( \begin{array}{c} rq \\ q^2c^2 + 2 S_a r q + r^2 b^2 \\ \frac{rp}{c^2p^2 + 2 S_a r p + a^2 r^2} \\ \frac{qp}{b^2p^2 + 2 S_a q p + q^2a^2} \end{array} \right)
\]

When \(P\) is outside \(ABC\), the simplest method is to revert the process and define \(P\), and therefore \(C\), using the given formula and, thereafter, check that lines \(F \Omega \pm \) are tangents to \(C\).

**Remark 12.17.2.** When substituting \(F = p : q : r\) by an umbilic \(\Omega \pm\), the \(\omega\) formula gives \(0 : 0 : 0\). Therefore, umbilics are expected to appear as artifacts when trying to revert the \(\omega\) formula to obtain the foci.

**Remark 12.17.3.** The Moebius-Steiner-Cremona transform (Section 10.6) provides another point of view... but computations aren’t easier (nor worse).

**Proposition 12.17.4.** The focus of an inconic can be obtained from the perspector by two successive second degree equations (ruler and compass construction). Let \(P = p : q : r\) be the perspector. Then the four focuses can be written as:

\[
F_i \simeq \left( \begin{array}{c} p(r + q) + \sqrt{K} \\ q(r + p) + t\sqrt{K} \\ r(q + p) - (1 + t)\sqrt{K} \end{array} \right)
\]

Then \(t\) is homographic in \(K\), while \(K\) is solution of a second degree equation. The converse is true : \(K\) is homographic in \(t\), while \(t\) is solution of a second degree equation (with same discriminant).

Obviously, the two solutions in \(t\) lead to orthogonal directions (orthopoints at infinity).

**Proof.** Straightforward elimination.

**Example 12.17.5.** Table 12.2 gives some examples of perspectors and focuses. In this table, expressions like \(X5 \pm X30\) are not "up to a proportionality factor", but are addressing the usual simplified values. All expressions are "centered" at the center, the \(\pm\) term being at infinity. Imaginary focuses associated with perspector \(X(80)\) have a very simple expression, namely \(a : bj : cj^2\) and \(a : bj^2 : cj\) where \(j\) is the third of a full turn.
\[ W_{673} = \sqrt{a^2 + c^2 + b^2 - 2bc - 2ac - 2ba} \]

\[ W_{694} = a^2b^2c^2 \left( \frac{1}{a} + \frac{1}{b} + \frac{1}{c} \right) \left( \frac{1}{a} + \frac{1}{b} - \frac{1}{c} \right) \left( \frac{1}{a} - \frac{1}{b} + \frac{1}{c} \right) \]

12.18  Focuses of the Steiner inconic

**Definition 12.18.1.** The Steiner in-ellipse \( \mathcal{S} \) is what happen to the incircle of an equilateral triangle when this triangle is transformed into an ordinary triangle by an affinity. One has center \( \text{perspector} = X_2 \). Moreover, the Steiner inconic is the envelope of the lines whose tripoles are at infinity.

12.18.1 Using barycentrics

**Notation 12.18.2.** In this section, the following radicals will be used:

\[ W = \sqrt{a^4 + b^4 + c^4 - a^2b^2 - b^2c^2 - c^2a^2} \]

\[ W_a = \sqrt{(a^2 + W)^2 - b^2c^2}; W_b = \sqrt{(b^2 + W)^2 - c^2a^2}; W_c = \sqrt{(c^2 + W)^2 - a^2b^2} \]

**Proposition 12.18.3.** Equation of the Steiner inconic \( \mathcal{S} \) is:

\[ x^2 + y^2 + z^2 - 2xy - 2yz - 2zx = (x, y, z) \begin{pmatrix} +1 & -1 & -1 \\ -1 & +1 & -1 \\ -1 & -1 & +1 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = 0 \]

When point \( P \cong p : q : r \) is inside the medial triangle, then \( Q \cong p^2 : q^2 : r^2 \) is inside \( \mathcal{S} \). Moreover point \( M_a \cong (q + r)^2 : q^2 : r^2 \) is the intersection of \( \mathcal{S} \) and segment \([AQ]\).

**Proof.** Direct computation.

**Exercise 12.18.4.** Applying the mksigpi formulas, we have:

\[ \tilde{a} = \sqrt{S_\omega + W/18}, \tilde{b} = \sqrt{S_\omega - W/18}, f = \sqrt{W/9} \]

**Proposition 12.18.5.** The four radicals \( W, W_j \) are real. Moreover, assuming \( c > a, c > b \), we have:

\[ (c^2 - b^2) W_b = W W_a + (b^2 - a^2) W_a \]
\[ W_c = W_a + W_b \]

**Proof.** For each of the \( W_j \), let \( \tilde{W}_a = \sqrt{(a^2 - W)^2 - b^2c^2} \). Then (1) \( W_a^2, \tilde{W}_a^2 \) are real; (2) \( W_a^2 \times \tilde{W}_a^2 = -16S^2(b^2 - c^2)^2 < 0 \); (3) \( W_a^2 > \tilde{W}_a^2 \). Therefore \( W_a^2 > 0 \) and, since \( W_a \) is real, we can assume that \( W_a \geq 0 \).

Consider now the expression \( Q_1 \cong \)

\[ (W_a + W_b + W_c) (-W_a + W_b + W_c) (W_a - W_b + W_c) (W_a + W_b - W_c) \]

Dependent only on the \( W_i^2 \), quantity \( Q \) is intended to be rational in \( W \). Substituting the value of \( W^2 \), one obtains \( Q = 0 \) so that one of the \( W_i \) is the sum of the other two. If \( c \) is the greatest side, this leads to \( W_c = W_a + W_b \).

In the same manner, the product \( Q_2 \cong \)

\[ (c^2 - b^2) W_c + (c^2 - a^2 + W) W_a + (c^2 - a^2 + W) W_a \]

simplifies to 0. If \( c \) is the greatest side, the first factor cannot vanish. And the \( W_b \) formula results using \( W_c = W_a + W_b \).
Proposition 12.18.6. The foci of the Steiner inconic are given by $F_{\pm} = Q X_2 \pm X_{3413}$ where:

$$X_2 = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} ; \quad X_{3413} = \begin{pmatrix} (b^2 - c^2) (a^4 - b^2c^2 - a^2 W) \\ (c^2 - a^2) (b^4 - a^2c^2 - b^2 W) \\ (a^2 - b^2) (c^4 - a^2b^2 - c^2 W) \end{pmatrix}$$

$$Q = \sqrt{2a^2b^2c^2 W^3 - 16 (a^4b^4 + b^4c^4 + c^4a^4) S^2 + a^2b^2c^2 (a^2b^2 + b^2c^2 + c^2a^2) (a^2 + b^2 + c^2)}$$

Proof. Isotropic lines through a focus are tangent to the curve. Write that $F_{\Omega^+}$ is tangent to $C$ and separate real and imaginary parts. Eliminate one of the coordinates of $F$ from this system. It remains a fourth degree equation (E) giving the two real and two imaginary foci. The discriminant of this equation contains $W^4$ in factor. Using this indication, we factorize (E) over $\mathbb{R}(W)$ and obtain:

$$(c^2v^2 - (2wa^2 + 2wW)v + b^2w^2) (c^2v^2 - (2wa^2 - 2wW)v + b^2w^2) = 0$$

The discriminants of these second degree factors are $W_a^2$ and $(\hat{W}_a)^2$. And we obtain the non symmetric expression:

$$F_+ \simeq \begin{pmatrix} (W + b^2) (b^2 - c^2) + (W + b^2 - a^2) W_a \\ (b^2 - c^2) (W_a + W + a^2) \\ (b^2 - c^2) c^2 \end{pmatrix}$$

In order to obtain a more symmetric expression, one can compute $U = \mathcal{L}_\infty \wedge (F_+ \wedge F_-)$, i.e. the point at infinity of the focal line. This point happens to be $X_{3413}$, the first Kiepert infinity point. The existence of $Q$ is obvious since $X_2$ is the middle of the foci. A straightforward computation leads to the given formula.

Figure 12.8: The Steiner in ellipse

Figure 12.8 summarizes these properties. The hyperbola is Kiepert RH, the Tarry point $X(98)$ is the gudulic point of the KRH axes, while it’s circumcircle antipode, the Steiner point $X(99)$, is the gudulic point of both the KRH asymptotes and the Steiner axes.

12.18.2 Using Morley affixes

Lemma 12.18.7. Using Morley affixes, the tangential equation of the Steiner inconic is:

$$C^* \simeq \begin{pmatrix} 2\sigma_2\sigma_3 & 2\sigma_1\sigma_3 & \sigma_2\sigma_1 - 3\sigma_3 \\ 2\sigma_1\sigma_3 & 6\sigma_3 & 2\sigma_2 \\ \sigma_2\sigma_1 - 3\sigma_3 & 2\sigma_2 & 2\sigma_1 \end{pmatrix}$$

— pldx : Translation of the Kimberling’s Glossary into barycentrics ——
Proof. Start from barycentric equation and transmute. The ordinary equation is not so handy, and we know that the adjoint matrix will look better. □

**Proposition 12.18.8** (Marden’s theorem). *(1945)*. When the vertices aren’t aligned, the foci of the Steiner in-ellipse relative to the triangle ABC are the roots of the derivative polynomial, i.e. the roots of \( \frac{\partial}{\partial Z} (Z - \alpha)(Z - \beta)(Z - \gamma) \). Therefore, the four foci are given by:

\[
F_j \simeq \left( \frac{\sigma_i \pm W_0 W_f}{3 \sigma_3} \right)
\]

where \( W_0 = \sqrt{\sigma_3}, W_f = \frac{\sqrt{\sigma_1^2 - 3 \sigma_2}}{\sigma_3}, W_g = \frac{\sqrt{\sigma_2^2 - 3 \sigma_1 \sigma_3}}{\sigma_3} \).

Proof. Write that isotropic lines \( \Omega_{\pm} F_j \) are tangent to the ellipse. The only difficulty is a sound management of the conjugacies: the conjugate of \( W_0 \) is \( 1/W_0 \), while \( W_f, W_g \) are the conjugate of each other. From all the four possibilities for the \( \pm \), two of them lead to visible points (the real focuses), the other two lead to non visible points (the analytical focuses). □

### 12.19 The Brocard ellipse, aka the K-ellipse

**Remark 12.19.1.** Since the K-circumconic, i.e. \( \text{CC}(X(6)) \), is nothing but the circumcircle, the name "K-ellipse" applies only to the K-inconic.

![Diagram of the K-ellipse](image.png)

**Figure 12.9: The K-ellipse**

1. Equation of the K-ellipse is:

\[
\frac{x^2}{a^2} + \frac{y^2}{b^4} + \frac{z^2}{c^4} - 2 \frac{xy}{a^2 b^2} - 2 \frac{yz}{b^2 c^2} - 2 \frac{zx}{a^2 c^2} = 0
\]

perspector is \( X(6) = a^2 : b^2 : c^2 \), center \( U \) is \( X(39) = a^2 (b^2 + c^2) \), etc.

2. Draw the circumcircle of the contact points \( A_K B_K C_K \) and obtain:

\[
a^2 yz + b^2 xz + xyc^2 - (x + y + z) \sum \frac{xb^2 c^2 (b^4 + c^4 + a^2 b^2 + b^2 c^2 + a^2 c^2 - a^4)}{2 (b^2 + c^2) (a^2 + b^2) (a^2 + c^2)} = 0
\]
3. Compute the fourth intersection of this circle with the conic and obtain:

\[
Q = a^2 \left( b^2 - c^2 \right)^2 \left( a^4 + a^2 b^2 + a^2 c^2 - b^4 - b^2 c^2 - c^4 \right), \quad \text{etc}
\]

4. The axes are the lines through the center that are parallel to the bisectors of \( A_K \overline{C_K}, B_K \overline{Q} \).
Therefore, compute:

\[
T = \tan \left( \overline{A_K C_K}, \overline{A_K B_K} \right),
\]
\[
t = \tan \left( \overline{A_K C_K}, \overline{U V} \right),
\]

where \( V \approx \rho : 1 : -1 - \rho \) is an unknown point at infinity, substitute into \( T = 2t/(1 - t^2) \) and solve. Solutions are rational, leading to \( V_1 = a^2 \left( b^2 - c^2 \right) \), etc and \( V_2 = a^2 \left( a^2 b^2 + a^2 c^2 - b^4 - c^4 \right), \) etc \( = X(511) \).

5. Compute the axes as \( U \cap V_1 = \left( a^4 - b^2 c^2 \right) \div a^2, \) etc and \( U \cap V_2 = \left( b^2 - c^2 \right) \div a^2, \) etc: the Brocard axis \( X(3)X(6) \).
Having the perspector on an axis is special.

6. The sideline \( AC \) and the perpendicular to \( AC \) through \( B_K \) cut the first axis in \( P_1, Q_1 \) and the second in \( P_2, Q_2 \). The idea is to draw circle having diameter \([P_1, Q_1]\), then the circle centered at \( U \) orthogonal to the former and obtain the focuses by intersection with the axis.

7. More simpler, write \( F_i = \mu P_i + (1 - \mu) Q_1 \) and find \( \mu \) such that \( \left( \overline{UF_i}/\overline{UP_i} \right) \div \left( \overline{UF_i}/\overline{UQ_1} \right) = 1 \).
These ratios involve vectors that all have the same direction, and no radicals are appearing.
In our special case, the equation factors, leading to a well-known result (the Brocard points):

\[
F_1 = a^2 b^2 : b^2 c^2 : c^2 a^2; \quad F_2 = c^2 a^2 : a^2 b^2 : b^2 c^2
\]

8. Proceed the same way with the other axis. Obtain an equation that doesn’t factors directly, but whose discriminant splits nevertheless when using the Heron formula (7.5). Finally,

\[
F_3, F_4 = \begin{cases} 
4 \left( a^2 b^2 + a^2 c^2 \right) S + i \left( b^4 + c^4 - a^2 c^2 - a^2 b^2 \right) a^2 \\
4 \left( b^2 c^2 + b^2 a^2 \right) S + i \left( a^4 + c^4 - a^2 b^2 - b^2 c^2 \right) b^2 \\
4 \left( c^2 a^2 + c^2 b^2 \right) S + i \left( a^4 + b^4 - a^2 c^2 - b^2 c^2 \right) c^2
\end{cases}
\]

9. To summarize, \( F_1, F_2 \) \( = X(39) \pm X(511) \), \( F_3, F_4 \) \( = X(39) \pm i X(512)/4S \). As it should be, the focal distance (from center to a focus) is the same since \( X(511) \) and \( X(512)/4S \) are obtained by a rotation (in space \( V \)).

## 12.20 Parabola

For the sake of completeness, let us recall the definition.

**Definition 12.20.1.** A parabola is a conic tangent to the infinity line. Two parallel lines make a non proper parabola. The union of line at infinity and another line is ... some kind of circle rather than a "special special" parabola.

**Corollary 12.20.2.** The conic defined by matrix \( \begin{bmatrix} \mathbf{C} \end{bmatrix} \) is a parabola when

\[
\mathcal{L}_\infty \cdot \text{Adjoint} \left( \begin{bmatrix} \mathbf{C} \end{bmatrix} \right) \cdot \dagger \mathcal{L}_\infty = 0
\]

**Definition 12.20.3.** The directrix of a parabola is the polar line of the focus.

**Remark 12.20.4.** This directrix is also the orthoptical cycle of the parabola. See Section 12.24.
12.20.1 Inscribed parabola

Proposition 12.20.5. The focus $F$ of an inscribed parabola is the isogonal conjugate of its point at infinity $U$ (and is therefore on the circumcircle), while the perspector $P$ is the isotomic conjugate of $U$ (and is therefore on the Steiner circumconic). Moreover line $FP$ goes through $X(99)$, the fourth intersection of the circumcircle and the outSteiner ellipse. Finally, the directrix is the Steiner line of $F$ and therefore goes through the orthocenter $H = X(4)$.

Proof. The first part is from Proposition 12.17.1, the last one is detailed at Section 12.24. And the $X(99)$ part is easy to compute.

12.20.2 Circumscribed parabola

Remark 12.20.6. Let $T_0 = u : v : w$ be the barycentrics of the point at infinity of a circumparabola. Then, from Proposition 12.21.9, its perspector is $P = u^2 : v^2 : w^2$ and lies on the inSteiner ellipse.

Remark 12.20.7. The perspectors of the two circumparabolas through the four points $A, B, C, D$ are the intersection of tripolar $D$ and the inSteiner conic.

Proposition 12.20.8. Using Morley affixes, let $\kappa : 0 : 1$ be the point at infinity of a given circumparabola. Then equation, perspector and focus are:

$$C_u \simeq \begin{pmatrix} 2\sigma_3 & -\kappa^2 - \sigma_1\sigma_3 & -2\sigma_3\kappa \\ -\kappa^2 - \sigma_1\sigma_3 & 2\sigma_1\kappa^2 + 4\sigma_3\kappa + 2\sigma_2\sigma_3 & -\sigma_3\kappa^2 - \sigma_3^2 \\ -2\sigma_3\kappa & -\sigma_2\kappa^2 - \sigma_3^2 & 2\sigma_3\kappa^2 \end{pmatrix}$$

$$P \simeq \begin{pmatrix} (3\sigma_2 - 4\sigma_1^2) + \frac{\sigma_1\sigma_2^2}{\sigma_3^2} & \frac{4\sigma_2^2}{\sigma_3} - 12\sigma_1 + (\sigma_2\sigma_1 - 9\sigma_3) \frac{1}{\kappa} \\ 2\sigma_2^2 - \frac{6\sigma_1}{\sigma_3} & \frac{2\sigma_2}{\sigma_3} - \frac{1}{\kappa} & -18 + (2\sigma_1^2 - 6\sigma_2) \frac{1}{\kappa} \\ \frac{5\sigma_3\sigma_1 - 9}{\sigma_3} & 4\sigma_1^2 - 12\sigma_2 & 3\sigma_1 + \frac{\sigma_1^2\sigma_2 - 4\sigma_2^2}{\sigma_3} \frac{1}{\kappa} \end{pmatrix}$$

$$F \simeq \begin{pmatrix} \frac{4\sigma_1}{\sigma_3} & \frac{\sigma_2}{\sigma_3} & \frac{\sigma_3}{\sigma_3} & \kappa^2 + 8\kappa + 2\sigma_2 - \frac{\sigma_3^2}{\kappa^2} \\ \frac{1}{\sigma_3} & \frac{\sigma_2}{\sigma_3} + \frac{\sigma_3}{\kappa} & \frac{1}{\sigma_3} & \frac{1}{\kappa} \\ -1 & \frac{2\sigma_1}{\sigma_3} & \kappa + 8 + (4\sigma_2 - \sigma_1^2) \frac{1}{\kappa} \end{pmatrix}$$

Proof. Use point $\kappa + i\kappa h$ with $h \to 0$ as the fifth point of the conic and compute the determinant. Thereafter, compute the polar triangle and its perspector. Finalize by writing that $\Omega \pm F$ are tangent to the conic.

Proposition 12.20.9. The locus of the foci of all the circumscribed parabola is a circular quintic. Singular focus (not on the curve) is $X(143)$, the nine points center of the orthic triangle. Other asymptotes are through points whose barycentrics are respectively, $2 : 1 : 1$, $1 : 2 : 1$, $1 : 1 : 2$. Its equation is:
Proposition 12.21.4. Share a given pair of asymptotes.

Corollary 12.21.3. An 12.21 Hyperbola

Proof. The first equality is the general formula for the tangent, while the second part is easily chosen for the asymptotes. It can be seen that $\tan^2(\Delta_1, \Delta_2) = \left( \frac{\Delta_1 \cdot \mathbf{W} \cdot \Delta_2}{\Delta_1 \cdot \mathbf{M} \cdot \Delta_2} \right)^2 = (-4) \frac{\mathbf{L}_\infty \cdot \text{Adjoint} \mathbf{C}}{\mathbf{M} \cdot \mathbf{C}^2} \cdot \Delta_2$ (12.12)

Proof. Elimination is straightforward. The real asymptotes are parallel to the sidelines.

12.21 Hyperbola

Definition 12.21.1. An hyperbola is a conic that intersects the line at infinity in two different points. An ellipse is a special hyperbola (the intersection points are not visible) and a parabola is not an hyperbola.

Proposition 12.21.2. Let $\Delta_1 \simeq (\rho, \sigma, \tau)$ and $\Delta_2 \simeq (u, v, w)$ be the asymptotes of an hyperbola $C$. Then equation of $C$ can be written as:

$$(\rho x + \sigma y + \tau z)(ux + vy + wz) - k(x + y + z)^2 = 0$$

Proof. Consider the line $\Delta_1 \simeq (\rho, \sigma, \tau)$ and its point at infinity $T_1 = \sigma - \tau : \tau - \rho : \rho - \sigma$. The matrix of the quadratic form is:

$$\mathbf{C} = \frac{1}{2} \left( \begin{array}{ccc} \Delta_1 \cdot \Delta_2 & 1 & \Delta_1 \cdot \Delta_2 \end{array} \right) - k \left( \begin{array}{c} \mathbf{L}_\infty \cdot \mathbf{C} \end{array} \right)$$

It can be seen that $\mathbf{T}_1 \cdot \mathbf{C} \cdot \mathbf{T}_1 = 0$ ($T_1$ belongs to conic) while $\mathbf{T}_1 \cdot \mathbf{C} = \Delta_1$ (the tangent to the conic at $T_1$ is line $\Delta_1$). Another method is $\Delta_1 \cdot \text{Adjoint} \mathbf{C} = \Delta_1 = 0$ (line $\Delta_1$ is tangent to the conic) while $\mathbf{C} \cdot \mathbf{T}_1 = T_1$ (the contact point of $\Delta$ is $T$).

Corollary 12.21.3. Equation (12.10) is the parametrization in $k$ of the pencil of hyperbola that share a given pair of asymptotes.

Proposition 12.21.4. The angle between the asymptotes $\Delta_1$, $\Delta_2$ of a conic is characterized by:

$$\tan^2(\Delta_1, \Delta_2) = \left( \frac{\Delta_1 \cdot \mathbf{W} \cdot \Delta_2}{\Delta_1 \cdot \mathbf{M} \cdot \Delta_2} \right)^2 = (-4) \frac{\mathbf{L}_\infty \cdot \text{Adjoint} \mathbf{C}}{\mathbf{M} \cdot \mathbf{C}^2} \cdot \Delta_2$ \tag{12.12}$$

See some additional comments at Corollary 12.22.6.
Remark 12.21.5. This amounts to restate (12.10) as searching \( k \) so that
\[
det \left( \mathcal{C} - k^t \mathcal{L}_\infty \cdot \mathcal{L}_\infty \right) = det \left( \mathcal{C} - k \left( \mathcal{L}_\infty \cdot \text{Adjoint} \left( \mathcal{C} \right) \cdot \mathcal{L}_\infty \right) \right)
\]
vanishes. Solution is unique... except when \( \mathcal{C} \) is a parabola.

Proposition 12.21.6. A rectangular hyperbola is an hyperbola with orthogonal asymptotes. Such an RH is characterized among all the conics by:
\[
\left( \mathcal{C} | \mathcal{M} \right) = \text{trace} \left( \mathcal{C} \cdot \mathcal{M} \right) = 0
\]
(12.13)

Proof. Obvious from the previous proposition. Another method: for any matrix \( Q \), we have \( \text{trace} \left( \mathcal{L}_\Delta \cdot \mathcal{L}_\Delta^* \right) = \mathcal{L}_\Delta \cdot \mathcal{M} \cdot \mathcal{L}_\Delta^* \). Since matrix \( \mathcal{M} \) is symmetric, \( \text{trace} \left( \mathcal{C} \cdot \mathcal{M} \right) \) equals \( \Delta_2 \cdot \mathcal{M} \cdot \Delta_1 \) and the result follows.

12.21.1 Circum-hyperbolas

Proposition 12.21.7. A circumconic \( C \) can be characterized by one asymptote \( \Delta_1 \simeq (\rho, \sigma, \tau) \). Then the second asymptote \( \Delta_2 \) is \( k/\rho : k/\sigma : k/\tau \) where \( k \) is the constant appearing in (12.10). Perspector \( P \), center \( U \), points at infinity \( T_1, T_2 \) are given by:

\[
\begin{align*}
P &= \rho (\sigma - \tau)^2 : \sigma (\tau - \rho)^2 : \tau (\rho - \sigma)^2 \\
U &= \rho (\sigma^2 - \tau^2) : \sigma (\tau^2 - \rho^2) : \tau (\rho^2 - \sigma^2) \\
T_1 &= \sigma - \tau : \tau - \rho : \rho - \sigma \\
T_2 &= \rho (\sigma - \tau) : \sigma (\tau - \rho) : \tau (\rho - \sigma)
\end{align*}
\]

while the equation of the conic can be rewritten into \( T_1 \ast T_2 \div X \in L_\infty \), and asymptotes as \( T_1 \ast X \div T_2 \in L_\infty \) and \( T_2 \ast X \div T_1 \in L_\infty \). Moreover, \( P = T_1 \ast T_2 \) and \( U = T_1 \ast T_2 \ast \text{polarmul} (T_1, T_2) \).

Proof. Direct examination.

Proposition 12.21.8. Consider a circumconic and its perspector \( P \). The points at infinity are given by:

\[
\begin{pmatrix}
(q^2 + r^2 - pq - pr) p + p (q - r) \ IST \\
(r^2 + p^2 - qr - qp) q + q (r - p) \ IST \\
(p^2 + q^2 - rp - qr) r + r (p - q) \ IST
\end{pmatrix}
\]

where \( IST^2 = p^2 + q^2 + r^2 - 2pq - 2qr - 2rp \) is the equation of the Steiner in-ellipse.

Proof. Direct inspection.

12.21.2 Circum-rectangular-hyperbolas

Proposition 12.21.9. When a circumscribed conic is a rectangular hyperbola, its perspector is on the tripolar of \( H = X(4) \) - the so-called orthic axis -, while its center \( C = \text{cevadiv} (X_2, P) \) is on the Euler circle (see also Corollary 12.22.8).

Proof. Write that \( \text{trace} \left( \mathcal{M} \mathcal{C} \right) = 0 \) and obtain a first degree equation for the perspector. Then substitute \( P = \text{cevadiv}(X_2, U) \). Even better: write \( U = 2G - V \) and see that \( V \in \Gamma \).

Proposition 12.21.10. Let \( M_1, M_2, M_3 \) be three distinct points on a RH. Then the Euler circle of \( M_1M_2M_3 \) goes through the center of the RH.

Proof. Obvious from preceding proposition.

January 3, 2024 21:08 published under the GNU Free Documentation License
Fact 12.21.11. The perspector of a circumRH can be written as:

\[
\frac{1}{S_a} - \frac{\mu}{S_b} : \frac{\mu - 1}{S_c}
\]

on the tripolar of \(X(4)\), while the RH itself is:

\[-S_a x (S_b y - z S_c) \mu + S_b y (S_a x - z S_c) = 0\]

Kiepert RH is \(\mu = (a^2 - c^2) S_b \div (b^2 - c^2) S_a\).

Proof. A direct proof is that \(\mathcal{H}\) belongs to the pencil generated by \(BC \cup AH\) and \(AC \cup BH\). □

Example 12.21.12. JRH, the Jerabek hyperbola, is the circumscribed RH through \(O\). Its perspector is \(X(647)\), on the \(H\) tripolar, its center is \(X(125)\), on the NPC. Points at infinity are \(X(2574)\) and \(X(2575)\). Example 12.21.13. FRH, the Feuerbach hyperbola, is the circumscribed RH through \(I\). Its perspector is \(X(650)\), on the \(H\) tripolar, its center is \(X(11)\), on the NPC, \(Gu = X(104)\). Points at infinity are \(X(3307)\) and \(X(3308)\), characterized by \(\omega^4 = s_3 s_2 / s_1\).

12.21.3 Inscribed hyperbolas

Proposition 12.21.14. An inconic \(C\) can be characterized by one asymptote \(\Delta_1 \simeq (\rho, \sigma, \tau)\). Then:

\[
\begin{align*}
\Delta_1 & \simeq \rho & \sigma & \tau \\
T_1 & = \sigma - \tau & \tau - \rho & \rho - \sigma \\
N_1 & \simeq \rho \sigma + \rho \tau - \sigma \tau & \rho \sigma + \sigma \tau - \rho \tau & \rho \tau + \sigma \tau - \rho \sigma \\
\Delta_2 & \simeq (\sigma - \tau) f^2 & (\tau - \rho) g^2 & (\rho - \sigma) h^2 \\
T_2 & = (\sigma - \tau) f^2 & (\tau - \rho) g^2 & (\rho - \sigma) h^2 \\
N_2 & \simeq (\rho \sigma + \rho \tau - \sigma \tau)^{-1} & (\rho \sigma + \sigma \tau - \rho \tau)^{-1} & (\sigma \tau + \rho \tau - \rho \sigma)^{-1} \\
P & = 1 \div ((\sigma - \tau) \rho^2) & 1 \div ((\tau - \rho) \sigma^2) & 1 \div ((\rho - \sigma) \tau^2) \\
C & = (\sigma - \tau) f & (\tau - \rho) g & (\rho - \sigma) h
\end{align*}
\]

where \((f, g, h) \simeq N_1 \simeq \text{anticomplem}(\text{isot}(\Delta))\) is the Newton line associated with line \(\Delta_1\) (cf Proposition 12.26.5). Therefore:

\[
\Delta_2 = \Delta_1 \div_b N_1; T_2 = T_1 \ast_b N_1 \ast_b N_1; N_2 = G \div_b N_1 \\
C = T_1 \ast_b N_1 \ast T_2 \ast_b N_2 = \text{crossmul}(G, P); P = \text{crossdiv}(G, C)
\]

Proof. Let \(\mathbf{M}\) be the matrix \([C](p : q : r)\) of the general inconic with perspector \(p : q : r\). Then formula giving \(P\) from \(\Delta\) is obtained by elimination from \(\Delta \cdot \text{Adj}(\mathbf{M}) \cdot \Delta = 0\) (tangency) and \(L_\infty \cdot \text{Adj}(\mathbf{M}) \cdot \Delta = 0\) \((T \in L_\infty)\). Thereafter, all formulas are proven by direct computing from matrix \([C] = [IC](P)\) where \(P\) is as given (cf. Stothers, 2003a). □

Proposition 12.21.15. Consider an inconic and its center \(U\). The points at infinity are given by:

\[
\begin{pmatrix}
  u^2 & (v^2 + w^2 - u^2) & (w^2 - u^2) \\
  v^2 & (u^2 + w^2 - v^2) & (w^2 - v^2) \\
  w^2 & (u^2 + v^2 - w^2) & (u^2 - w^2)
\end{pmatrix}
\pm \text{OST}(v + w - u)(w + u - v)(u + v - w)
\begin{pmatrix}
  (u^2 - w^2) & u^2 \\
  (w^2 - u^2) & v^2 \\
  (u^2 - v^2) & w^2
\end{pmatrix}
\]

where \(\text{OST}^2 = -qr - rp - pq\) is the equation of the Steiner out-ellipse.

Proof. Direct inspection. □
Proposition 12.21.16. Consider point \( U = v - w : w - u : u - v \in \mathcal{E}_\infty \) and its tripolar \( \Delta_0 \simeq [1/(v-w), 1/(w-u), 1/(u-v)]. \) This line is tangent to Steiner in-ellipse and the contact point is \( T_0 = (v-w)^2 : (w-u)^2 : (u-v)^2. \) Define (index i=inscribed, c=circumscribed) lines \( \Delta_i = \text{tripole}(T_0), \Delta_c \simeq [v-w, w-u, u-v] \) and point
\[
C = TG(U) = (v-w)^2 (v+w-2u) : (w-u)^2 (u+w-2v) : (u-v)^2 (u+v-2w)
\]
Then \( C \) is the common center of a circum-hyperbola with asymptotes \( \Delta_0, \Delta_c \) and an in-hyperbola with asymptotes \( \Delta_0, \Delta_i. \) The locus of \( C \) (and also of the circum-perspector) is \( K219 : \)
\[
\sum_3 x^3 - \sum_6 x^2 y + 3xyz = 0
\]
while the locus of the in-perspector is : \( \sum_6 x^2 y - 6xyz = 0. \) All lines \( \Delta_c \) contain \( G = X_2 \) while envelope of the \( \Delta_i \) is cubic : \( \sum_3 x^3 + 3\sum_6 x^2 y - 21xyz = 0. \)

Proof. In all these cases, \( G \) is an isolated point (and don’t belong to the locus). Otherwise, computing as usual. \( \square \)

12.21.4 Inscribed-rectangular-hyperbolas

Proposition 12.21.17. An inscribed conic is a rectangular hyperbola if, and only if, its auxiliary point is on the Longchamps circle. Equivalently, its center is on the polar circle. See Section 13.7 and Section 13.8. As a result, such conics are visible only for obtuse triangles.

Proof. Use Proposition 12.21.6 and write that trace \( \begin{bmatrix} M & C \end{bmatrix} = 0. \) \( \square \)

12.22 Metric elements

Proposition 12.22.1. When a conic \( \mathcal{C} \) degenerates in the reunion of two lines \( \Delta_j \) then
\[
\cos^2(\Delta_1, \Delta_2) = \frac{ \begin{bmatrix} M & C \end{bmatrix}^2 }{ 2 \begin{bmatrix} M & C \end{bmatrix} \begin{bmatrix} M & C \end{bmatrix}^T - \begin{bmatrix} M & C \end{bmatrix}^2 } \quad (12.14)
\]

Proof. Use \( \Delta_1 = [f,g,h] ; [u,v,w], \) obtain tan \( (\Delta_1, \Delta_2) \) from 7.20 and then use \( \cos^2 = 1/(1 + \tan^2) \).

This gives a 4,4, rational fraction in \( u,v,w,f,g,h. \) What is given is the matrix \( (m_{jk}) \) of the bi-linear form \( (fx + gy + hz) \times (ux + vy + wz). \) An identification leads to a 2,2 rational fraction in the \( m_{jk}. \) And the \( \cos^2 \) formula follows. \( \square \)

Definition 12.22.2. The metric elements of a proper conic are the \( a \) and \( b \) that are used to write the "standard equation relative to the standard axes"\(^8\)
\[
P \in \mathcal{C} \iff x^2/a^2 + y^2/b^2 = 1
\]
when the only allowed transforms are isometries. Due to their symmetry properties, the following quantities will remain useful in every context:
\[
\sigma = a^2 + b^2 ; \quad \pi = a^2 b^2
\]

Remark 12.22.3. In the elementary situation, were we have an ellipse and \( b < a \) can be assumed, it is convenient to introduce \( f \) as the focal distance \( |UF|, \) \( \delta \) as the distance center-directrix, \( e \) as the eccentricity i.e. the constant ratio \( \text{dist}(M,F) / \text{dist}(M,\Delta) \) and \( p, \) the parameter, as \( e(\delta - f), \) all these quantities being ruled by:
\[
f = \sqrt{a^2 - b^2} ; \quad \delta = a^2/f ; \quad e = f/a ; \quad p = b^2/a = e(\delta - f)
\]

(12.15)

In a context of dynamic geometry, the elementary quantities \( a, b, f, \delta, e, p \) have to be replaced because we can only assume \( a^2, b^2 \in \mathbb{R} \) while \( |b| < |a| \) is not granted. On the contrary, using \( \sigma = a^2 + b^2 ; \pi = a^2 b^2 \), don’t require to know who is the "main" axis and who is the "other" axis to be able to even define them.
12. About conics

Proposition 12.22.4. Using \(|\rangle\rangle\) to denote the scalar product of two matrices (the trace of their product), we have:

\[
\begin{align*}
\sigma_b &= (-2S) \left( \begin{array}{c|c}
\mathcal{M} & \text{Adjoint} \mathcal{C}_b^o \\
\mathcal{L}_\infty & \mathcal{C}_b^o \end{array} \right) \ \ ; \ \ \pi_b = (-2S)^2 \frac{\det \mathcal{C}_b^o}{(\mathcal{L}_\infty \cdot \text{Adjoint} \mathcal{C}_b^o \cdot \mathcal{L}_\infty)^3} \\
\sigma_z &= \left( \frac{iR^2}{2} \right) \left( \begin{array}{c|c}
\mathcal{M} & \text{Adjoint} \mathcal{C}_z^o \\
\mathcal{L}_z & \mathcal{C}_z^o \end{array} \right)^2 \ \ ; \ \ \pi_z = \left( \frac{iR^2}{2} \right)^2 \frac{\det \mathcal{C}_z^o}{(\mathcal{L}_z \cdot \text{Adjoint} \mathcal{C}_z^o \cdot \mathcal{L}_z)^3}
\end{align*}
\]

Quantities \( f, e \) are only accessible through:

\[
f^4 = \sigma^2 - 4\pi : \ \pi = \phi(e) = \frac{1 - e^2}{(2 - e^2)^2}
\]

so that we have four focuses for a conic (not a parabola), while \( \phi(e) = \phi(e') \) leads to \( e' = \pm e \) but also to \( e' = \pm e/\sqrt{c^2 - 1} \) for hyperbolas.

Proof. Formulas for \( f^4 \) and \( \phi(e) \) come from elimination between equations \((12.15)\). Concerning \((12.16)\), one can establish them for \( x^2/a^2 + y^2/b^2 - 1 \) in the Morley context and extend them to the barycentrics. Another method is the following algorithm.

Algorithm 12.22.5. Consider a conic \( \mathcal{C}^o \) not tangent to the infinity line (i.e. not a parabola). Its center is \( U \approx \mathcal{C}^o \mathcal{L}_\infty \). Consider the point at infinity \( \delta \approx 1: t = -1 : t \), and draw line \( \Delta = U\delta \). Then consider line \( \Delta' \approx U\delta + \mu \mathcal{L}_\infty \) and determine \( \mu \) such that \( \Delta' \) is tangent to the conic. This results in an equation that can be written as:

\[
\left( \mathcal{L}_\infty \cdot \text{Adjoint} \mathcal{C}_e^o \cdot \mathcal{L}_\infty \right) \left( \mu^2 + \text{poly}_2(t) \right) = 0
\]

The first factor is the condition for \( U \in \mathcal{L}_\infty \) (parabola). The other factor gives \( \mu^2 \). Then we consider \( D^2(t) = \text{dist}^2(U, \Delta') \), that occurs to be a rational fraction of degrees \((2, -2)\).

Condition \( \partial \left( D^2(t) \right)/\partial t = 0 \) is second degree in \( t \). Let us call \( s, t \) its two roots. Then \( s + t \) and \( st \) are known from \( \mathcal{C}^o \). And therefore \( \sigma = D^2(s) + D^2(t) \) and \( \pi = D^2(s) \times D^2(t) \) are accessible.

Corollary 12.22.6. The formula \((12.12)\) that gives the angle between the asymptotes \( \Delta_1, \Delta_2 \) of a conic can be completed as:

\[
\tan^2(\Delta_1, \Delta_2) = -4 \frac{\mathcal{L}_\infty \cdot \text{Adjoint} \mathcal{C} \cdot \mathcal{L}_\infty}{\langle \mathcal{M} | \mathcal{C} \rangle^2} = -4\phi(e) = -4\pi \frac{1 - e^2}{(2 - e^2)^2}
\]

Proof. The first part is \((12.12)\), the others are immediate from preceding results.

Remark 12.22.7. The eccentricity of the conic is controlled by this quantity \( \phi(e) \), which is the quotient of \( \mathcal{L}_\infty \cdot \text{Adjoint} \mathcal{C} \cdot \mathcal{L}_\infty \) which is null when \( \mathcal{C} \) is a parabola, and of \( \langle \mathcal{M} | \mathcal{C} \rangle^2 \) which is null when \( \mathcal{C} \) is a rectangular hyperbola RH.

Therefore, \( \phi(e) \) can be perceived as the quotient of the measure of the parabolic character, by the measure of the RH character. The square at denominator allows for a formula that remains homogeneous in \( \mathcal{C} \). The above formula doesn’t depend on \( k \) since all hyperbolas that share the same asymptotes are similar to each other.

Corollary 12.22.8. Applied to the circumscribed conic with perspector \( P \approx p : q : r \), we obtain:

\[
\phi(e) = \frac{\pi}{\sigma^2} = \frac{(1 - e^2)}{(2 - e^2)^2} = \frac{S^2(p^2 + q^2 + r^2 - 2pq - 2qr - 2rp)}{(S_ap + S bq + S_c r)^2}
\]

We re-obtain that perspectors of the circumscribed parabolas stay on the in-Steiner conic, while those of the RH stay on the orthic axis.
Corollary 12.22.9. Applied to the inscribed conic with perspector \( P \simeq p : q : r \), auxiliary point \( \text{isot} \) \( P \simeq f : g : h \) and center \( U \simeq u : v : w \) we obtain:

\[
\begin{align*}
\phi (e) &= \frac{\pi}{\sigma^2} = \frac{1 - e^2}{1 - e^2} = \frac{16 S^2 p^2 q^2 r^2 (pq + pr + rq)}{(a^2 q^2 r^2 + b^2 p^2 r^2 + c^2 p^2 q^2 + 2 S_a p^2 q r + 2 S_b p q^2 r + 2 S_c p q r^2)^2} \\
&= \frac{16 S^2 f g h (f + g + h)}{(a^2 f^2 + b^2 g^2 + c^2 h^2 + 2 S_a g h + 2 S_b h f + 2 S_c f g)^2} \\
&= \frac{S^2 (v + w - u) (w + u - v) (u + v - w) (w + v + u)}{(S_a u^2 + S_b v^2 + S_c w^2)^2}
\end{align*}
\]

We re-obtain that perspectors of the inscribed parabolas stay on the out-Steiner conic, auxiliary points and centers at infinity. On the contrary, auxiliary points of the inscribed RH are on the on Longchamps circle and centers on the polar circle.

12.23 Diagonal conics

Triangle \( ABC \) is autopolar wrt conic \( C \) if, and only if, the non-diagonal coefficients vanish. Such conic is called either autopolar or diagonal.

Remark 12.23.1. The only autopolar circle is the polar circle (see Section 13.7).

Remark 12.23.2. An autopolar parabola is tangent to \([1,1,1]\) (the line at infinity) and therefore to \([\pm 1, \pm 1, \pm 1]\) the sidelines of the medial triangle \( A'B'C' \). If \( U \) is its point at infinity, its focus \( F \) is the \( A'B'C' \)-isogonal of \( U \) (on the Euler circle) while the \( A'B'C' \)-perspector is the \( A'B'C' \)-isotomic of \( U \).

12.23.1 Pencils of diagonal conics

Proposition 12.23.3. Let \( \gamma (\mu) \) be a pencil of diagonal conics:

\[
\gamma (\mu) = (1 - \mu) (a_1 x^2 + b_1 y^2 + c_1 z^2) + \mu \{a_2 x^2 + b_2 y^2 + c_2 z^2\} = 0
\]

and \( U = u : v : w \) a point, not a vertex of \( ABC \). Then polar lines of \( U \) wrt all the conics of the pencil are concurring at a point \( U^* \) that will be called the isoconjugate of \( U \) wrt the pencil. In fact, \( U^* = U_p \), cf \( (17.2) \) where:

\[
P = \beta_1 \gamma_2 - \gamma_1 \beta_2 : \gamma_1 \alpha_2 - \alpha_1 \gamma_2 : \alpha_1 \beta_2 - \beta_1 \alpha_2
\]

The four fixed points (real or not) of the conjugacy, i.e., the points \( \pm \sqrt{p} : \pm \sqrt{q} : \pm \sqrt{r} \), are the points common to all conics of the pencil.

When a pair of isoconjugates \( U_1 \) and \( U_2 \) is known, \( P \) is known and therefore the isoconjugacy. The pencil contains the conic \( \gamma_1 \) through \( U_1 \), \( \text{cevadiv} (U_2, U_1) \) and the vertices of their respective anti-cevian triangles. Conic \( \gamma_2 \) is defined cyclically. Both conics are tangent to \( U_1 U_2 \).

Circumconic \( CC (P) \) is together the \( P \)-isoconjugate of \( L_\infty \), the locus of centers of the conics of the pencil and the conic that contains the six midpoints of the quadrangle formed by the four fixed points.

12.24 Orthoptic cycle

Proposition 12.24.1. The orthoptic cycle \( \varnothing \) of a given conic \( C \) is the locus of points \( m \) where the tangents from \( m \) to \( C \) are orthogonal to each other. When \( C \) is a parabola, \( \varnothing \) is nothing but its directrix. Otherwise, \( \varnothing \) is concentric with \( C \) and its radius is given by:

\[
p^2 = (\cdots) \det \left[ \begin{array}{c}
\mathcal{M}^	op \\
L_\infty \cdot \text{Adj} \left[ C \right] \end{array} \right] = (\cdots) \left[ \begin{array}{c}
\mathcal{M}^	op \\
L_\infty \cdot \text{Adj} \left[ C \right]
\end{array} \right] = a^2 + b^2
\]
Corollary 12.24.2. The orthoptic cycle of a parabola is its directrix.

Corollary 12.24.3. The orthoptic cycle of a parabola is its directrix.

Proof. This can be seen as an obvious corollary of 12.24.7. A direct proof is obtained by using (12.3) to describe the reunion \( \mathcal{D}_0 \) of the two tangents issued from \( m \) as a (degenerate) RH. We have:

\[
\begin{align*}
\mathbf{t} \cdot \mathcal{D}_0 \cdot \mathbf{M} & = (\mathbf{t} \cdot \mathcal{C} \cdot \mathbf{M}) (\mathbf{t} \cdot \mathcal{C} \cdot \mathbf{M}^2) - (\mathbf{t} \cdot \mathcal{C} \cdot \mathbf{M})^2 \\
\mathcal{D}_0 & = \mathcal{C} (\mathbf{t} \cdot \mathcal{C} \cdot \mathbf{M}) - \mathbf{C} \cdot \mathbf{M} \cdot \mathbf{t} \cdot \mathcal{C}
\end{align*}
\]

The required condition is \( \langle \mathbf{M} \mid \mathcal{D}_0 \rangle = 0 \). One can check that the result is not only a conic, but actually a cycle. In the general case (not a parabola), center and \( \rho^2 \) are straightforward. The \( \mathcal{C}^* \) part comes from Adjoint (Adjoint \( \mathcal{M} \)) = (det \( \mathbf{M} \)) \( \mathcal{M} \), while the \( a^2 + b^2 \) part comes from 12.16.

Corollary 12.24.2. The orthoptic cycle of an RH is the point-circle concentric with \( \mathcal{C} \).

Proposition 12.24.4. Orthoptic cycle of an inscribed conic. Let \( P \simeq p : q : r \) be the perspector of an inscribed conic. Its auxiliary point is \( Q \equiv \text{isotom} P \equiv f : g : h \), while the center is \( U \equiv p(h + r) \), etc. The orthoptic circle is described by:

\[
\mathcal{C}^* \simeq \begin{pmatrix} 0 & h & g \\ h & 0 & f \\ g & f & 0 \end{pmatrix}; \quad \mathfrak{D} \simeq \begin{pmatrix} f S_a \\ g S_b \\ h S_c \\ f + g + h \end{pmatrix}; \quad \mathfrak{D} \simeq \begin{pmatrix} (v + w - u) S_a \\ (w + u - v) S_b \\ (u + v - w) S_c \\ w + v + u \end{pmatrix}; \quad \mathfrak{D} \simeq \begin{pmatrix} qr S_a \\ pr S_b \\ pq S_c \\ qr + pr + pq \end{pmatrix}
\]

(12.17)

and is orthogonal to \( \gamma \), the polar circle of \( ABC \).

When \( \mathcal{C} \) is not a parabola, circle \( \mathfrak{D} \) and conic \( \mathcal{C} \) are concentric at \( u : v : w \equiv g + h : h + f : f + g \), while the radius of \( \mathfrak{D} \) is given by:

\[
\rho^2 = \frac{a^2 f^2 + b^2 g^2 + c^2 h^2 + 2 S_a g h + 2 S_b h f + 2 S_c f g}{4(h + g + f)^2} = \frac{S_a u^2 + S_b v^2 + S_c w^2}{(w + v + u)^2}
\]

(so that the center of any inscribed RH is on the polar circle).

When \( \mathcal{C} \) is an inscribed parabola, \( Q \in \mathcal{L}_\infty \) and the orthoptic circle becomes a line \( (\rho = \infty) \) and this line is the directrix of the conic.

Proof. Since \( \gamma \simeq [S_a, S_b, S_c, 1] \), orthogonality is straightforward. For a RH, \( \rho = 0 \) implies \( U \in \gamma \). For a parabola (see Proposition 12.20.5), one has \( Q \in \mathcal{L}_\infty \), while the center is \( U \simeq f^2 : g^2 : h^2 \) (on in-Steiner), the focus is isogon \( Q \) (on the circumcircle) and its polar \( - \)the directrix, going through \( X(4) \)– is \( [f S_a, g S_b, h S_c] \), as required.

Proposition 12.24.5. Circle \( \mathfrak{D} \) is the locus of points \( M \) such that:

\[
\mathbf{f M B} \cdot \mathbf{M C} + g \mathbf{M C} \cdot \mathbf{M A} + h \mathbf{M A} \cdot \mathbf{M B} = 0
\]

Construction 12.24.6. Construct the orthoptic circle of an inconic. Let \( P \) be the perspector and \( A_P B_P C_P \) its cevians. Circle \( \delta \equiv [A A_P P] \) cuts circle \( \epsilon \equiv [B C] \) in two points, etc. These six points belong to the required circle.

Proof. One has \( \delta \simeq S_a : 0 : 0 : 1 ; \epsilon \simeq 0 : r S_b : q S_c : q + r \). And \( x \delta + (1 - x) \epsilon \equiv \mathfrak{D} \) holds when \( x = qr/(pq + pr + qr) \). Remark: all these circles are orthogonal to the polar circle.
Theorem 12.24.7. Joint-orthoptic circle. When $C_t$ and $C_s$ are two confocal conics, but not circles, the punctual pencil they generate contains exactly one circle $\Omega$. Then $\Omega, C_s, C_t$ are concentric while $\Omega$ is the locus of points from which one can issue a tangent $\Delta_t$ to $C_t$ and a tangent $\Delta_s$ to $C_s$ so that $\Delta_t \perp \Delta_s$. As special cases, using $C_s = C_t$ leads to previously described orthoptic cycle, while using $C_F = \{F_1, F_2\}$ as conic $C_s$ leads to the auxiliary circle of $C_t$. Moreover, using obvious notations, we have:

$$2\rho^2(C_s, C_t) = \rho^2(C_s) + \rho^2(C_t) \quad ; \quad \rho^2(C_t) = a^2 + b^2 \quad ; \quad \rho^2(C_F) = f^2$$

Proof. The conics having their foci at $F_1 (z = -1)$ and $F_2 (z = +1)$ are:

$$C_s' \simeq (1 - t) \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix} + \begin{pmatrix} -1 & 0 & -1 \\ 0 & 1 & 0 \\ -1 & 0 & -1 \end{pmatrix} = \begin{pmatrix} -1 & 0 & -t \\ 0 & 1 & 0 \\ -t & 0 & -1 \end{pmatrix} : C_t \simeq \begin{pmatrix} -1 & 0 & t \\ 0 & 1 & 0 \\ t & 0 & -1 \end{pmatrix}$$

Thus the circle that belongs to the punctual pencil generated by $C_t$ and $C_s$ is $C_t - C_s$, whose center is $z = 0$ while $2\rho^2 = s + t$. Consider now a generic point $m \simeq z_0 : t_0 : \zeta_0$. Using (12.3) to describe the tangents issued from $m$ to $C_t$, we obtain:

$$\left( ^t m \cdot C_t \cdot M \right) \left( ^t m \cdot C_t \cdot m \right) - \left( ^t m \cdot C_t \cdot M \right)^2 = 0$$

where $M \simeq Z : T : \overline{Z}$ is the generic point. The $\omega^2$ of the tangents are obtained by substituting $T = 0$, leading to

$$\left( \frac{Z}{\overline{Z}} \right)^2 - 2 \left( t \times t_0^2 - z_0 \zeta_0 \right) \left( \frac{Z}{\overline{Z}} \right) + \left( \frac{t_0^2 - \zeta_0^2}{t_0^2 - \zeta_0^2} \right) = 0$$

Their product is constant, well known property: the tangents are reflected by the bisectors of $(mF_1, mF_2)$. Thus the orthogonality involves also the other pair of tangents, so that the sums of the $\omega^2$ are opposite. But this gives $(t + s) t_0^2 - z_0 \zeta_0 = 0$, i.e. the joint circle.

12.25 PTPT, the bitangent pencil

PTPT means point,tangent,point,tangent.

Remark 12.25.1. Many properties are better stated when using complex projective coordinates (the Morley frame) and therefore, it could be better to read the corresponding chapter before the present section.
12.5 The focal cubic

**Definition 12.25.2.** All of the conics that are tangent to two fixed lines at two given points form a linear pencil, called the bitangent pencil \( F \). We define \( B, C \) as the contact points and \( A \) as the intersection of the tangents.

**Definition 12.25.3.** \( A', R, F_s \). Midpoint \( A' \) is defined by \( A' = (B + C)/2 \). Then \( A' \simeq 0 : 1 : 1 \). Gudulic point \( R \) is the second intersection of the \( ABC \) circumcircle and the \( A \)-symmedian of this triangle. We have:

\[
R \simeq a^2 : -2b^2 : -2c^2 ; R \simeq \frac{2a - \gamma - \beta}{\alpha + \alpha \gamma - 2\gamma \beta} : \frac{2a - \gamma - \beta}{\alpha + \alpha \gamma - 2\gamma \beta} : 1
\]

Lastly, we define \( F_s \) as \( (A + R)/2 \).

**Proof.** One can verify that:

\[
\omega_{AR} \times \omega_{AM} = \frac{\alpha (\gamma \alpha + \alpha \beta - 2\beta \gamma)}{\beta + \gamma - 2\alpha} \times \frac{\alpha \beta \gamma (\beta + \gamma - 2\alpha)}{\gamma \alpha + \alpha \beta - 2\beta \gamma} = (-\alpha \beta) \times (-\alpha \gamma) = \omega_{AB} \times \omega_{AC}
\]

**Theorem 12.25.4.** The punctual and tangential equation of the conics \( C_\lambda \) of the tangential pencil \( F \) are:

\[
C_b \simeq \begin{pmatrix} \lambda & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix} ; \quad C_b^* = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & \lambda \\ 0 & \lambda & 0 \end{pmatrix}
\]

Their centers are \( O_\lambda \simeq 1 : \lambda : \lambda \) (on the median line \( AA' \)). Their two foci \( F_\lambda \) lie on the cubic \( K_b \), whose barycentric equation is:

\[
K_b(x, y, z) = (c^2 y^2 - b^2 z^2) x + 2 S_b y^2 z - 2 S_c z^2 y = (z - y)(a^2 yz + b^2 zx + c^2 xy) + (b^2 - c^2)yz(x + y + z) \quad (12.18)
\]

This curve goes through the umbilics, while the singular focus \( F_s = (A + R)/2 \simeq 2S_a : b^2 : c^2 \) belongs to the curve. The curve goes also through points \( B, C \) and twice the point \( A \). The asymptote \( \Delta_\infty \) is the parallel to the median \( AM \) through point \( \Omega = (3A - R)/2 \).

As a consequence, the curve is unicursal, i.e. has rational parametrizations.

![Figure 12.11: The tangential pencil](image-url)
Proof. Usual computations, using the Plucker’s method, i.e. \((F \wedge \Omega_\alpha) \cdot \vec{C}_\alpha = 0\) and then taking real and imaginary parts. Other properties are straightforward from the gradient. \(\Box\)

**Proposition 12.25.5.** Line \(FA\) is a bisector of angle \(FB, FC\). This comes from the following relations between parameter \(\lambda\) and focuses \(F_\lambda\) described as \(Z : T : \Omega\) in the Morley frame:

\[
\frac{-1}{2\lambda} = \frac{(Z - \beta T)(Z - \gamma T)}{(Z - \alpha T)^2} = \frac{(\Omega - T/\beta)(\Omega - T/\gamma)}{(Z - T/\alpha)^2} \tag{12.19}
\]

Moreover, the Morley equation of the "focal cubic" is \(K_z:\)

\[
\left( \frac{2}{\alpha} - \frac{1}{\beta} - \frac{1}{\gamma} \right) Z^2 \Omega + (\gamma + \beta - 2\alpha) Z \Omega^2 + \left( \frac{1}{\beta \gamma} - \frac{1}{\alpha^2} \right) Z^2 T + (\alpha^2 - \beta \gamma) \Omega^2 T + 2 \left( \frac{\alpha + \alpha - \beta + \gamma}{\alpha} \right) Z \Omega T + \left( \frac{\beta + \gamma - 2\alpha}{\alpha^2} \right) Z T^2 + \left( \frac{2\beta \gamma}{\alpha} - \frac{\alpha^2}{\gamma} - \frac{\alpha^2}{\beta} \right) Z T^2 + \left( \frac{\alpha^2 - \beta \gamma}{\alpha^2} \right) T^3
\]

Proof. Use \(\vec{C}_\alpha = \vec{L}_\alpha^{-1} \cdot \vec{C}_0 \cdot \vec{L}_\alpha^{-1} \cdot \vec{L}_\tau \cdot \vec{C}_\alpha \cdot \vec{L}_\tau\) to obtain the matrices, and then use the Plucker’s equations. A separation of the variables occurs, giving one equation in the upper view \(Z : T\) and another in the lower view \(\Omega : T\). Equation of \(K_z\) is easily obtained from \(K_{\lambda}\) and even more easily by subtracting both sides of \((12.19)\). \(\Box\)

**Proposition 12.25.6.** In the Morley frame, the focal cubic can be parametrized by a turn \(\tau:\)

\[
F_{\tau} \simeq \begin{bmatrix}
(\alpha - \gamma)(\alpha - \beta)\tau^2 + \alpha (2\beta \gamma - \alpha \beta - \alpha \gamma) \tau + \beta \gamma \left(\alpha^2 - \beta \gamma\right) \\
(2 \beta \gamma - \alpha \beta - \alpha \gamma)\tau + \beta \gamma (2 \alpha - \beta \gamma) \\
1 \\
(2 \beta \gamma - \alpha \beta - \alpha \gamma)\tau + \beta \gamma (2 \alpha - \beta \gamma)
\end{bmatrix}
\]

Proof. Let \(K\) be the point \(\tau : 1 : 1/\tau\). Cut the cubic by line \(AK\), and obtain \(A\) (twice) and \(F\). \(\Box\)

**Proposition 12.25.7.** Three points \(F(\tau), F(\kappa), F(\delta)\) on the focal cubic are aligned when:

\[
(\beta \gamma + \delta \kappa + \delta \tau + \kappa \tau)(2 \beta \gamma - \alpha \beta - \alpha \gamma) + \beta \gamma (\delta + \kappa + \tau + \tau \kappa \delta)(2 \alpha - \beta - \gamma) = 0 \tag{12.20}
\]

Therefore, the tangential of \(F(\tau)\), i.e. the point where the tangent of \(F(\tau)\) cuts again the curve, is \(F(\delta)\) where:

\[
\delta = -\frac{\beta \gamma (\beta + \gamma - 2 \alpha) \tau + (\beta \gamma + \tau^2)(\alpha \beta + \alpha \gamma - 2 \beta \gamma)}{2 (\alpha \beta + \alpha \gamma - 2 \beta \gamma) \tau + (\beta \gamma + \tau^2)(\beta \gamma + 2 \alpha)}
\]

while points \(F(\tau)\) and \(F(\tau')\) have the same tangential when \(\tau \tau' = \beta \gamma\).

<table>
<thead>
<tr>
<th>point</th>
<th>(A)</th>
<th>(B)</th>
<th>(C)</th>
<th>(\Omega_x)</th>
<th>(\Omega_y)</th>
<th>(\infty)</th>
<th>(F_x)</th>
</tr>
</thead>
<tbody>
<tr>
<td>(\pm \sqrt{3}_{\text{F}})</td>
<td>(\beta)</td>
<td>(\gamma)</td>
<td>(\infty)</td>
<td>0</td>
<td>(\beta \gamma (2 \alpha - \beta - \gamma))</td>
<td>(z_R)</td>
<td></td>
</tr>
<tr>
<td>(\mp \sqrt{3}_{\text{F}})</td>
<td>(-\alpha)</td>
<td>(z_R = \frac{\alpha + \alpha \gamma - 2 \beta \gamma}{2 \alpha - \beta - \gamma})</td>
<td>(-z_R)</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>tang.</td>
<td>(T_{\Omega_{BC}})</td>
<td>(F_x = (A + R)/2)</td>
<td>(T_{\infty})</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

Proof. Compute the determinant of the three points. \(\Box\)

### 12.25.2 More constructions of the focal cubic

The focal cubic can be constructed in many ways (apart from the parametrization given at Proposition 12.25.6).
12. About conics

Figure 12.12: Two constructions of the focal cubic

Proposition 12.25.8. Using the tau+kappa property. When points $F(\tau)$ and $F(\kappa)$ are aligned with $F_s$ then $\tau + \kappa = 0$ and circle with diameter $[F(\tau), F(\kappa)]$ goes through $A$. This gives a construction of the focal cubic (see Figure 12.12): choose a point $J \in AA'$. The circle $(J, A)$ cuts line $JF_s$ in two points $L_j$ that belongs to the cubic.

Proof. Use the (12.20) formula and then $J = (F(\tau) + F(-\tau))/2$. □

Proposition 12.25.9. Using the median pencil. Draw a circle $\gamma_k$ tangent at $A$ to the median $AM$ (and center $K$, see Figure 12.12). Draw the tangents from $F_s$ to this circle. The contact points $M_j$ are on the focal cubic, while $M_1M_2$ goes through $T_i$, the common tangential of $F_s$ and $F_\infty$.

Proof. The equation of the circles $\gamma_k$ are parametrized by:

$$
\begin{pmatrix}
-1 \\
2\alpha \\
-\alpha^2 \\
\alpha
\end{pmatrix} + k
\begin{pmatrix}
(\alpha + \alpha' - 2\beta') \\
(\beta + \gamma) (\beta' - \alpha^2) \\
\alpha \beta' (2\alpha - \beta - \gamma) \\
0
\end{pmatrix}
$$

Multiply by the Veronese of $F(\tau)$ and obtain the condition $F(\tau) \in \gamma_k$: first degree in $k$, second degree in $\tau$. And conclude, since this condition divides the condition ensuring that $F_sF(\tau)$ is tangent to $\gamma_k$. Then substitute the $k$ values of $\tau + \kappa$ and $\tau \kappa$ into (12.20). □

Proposition 12.25.10. Using the circumcircle. Start from the variable point $K = \tau : 1 : 1/\tau$ on the circumcircle It defines a variable line $AK$. Reflect $B, C$ into $AK$ and obtain $B_k, C_k$. Then point $F = BC_k \cap CB_k$ is on the cubic.

Proof. The idea comes from the bisector property of the last proposition. □

Proposition 12.25.11. Using the A-Apollonian circle. The isogonal $K^*$ of the focal cubic is the circle:

$$K^* \simeq a^2 (y^2 - zb^2) (x + y + z) + (b^2 - c^2) (a^2yz + b^2zx + c^2xy)$$

i.e. the A-Apollonian circle, centered at $0 : b^2 : -c^2$ and going through $A$ (a diameter is given by the feet of the A-bisectors).

Proof. Obvious from (12.18). Using the Morley isogonal formula (17.3), one obtains:

$$F^* \simeq \left[ \frac{\alpha (\alpha + \alpha' - 2\beta')}{\alpha^2 - \beta \gamma} - \frac{(\alpha - \gamma)(\alpha - \beta) \beta \gamma}{(\alpha^2 - \beta \gamma)} \frac{\beta \gamma}{\tau} \right]$$

\[\begin{align*}
\frac{2\alpha - \beta - \gamma}{\alpha^2 - \beta \gamma} + \frac{1}{(\alpha^2 - \beta \gamma)} \frac{(\alpha - \gamma)(\alpha - \beta) \tau}{\beta \gamma} \quad \Box
\end{align*}\]
**Proposition 12.25.12. Knowing the center.** When center $O$ is given on the median $AA'$, the focuses $F, F'$ can be constructed as follows (see Figure 12.14). Draw the bisectors $Δ_1, Δ_2$ of $O_A, O_R$. Cut them at $H_1, H_2$ by the perpendicular bisector of $[A, R]$. Draw circle $γ_1 (H_1, A)$ and cut $Δ_2$. Additionally, draw circle $γ_2 (H_2, R)$ and cut $Δ_1$. This gives the four focuses.

**Proof.** This comes from the involutory homography $ψ$.

**Proposition 12.25.13. Using the cissoidal property.** Consider a point $P$ on the circle $γ$ through $A$ and centered at $Ω' = \frac{(3A - R)}{2}$. Define $Q$ as the intersection of $AP$ with $Δ_∞$. Then $F = A + Q - P$ belongs to the cubic. The cissoidal property is the relation: $\overrightarrow{AF} = -\overrightarrow{PQ}$.

**Proof.** A possible parametrization of the cissoidal circle is:

$$P_μ \simeq \begin{bmatrix} 3μ^2 + 2μγ - 2μ(β + γ) \\ -2(μ - β - γ) \end{bmatrix} + \begin{bmatrix} (α - γ)(α - β) \\ αβ + αγ - 2βγ \end{bmatrix} \begin{bmatrix} 1 \\ (α - β)(α - γ) \end{bmatrix} \begin{bmatrix} 1 \\ 2(α - β - γ)α \end{bmatrix}$$

Here $A = μ = (αβ + αγ - 2βγ) ÷ (β + γ - 2α)$. Everything else is straightforward.

**Proposition 12.25.14.** The two visible focuses $F, F'$ of a given conic $C_λ$ are exchanged:

1. in the parametrization of Proposition 12.25.6 by $ττ' = βγ$.

2. in the construction of Proposition 12.25.10 by using lines $AK$ and $AK'$ that are equally inclined on lines $AB, AC$.

3. in the upper Riemann sphere by the involutory homography $ψ$:

$$ψ : \begin{pmatrix} Z' \\ T' \end{pmatrix} \simeq \begin{pmatrix} βγ - α^2 \\ β + γ - 2α \end{pmatrix} \begin{pmatrix} α^2β + γ(α^2 - 2αβγ) \\ α^2β + γ(α^2 - 2αβγ) \end{pmatrix} \begin{pmatrix} Z \\ T \end{pmatrix}$$
The fixed points of $\psi$ are $A$ and $R$ (the second intersection of the $A$-symmedian with the circumcircle).

4. in the isogonal parametrization of Proposition 12.25.11, points $F^*, F'^*$ are symmetric with respect to $BC$.

5. in the construction Proposition 12.25.13, $P'$ is the reflection of $P$ into the asymptote $\Delta_\infty$ while $Q'$ is the reflection of $Q'$ in the cissoidal circle.

**Proof.**

(1) One has

$$\frac{-1}{2\lambda} = \frac{(\tau - \gamma)(\tau - \beta)}{(\tau - \gamma)(\tau - \beta)} \cdot \frac{(\tau - \alpha - \beta)}{(\tau - \alpha - \beta)}$$

Then $\lambda(\tau) - \lambda(\tau') = 0$ gives four values for $\tau$, since there are four focuses. But, apart $\tau$ itself, only $\tau' = \beta\gamma/\tau$ is unimodular. The other two are not turns (and lead to both non visible focuses). (2) $\kappa\tau = \beta\gamma$.

(3) Write and factor $\lambda(Z : T) - \lambda(z : t)$ from (12.19). This gives $(zT - tZ)$ together with another first degree factor with respect to $Z, T$ and also with respect to $z, t$: this is our homography. The fixed points of $\psi$ are $\alpha$ and $z_R = (\alpha\beta + \alpha\gamma - 2\beta\gamma)/(2\alpha - \beta - \gamma)$.

(4) Reflection into the asymptote is obvious from parametrization. Reflection into the circle comes from $\Omega^2 = |\Omega'A|^2$.

---

### 12.26 LLLL, the Miquel pencil

**Definition 12.26.1.** All of the conics that are tangent to four fixed lines form a pencil, called the Miquel pencil $F$ of these four lines.

**Lemma 12.26.2.** Let us use describe our four lines as the three sidelines of $ABC$ and the transversal whose tripole is $p : q : r$. In other words, $\mathcal{L}_0 \simeq [qr, rp, pq]$. See Theorem 3.5.5 for more details.

In the Lubin frame, the transversal points, the Newton line, the Miquel point $M_q$ of the quadrilateral and the Clawson-Schmidt homography $\Psi$ are:

$$A', B', C' \simeq z \begin{pmatrix} \beta q - \gamma r \\ q - r \\ \beta - \gamma \end{pmatrix}, \begin{pmatrix} \gamma r - \alpha p \\ r - p \\ \gamma - \alpha \end{pmatrix}, \begin{pmatrix} \alpha p - \beta q \\ p - q \\ \alpha - \beta \end{pmatrix}$$

Newton $\simeq z [2c_2, -s_2 c_1 - c_3, 2 s_3 c_1]$
\[ M_q \cong \begin{pmatrix} c_2/c_1 \\ 1 \\ c_1/c_2 \end{pmatrix} ; \quad \Psi \begin{pmatrix} Z \\ T \\ \overline{Z} \end{pmatrix} \cong \begin{pmatrix} Zc_2 - Tc_3 \\
cline{Zc_1 - Tc_2} \\
cline{c_2\overline{Z} - c_1T} \end{pmatrix} \]

where

\[
\begin{align*}
c_3 &= \alpha^2 (\gamma - \beta) p + \beta^2 (\alpha - \gamma) q + \gamma^2 (\beta - \alpha) r \\
c_2 &= \alpha (\gamma - \beta) p + \beta (\alpha - \gamma) q + \gamma (\beta - \alpha) r \\
c_1 &= (\gamma - \beta) p + (\alpha - \gamma) q + (\beta - \alpha) r \\
c_0 &= \frac{1}{\alpha} (\gamma - \beta) p + \frac{1}{\beta} (\alpha - \gamma) q + \frac{1}{\gamma} (\beta - \alpha) r
\end{align*}
\]

Remark 12.26.3. The four quantities \( c_j \) are bound by relations:

\[
c_2 \frac{\sigma_1}{\sigma_3} + c_0 = \frac{1}{\sigma_3} + c_1 \frac{\sigma_2}{\sigma_3} \in i\mathbb{R}
\]

\[
\frac{c_k}{\sigma_3} = -\frac{c_{3-k}}{\sigma_3}
\]

Proposition 12.26.4. The four fixed points of the isoconjugacy \( \Psi \) are given by:

\[
\Phi \cong \begin{pmatrix} c_2/c_1 + W_u/c_1 \\ 1 \\ c_1/c_2 + W_d/c_2 \end{pmatrix}
\]

where the up and down radicals are given by:

\[
\begin{align*}
W_u^2 &= (\alpha - \beta) (\beta - \gamma) (\gamma - \alpha) \times d_1 \\
W_d^2 &= (\alpha - \beta) (\beta - \gamma) (\gamma - \alpha) \times d_2
\end{align*}
\]

where

\[
\begin{align*}
d_2 &= \alpha (\gamma - \beta) qr + \beta (\alpha - \gamma) rp + \gamma (\beta - \alpha) pq \\
d_1 &= (\gamma - \beta) qr + (\alpha - \gamma) rp + (\beta - \alpha) pq
\end{align*}
\]

They can be constructed as follows. Lines \( \Delta_1, \Delta_2 \) are the common bisectors of \( \Omega A, \Omega A', \Omega B, \Omega B', \Omega C, \Omega C' \) and \( \delta_A \) is the perpendicular bisector of \( [A, A'] \). Then \( H_1 = \Delta_1 \cap \delta_A \) (resp. \( H_2 = \Delta_2 \cap \delta_A \)) is the center of a circle by \( A, A' \) that cuts \( \Delta_2 \) (resp \( \Delta_1 \)) at the four \( \Phi_j \) points.

Proof. These points are characterized by

\[
c_3Z^2 - 2c_2ZT + c_3T^2 = 0 \quad ; \quad c_2\overline{Z}^2 - 2c_1\overline{Z}T + c_0T^2 = 0
\]

Proposition 12.26.5 (Newton). All the conics that are tangent to four given lines have their centers on a line, that goes through the midpoints \( M_j \) of the diagonal pairs \( AA', BB', CC' \) (called the Newton axis of the quadrilateral). When this center \( U \) is Newton is defined as \( K M_b + (1 - K) M_c \) the conic can be written as:

\[
\begin{pmatrix} c_0^* \end{pmatrix} \cong (r - p)
\begin{pmatrix} 0 & 0 & -p \\ 0 & 0 & q \\ -p & q & 0 \end{pmatrix}
\begin{pmatrix} 0 & p - q & r - p \\ p - q & 0 & q - r \\ r - p & q - r & 0 \end{pmatrix}
\]

Proof. Let \( \rho : \sigma : \tau \) be the "service point" of a conic \( C \in \mathcal{F} \). We have \( \begin{pmatrix} c_0^* \end{pmatrix} = [0, \sigma, \tau; 0, \rho, \sigma, \rho, 0] \) and therefore \( \rho p + q \sigma + r \tau = 0 \). Since \( U = \sigma + \tau : \tau + \rho : \rho + \sigma \), the Newton line is:

\[
[q + r - p, r + p - q, p + q - r]
\]

and the conclusions follow (remember that \( p : q : r \) is the tripole of the transversal).
Theorem 12.26.6. The Morley affixes \( Z : T : \bar{Z} \) of a focus of the conic \( C(K) \in \mathcal{F} \) are bound to the parameter \( K \) by:

\[
K = \frac{(p-r)(Z - \gamma T)((q-p)Z + (\alpha p - \beta q)T)}{pT(c_1Z - c_2T)}
\]

and the focus is located on the "focal cubic" \( \mathcal{K} \):

\[
\frac{c_1}{\sigma_3}Z^2Z + c_1Z\bar{Z}^2 - \frac{c_1}{\sigma_3}TZ^2 = \frac{c_1 + c_2}{\sigma_3}T^2Z^2 - c_2T\bar{Z}^2 + \frac{c_1\sigma_1}{\sigma_3}T^2Z + \frac{c_2\sigma_2}{\sigma_3}T^2\bar{Z} + \frac{c_3 - c_2\gamma}{\sigma_3}T^3
\]

Proof. Matrix \( C^2 \) is obtained as \( \begin{bmatrix} Lu & C'_b & Lu \end{bmatrix} \) and then Plucker method is used. Some factors \((q-p)\) are appearing during the elimination process, but not all the \((p-q)(q-r)(r-p)\). Nothing special occurs when \( P \) is on a median (but not at the centroid). One can check that \( K \) is turned into its opposite when taking the (complex) conjugate.

\[\square\]

Theorem 12.26.7. The focuses \( F_j \) of a given conic \( C(K) \in \mathcal{F} \) are exchanged by homographies \( \psi, \bar{\psi} \). They can be constructed as follows. Call \( \text{New}^+ \) the perpendicular to the Newton line at \( \Omega \). Draw the bisectors \( \Delta_1, \Delta_2 \) of \( U\Phi_1, U\Phi_2 \). Cut them at \( H_1, H_2 \) by \( \text{New}^+ \). Draw circle \( \gamma_1(H_1, \Phi_1) \) and cut \( \Delta_2 \). Additionally, draw \( \gamma_2(H_2, \Phi_2) \) and cut \( \Delta_1 \).

Proof. Write and factor \( K(Z : T : \bar{Z}) - K(z : t : \zeta) \) from (12.21). This gives \((p-q)(p-r)\) but not \((q-r)\), being smooth except at the centroid (i.e. when the fourth line at infinity). Otherwise, this gives \((zT - tZ)\) together with another first degree factor with respect to \( Z, T \) and also with respect to \( z, t \). In the upper view \( Z : T \), this induces the identity together with another homography. Since the later has to provide \( A \leftrightarrow A', B \leftrightarrow B', C \leftrightarrow C' \), it has to be \( \psi \). The same occurs in the lower view \( \bar{Z} : \bar{T} \), and the conclusion follows.

\[\square\]

Remark 12.26.8. In the Geogebrara Figure 12.15, the orange conic is drawn as follows. Reflect focuses \( F, F' \) into sideline \( AC \) and obtain \( F_b, F'_b \). Then point \( E_b = F F'_b \cap F'F_b \) on sideline \( AC \) belongs to the conic. In the same way, obtain the point \( E_c \) on \( AB \). We draw both conics, an ellipse and an hyperbola, with focuses \( F, F' \) that go through \( E_b \), but only the one that goes also through \( E_c \) is displayed.

Proposition 12.26.9. When the transversal is tangent to one of the inexcircles of triangle \( ABC \), the pencil contains one circle and the focal conic has a double point. This point is the center of the circle (see Figure 12.15b). When the transversal touches two of the inexcircles, the focal cubic degenerates into the Newton line and a circle having the corresponding inexcenters as antipodal points.

Proof. Only circles have equal focuses.

\[\square\]

12.27 Tg and Gt mappings

Definition 12.27.1. Tg and Gt mappings. Suppose \( U \) is a point not on a sideline of \( ABC \).

Let:
- \( gU \) : isogonal conjugate of \( U \), \( tU \) : isotomic conjugate of \( U \)
- \( tgU \) : isotomic conjugate of \( gX \), \( gtU \) : isogonal conjugate of \( tU \)
- \( GtU \) : intersection of lines \( U - tU \) and \( gU - gtU \)
- \( TgU \) : intersection of lines \( U - gU \) and \( U - tgU \)

If \( U = u : v : w \) (barycentrics), then:

\[
GtU = \frac{a^2(b^2 - c^2)}{(v^2 - w^2)u} : \frac{b^2(c^2 - a^2)}{(w^2 - u^2)v} : \frac{c^2(a^2 - b^2)}{(u^2 - v^2)w}
\]

\[
TgU = \frac{b^2 - c^2}{(u^2 - v^2)b} : \frac{c^2 - a^2}{(u^2 - v^2)a} : \frac{a^2 - b^2}{(v^2 - u^2)b}
\]
Figure 12.15: The Miguel pencil and its focal cubic

(a) The general case

(b) The unicursal case
12. About conics

Proposition 12.27.2. For any point $U$, not on a sideline of $ABC$, points $A$, $B$, $C$, $gU$, $tU$, $TgU$, $GtU$ are on a same conic (Tuan, 2006). The perspector of this conic is $X_{512} \div U$. This conic is the isogonal image of line $U - gtU$ and also the isotomic image of line $U - tgU$.

Proof. Straightforward computation.

Example 12.27.3. Points $X(3112)$ to $X(3118)$ are related to $Gt$ and $Tg$ functions. The $X(31)$-conic passes through $X(I)$ for

$I = 75, 92, 313, 321, 561, 1441, 1821, 1934, 2995, 2997, 3112, 3113$

The $X(32)$-conic passes through $X(I)$ for

$I = 76, 264, 276, 290, 300, 301, 308, 313, 327, 349, 1502, 2367, 3114, 3115$

The $X(76)$-conic passes through $X(I)$ for

$I = 6, 32, 83, 213, 729, 981, 1918, 1974, 2207, 2281, 2422, 3114, 3224, 3225$
Chapter 13

More about circles

13.1 General results

Let us start by recalling two key results.

**Theorem 13.1.1** (Already stated in Section 7.3 as Theorem 7.3.2). Let $\Omega$ be the circle centered at $P$ with radius $\omega$. The **power formula** giving the $\Omega$-power of any point $X = x:y:z$ from the power at the three vertices of the reference triangle is:

$$\text{power} \ (\Omega, X) = |PX|^2 - \omega^2 = \frac{ux + vy + wz}{x + y + z} - \frac{a^2yz + b^2xz + c^2xy}{(x + y + z)^2}$$  \hspace{1cm} (13.1)

where $u = \text{power} \ (\Omega, A)$, etc

**Definition 13.1.2** (Already stated in Section 7.3 as Definition 7.3.3). From $\text{power} \ (\Gamma, A) = 0$, etc, we have defined the standard equation of the circumcircle as:

$$\Gamma_{\text{std}}(x,y,z) = -a^2yz + b^2xz + c^2xy = 0$$  \hspace{1cm} (13.2)

**Proposition 13.1.3.** Let $\mathcal{C}$ be a conic, with matrix $[\mathcal{C}] = (m_{jk})$ (notations of Definition 12.3.1). Then $\mathcal{C}$ is a cycle if and only if, for a suitable factor $k$, we have:

$$[\mathcal{C}] - \frac{1}{2} \begin{pmatrix} 2m_{11} & m_{11} + m_{22} & m_{33} + m_{11} \\ m_{11} + m_{22} & 2m_{22} & m_{22} + m_{33} \\ m_{33} + m_{11} & m_{22} + m_{33} & 2m_{33} \end{pmatrix} = k \begin{pmatrix} 0 & c^2 & b^2 \\ c^2 & 0 & a^2 \\ b^2 & a^2 & 0 \end{pmatrix}$$

*Proof.* Obvious from (13.1).

**Proposition 13.1.4.** Four points at finite distance belong to the same cycle (aka circle or straight line) when their barycentrics $p_i : q_i : r_i$ are such that:

$$\det_{i=1}^{i=4} [p_i, q_i, r_i, \Gamma_{\text{std}}(p_i, q_i, r_i)] = 0$$  \hspace{1cm} (13.3)

*Proof.* Obvious from (13.1). Don’t forget how $\Gamma_{\text{std}}$ was defined in (13.2)!

*Computed Proof.* Denominators are a reminder of the fact that circles don’t escape to infinity. Write the Cartesian equation of the circle as:

$$\Delta_{\text{cart}} \equiv \det_{i=1}^{i=4} [\xi_i^2 + \eta_i^2, \xi_i, \eta_i, 1] = 0$$

where $\xi, \eta$ are the Cartesian coordinates of the points. Substitute these coordinates by:

$$\xi = \frac{x\xi_0 + y\xi_0 + z\xi_c}{x + y + z}, \eta = \frac{x\eta_0 + y\eta_0 + z\eta_c}{x + y + z}$$
and obtain another determinant $\Delta'(x, y, z)$. Then compute $F \cdot \Delta' \cdot T^{-1} \cdot G$ where $F$ is the diagonal matrix $diag(p_i + q_i + r_i)$ and

$$T = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & \xi_a & \eta_a & 1 \\ 0 & \xi_b & \eta_b & 1 \\ 0 & \xi_c & \eta_c & 1 \end{bmatrix}, \quad G = \begin{bmatrix} -1 & 0 & 0 & 0 \\ \xi_a^2 + \eta_a^2 & 1 & 0 & 0 \\ \xi_b^2 + \eta_b^2 & 0 & 1 & 0 \\ \xi_c^2 + \eta_c^2 & 0 & 0 & 1 \end{bmatrix}$$

Matrix $F$ acts on rows and kills all denominators, $T$ acts on the last three columns and goes back to barycentrics while $G$ acts on the first column to cut all square terms. After what everything simplifies nicely and leads to (13.3)

**Proposition 13.1.5.** The barycentric equation of circle with center $P = p : q : r$ and radius $\omega$ is :

$$(u_0x + v_0y + w_0z)(x + y + z) - \omega^2 (x + y + z)^2 - (a^2yz + b^2zx + c^2xy) = 0 \quad (13.4)$$

where quantities $u_0, v_0, w_0$ are defined as :

$$u_0 \doteq |PA|^2 = \frac{(c^2q^2 + b^2r^2 + (b^2 + c^2 - a^2)qr)}{(p + q + r)^2}$$

$$v_0 \doteq |PB|^2 = \frac{(a^2r^2 + c^2p^2 + (c^2 + a^2 - b^2)pq)}{(p + q + r)^2}$$

$$w_0 \doteq |PC|^2 = \frac{(b^2p^2 + a^2q^2 + (b^2 + a^2 - c^2)pq)}{(p + q + r)^2} \quad (13.5)$$

**Proof.** Obvious from (13.1). The added value here is the emphasis on center and $\omega$. It must be noticed that $u : v : w$ is not a point nor a line. Quantities $u, v, w$ are strongly defined objects and are not defined up to a proportionality factor. They are to be considered exactly as $\omega$, i.e. are of the same nature as a surface. It can be observed that $u$ (or $v$ or $w$) is zero-homogeneous wrt the barycentrics of point $P = p : q : r$. More details are given in Chapter 14

**Proposition 13.1.6.** Center. The center of a circle defined by its equation (13.1) is given by :

$$\text{center} \doteq \frac{1}{2vX(3)} \cdot \frac{\mathbf{K}^T \cdot \mathbf{U}}{\mathbf{U}} = \left( \begin{array}{c} a^2 S_a \\ b^2 S_b \\ c^2 S_c \end{array} \right) - \left( \begin{array}{ccc} a^2 & S_c & -S_b \\ -S_c & b^2 & S_a \\ S_b & -S_a & c^2 \end{array} \right) \cdot \left[ \begin{array}{c} u \\ v \\ w \end{array} \right] \quad (13.6)$$

$$\doteq \frac{1}{2S} \frac{a^2 S_a : b^2 S_b : c^2 S_c}{\mathbf{M}} - \frac{\mathbf{K}}{\mathbf{M}} \cdot \mathbf{U} \quad (13.7)$$

**Proof.** As stated in Definition 12.3.17, the center of a conic is the pole of the line at infinity wrt the conic. Computations are straightforward. It can be noticed that product $\frac{\mathbf{K}}{\mathbf{M}} \cdot \mathbf{U}$ gives the orthodir of line $U$ (i.e. the point at infinity in the orthogonal direction). Since the line of centers is orthogonal to the radical axis of $\Gamma$ and $\Omega$, this formula describes the coefficients to be used when the circumcenter $X(3)$ is described as in ETC by $vX(3) = a^2 \left( b^2 + c^2 - a^2 \right) \cdot U$. Let us recall the Al-Kashi formula $\mathbf{K} = 2S \mathbf{M}$

**Remark 13.1.7.** In both the center and the radius formulas, everything must be used exactly as written, and not to a proportionality factor. A more efficient formulation will be given later, with formula (14.15)

**Proposition 13.1.8.** Radius. The radius of a circle defined by its equation (13.1) is given by :

$$\omega^2 = \frac{1}{16 S^2} \left( U : \frac{\mathbf{K}}{\mathbf{M}} \right) U - U.vX(3) + a^2 b^2 c^2 \quad (13.8)$$

**Proof.** Subtract formula (13.1) from $|PX|^2$ obtained from (13.6) and Pythagoras formula

**Definition 13.1.9.** kitW. Some usual square roots are given in Table 13.1, and some other notations in Table 13.2.

**Example 13.1.10.** Table 13.3 describes some of the usual circles in triangle geometry. For further information on many circles, refer to http://mathworld.wolfram.com/Circle.html
13. More about circles

Table 13.1: Some usual square roots (kit W)

<table>
<thead>
<tr>
<th>name</th>
<th>#</th>
<th>value</th>
<th>where</th>
</tr>
</thead>
<tbody>
<tr>
<td>Lemoine</td>
<td>$W_1$</td>
<td>$\sqrt{a^2b^2 + a^2c^2 + b^2c^2}$</td>
<td>$\exp(i\omega) = \frac{a^2 + b^2 + c^2 + 4iS}{2W_1}$ (13.11)</td>
</tr>
<tr>
<td>Brocard</td>
<td>$W_2$</td>
<td>$\sqrt{a^2 + b^2 + c^2 - a^2b^2 - a^2c^2 - b^2c^2}$</td>
<td>$</td>
</tr>
<tr>
<td>Euler</td>
<td>$W_3$</td>
<td>$\sqrt{\sum_{i=1}^{3}a^6 - \sum_{i=1}^{6}a^4b^2 + 3a^2b^2c^2}$</td>
<td>$</td>
</tr>
<tr>
<td>Fuhrmann</td>
<td>$W_4$</td>
<td>$\sqrt{\sum_{i=1}^{3}a^4 - \sum_{i=1}^{6}a^2b^2 + 3abc}$</td>
<td>$</td>
</tr>
</tbody>
</table>

Table 13.2: Some usual notations (circle kit 2)

<table>
<thead>
<tr>
<th>#</th>
<th>value</th>
<th>name</th>
</tr>
</thead>
<tbody>
<tr>
<td>$s$</td>
<td>$(a + b + c)/2$</td>
<td>half-perimeter</td>
</tr>
<tr>
<td>$R$</td>
<td>$\frac{abc}{\sqrt{s(s-a)(s-b)(s-c)}}$</td>
<td>circumradius</td>
</tr>
<tr>
<td>$S$</td>
<td>$\frac{abc}{4R}$</td>
<td>area of triangle</td>
</tr>
<tr>
<td>$\omega$</td>
<td>$\exp(i\omega) = \frac{a^2 + b^2 + c^2}{2W_1} + i\frac{2S}{W_1}$</td>
<td>Brocard angle</td>
</tr>
<tr>
<td>$e$</td>
<td>$\sqrt{\frac{a^2 + b^2 + c^2}{abc} - 1}$</td>
<td>$\sqrt{1 - 4\sin^2\omega}$</td>
</tr>
<tr>
<td>$r$</td>
<td>$\frac{S}{s} = \frac{abc}{2R(a + b + c)}$</td>
<td>inradius</td>
</tr>
</tbody>
</table>

Definition 13.1.11. Pencil of cycles. When $C_1$ and $C_2$ are two circles, then $\lambda C_1 + \mu C_2$ is also a cycle. The family generated from two given circles is called a pencil. Then all centers are on the same line, which is orthogonal to the only line contained in the pencil (the so called radical axis). More details in Chapter 14

Definition 13.1.12. The radical trace of two non-concentric circles is the point of intersection of the radical axis of the circles and the line of the centers of the circles. (For examples, see X(I) for $I = 6, 187, 1570, 2021-2025, 2030-2032$.)

13.2 Inversion in a circle

Remark 13.2.1. All results relative to the inversion in a cycle have moved to Section 14.7.

13.3 Antipodal Pairs on Circles

Remark 13.3.1. Since the previous section has moved, the present section is now orphaned... and probably should be moved.

Proposition 13.3.2. Suppose $(O_1)$ and $(O_2)$ are circles and that $P, P'$ are antipodes on $(O_1)$. Let $U = \text{insim}(O_1, O_2)$ and $V = \text{exsim}(O_1, O_2)$ be the respective internal and external center of homothety of circles $(O_1)$ and $(O_2)$. Define $Q = PU \cap PV$ and $Q' = PU \cap PV$. Then $Q, Q'$ are antipodes on $(O_2)$. Moreover, the lines $PP'$ and $QQ'$ are parallel.

Proof. The result is quite obvious, but giving a non-circular proof is not so obvious... except from using $z : t : \zeta$ coordinates. Write the antipodal points as $P = z_1/t_1 + r_1\tau$ and $P' = z_1/t_1 - r_1\tau$. —— pldx : Translation of the Kimberling’s Glossary into barycentrics ——
where $\tau$ is a turn. Point $U, V$ are obtained as

$$U = \left( r_2 z_1 / t_1 + r_1 z_2 / t_2 \right) / (r_2 + r_1) ; \quad V = \left( r_2 z_1 / t_1 - r_1 z_2 / t_2 \right) / (r_2 - r_1)$$

Two wedges later, we have:

$$Q = z_2 / t_2 + r_2 \tau ; \quad Q' = z_2 / t_2 - r_2 \tau$$

and we are done.

In the following examples, suppose $P = p : q : r$ on the first circle.

### 13.4 Circumcircle

**Definition 13.4.1.** The circumcircle is the circle through $A, B, C$. Perspector is $X_6$ and center $X_3$. Equation, matrix, column are:

$$a^2 yz + b^2 zx + c^2 xy = 0 ; \quad \begin{pmatrix} Pyth \end{pmatrix} : \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix}$$

Its standard parametrization is (7.16), i.e.:

$$\frac{a^2}{\sigma - \tau} : \frac{b^2}{\tau - \rho} : \frac{c^2}{\rho - \sigma} | \rho : \sigma : \tau \neq L_\infty$$

**Lemma 13.4.2.** The distance $|PX_3|$ to center from any point of the plane is given by:

$$|PX_3|^2 = R^2 - \frac{a^2 qr + b^2 rp + c^2 pq}{(p + q + r)^2}$$

**Proof.** Direct inspection using Theorem 7.2.4. As it should be, $|X_3X_3| = 0$ while the equation of the circumcircle is $|PX_3|^2 = R^2$. 

---

Table 13.3: Some circles

<table>
<thead>
<tr>
<th>Name</th>
<th>Center</th>
<th>Radius</th>
</tr>
</thead>
<tbody>
<tr>
<td>circumcircle</td>
<td>X(3)</td>
<td>$R$</td>
</tr>
<tr>
<td>incircle</td>
<td>X(1)</td>
<td>$r$</td>
</tr>
<tr>
<td>nine-point circle</td>
<td>X(5)</td>
<td>$\frac{1}{2}R$</td>
</tr>
<tr>
<td>polar circle</td>
<td>X(4)</td>
<td>$\sqrt{-S_a S_b S_c} / 2S$</td>
</tr>
<tr>
<td>Longchamps circle</td>
<td>X(20)</td>
<td>$\sqrt{-S_a S_b S_c} / S$</td>
</tr>
<tr>
<td>Bevan circle</td>
<td>X(40)</td>
<td>$2R$</td>
</tr>
<tr>
<td>Spieker circle</td>
<td>X(10)</td>
<td>$r/2$</td>
</tr>
<tr>
<td>Apollonius circle</td>
<td>X(970)</td>
<td>$\frac{abc}{(a^2 + b^2 + c^2)}$</td>
</tr>
<tr>
<td>1st Lemoine</td>
<td>X(182)</td>
<td>$R \div 2 \cos \omega$</td>
</tr>
<tr>
<td>2nd Lemoine</td>
<td>X(6)</td>
<td>$abc / (a^2 + b^2 + c^2)$</td>
</tr>
<tr>
<td>Sin-triple-angle</td>
<td>X(49)</td>
<td>$R_{sta}$</td>
</tr>
<tr>
<td>Brocard circle</td>
<td>X(182)</td>
<td>$eR \div 2 \cos \omega$</td>
</tr>
<tr>
<td>Brocard second</td>
<td>X(3)</td>
<td>$eR$</td>
</tr>
<tr>
<td>Orthocentroidal</td>
<td>X(381)</td>
<td>$</td>
</tr>
<tr>
<td>Fuhrmann</td>
<td>X(355)</td>
<td>$</td>
</tr>
</tbody>
</table>

January 3, 2024 21:08 published under the GNU Free Documentation License
Proposition 13.4.3. For any finite point, other than the circumcenter $X_3$, the inverse-in-circumcircle of $P = p : q : r$ has barycentrics $u : v : w$ obtained cyclically from:

$$u = -p^2 + \frac{c^2 - a^2}{b^2} pq + \frac{b^2 - a^2}{c^2} pr + \frac{a^2(b^2 + c^2 - a^2)}{b^2c^2} qr$$

Proof. Points $X(3), P, U$ are on the same line, and distance from $X(3)$ to $U$ is $R^2/|PX_3|$. Therefore, in normalized barycentrics, we have: $u = x_3 + (p - x_3) R^2/|PX_3|^2$.

Remark 13.4.4. On ETC $n \leq 3587$, there are:

- 258 named points that belongs to $\Gamma$
- 47 pairs of "true" inverses that both are named
- 220 named points of $\Gamma$ that have a named isogonal conjugate (among the 229 points of $L_\infty$)
- 62 pairs of named antipodal points

13.5 Incircle

Definition 13.5.1. The incircle is one of the four circles that are tangent to the sidelines. This circle is inside the triangle, and also inside the nine point circle (these circles are tangent). Center $X(1)$, perspector $X(7)$, radius $r = S/s = abc / 2R(a + b + c)$, while equation and column are:

$$\Gamma_{std} + \frac{1}{4} \sum x (b - a + c)^2 : \begin{pmatrix} (-a + b + c)^2 \\ (+a - b + c)^2 \\ (+a + b - c)^2 \\ 4 \end{pmatrix}$$

(13.9)

Proof. well-known properties.

Remark 13.5.2. On ETC $n \leq 3587$, there are:

- 39 named points on the incircle
- 7 pairs of "true" inverses that both are named
- 10 pairs of named antipodal points

Proposition 13.5.3. Centers of homothety with the circumcircle are in=$X(55)$, and ex=$X(56)$. When $p : q : r \in \Gamma$, then $Q, Q'$ are antipodal points on the incircle:

$$Q = \left((b - c)^2 p + a^2 q + a^2 r \right) (b + c - a), \text{ etc}$$

$$Q' = \frac{(b + c)^2 p + a^2 q + a^2 r}{b + c - a}, \text{ etc}$$

Proof. Proposition 13.3.2.
**Proposition 13.5.4. Incircle transform.** Let \( U = u : v : w \) be a point other than the symmedian point, \( X_6 \). Then reflection of \( T \) (the intouch triangle) in the line \( UX_1 \) is perspective with triangle \( ABC \). The isogonal conjugate of the corresponding perspector is called the incircle transform of \( U \). Its barycentrics are:

\[
IT(u : v : w) = \frac{a^2(bw - cv)^2}{b + c - a} : \frac{b^2(cu - aw)^2}{a + c - b} : \frac{c^2(au - bv)^2}{b + a - c}
\]

and this point is on the incircle.

**Proof.** Line \( UX_1 \) is a diameter of the incircle and the reflected triangle \( T \) is also inscribed in the incircle. The barycentrics of \( UX_1 \) are \([bw - cv, cu - aw, av - wb]\). Reflection in this line is obtained using (7.26), and barycentrics of \( T \) are obtained. Perspectivity and perspector are easily computed and conclusion follows by substituting in the incircle equation. \(\square\)

**Remark 13.5.5.** In ETC another formula is given... and the point is also on the incircle:

\[
IT2(U) = \frac{a^2(b^2w - c^2v)^2}{b^2c^2(b + c - a)} : \frac{b^2(c^2u - a^2w)^2}{a^2c^2(c + a - b)} : \frac{c^2(a^2v - b^2u)^2}{a^2b^2(a + b - c)}
\]

One has \( IT(X) = IT(U) \) when \( U, X \) aligned with \( X_1 \) while \( IT2(X) = IT(U) \) when \( U, X \) aligned with \( X_6 \).

### 13.6 Nine-points circle

**Definition 13.6.1.** The nine point circle is the circumcircle of the orthic triangle. It goes also through the six midpoints of the orthocentric quadrangle \( ABCH \). Center \( X(5) \), radius \( R/2 \) (half the \( ABC \) circumradius), perspector \( X(3613) \):

\[
\frac{1}{b^2c^2 + 2Sa^2} : \frac{1}{a^2c^2 + 2Sb^2} : \frac{1}{b^2a^2 + 2Sc^2}
\]

while equation, matrix, column are:

\[
\Gamma_{std} + \frac{1}{2}(xSa + ySb + zSc) \begin{bmatrix} 2Sa & -c^2 & -b^2 \\ -c^2 & 2Sb & -a^2 \\ -b^2 & -a^2 & 2Sc \end{bmatrix} \begin{bmatrix} Sa \\ Sb \\ Sc \end{bmatrix} = \frac{1}{2}
\]

**Proposition 13.6.2.** Centers of homothety with \( \Gamma \) are \( in = X(2) \), and \( ex = X(4) \). \( \Omega \) intersects \( \Gamma \) when \( ABC \) is not acute. Radical trace \( X(468) \), direction of center axis (Euler line) \( X(30) \), direction of radical axis \( X(523) \). Standard parametrization (homothety from circumcircle):

\[
U \simeq \begin{pmatrix} (\sigma - \tau)(c^2\tau - b^2\sigma + b^2\rho - c^2\rho) \\ (\tau - \rho)(a^2\rho - c^2\tau + c^2\sigma - a^2\sigma) \\ (\rho - \sigma)(b^2\sigma - a^2\rho + a^2\tau - b^2\tau) \end{pmatrix}
\]

**Remark 13.6.3.** On ETC \( n \leq 3587 \), there are:

- 37 named points on the nine points circle
- 9 pairs of "true" inverses that both are named
- 12 pairs of named antipodal points

**Proposition 13.6.4 (Feuerbach).** The nine-point circle is tangent to the incircle and the three excircles. The contact with incircle is \( X(11) \), the so-called Feuerbach point.

**Proof.** Use (13.9) and (13.10) to obtain the radical axis. \(\square\)
13.7 Polar circle

Definition 13.7.1. The polar circle is the only circle whose matrix is diagonal (and triangle $ABC$ is autopolar). Center $X(4)$, the orthocenter, radius $\sqrt{-S_a S_b S_c} \div 2S$, equation and column are:

$$\frac{1}{x + y + z} \sum S_a x^2 = \Gamma_{std} + (x S_a + y S_b + z S_c) = 0; \begin{pmatrix} S_a \\ S_b \\ S_c \\ 1 \end{pmatrix}$$

Proposition 13.7.2. This circle belongs to the same pencil as the circum- and the nine-points circles. This circle is real only when triangle $ABC$ is not acute. Therefore, no named points can belong to this circle.

Proposition 13.7.3. The polar circle is the locus of the centers of the inscribed rectangular hyperbolas (cf Section 12.8).

13.8 Longchamps circle

Definition 13.8.1. The Longchamps circle of $ABC$ is the polar circle of the antimedial triangle. Center $X(20)$, the Longchamps point, radius $\sqrt{-S_a S_b S_c}/S$, equation, matrix and column are:

$$\Gamma_{std} + (x a^2 + y b^2 + z c^2) = 0; \begin{pmatrix} a^2 \\ b^2 \\ c^2 \\ 1 \end{pmatrix}$$

Proposition 13.8.2. The Longchamps circle is the locus of the auxiliary points of the inscribed rectangular hyperbolas (cf Section 12.8).

13.9 Bevan circle

Definition 13.9.1. The Bevan circle is the circumcircle of the excentral triangle. Perspector $X(57)$, center $X(40)$, radius $2R$, equation and column:

$$\Gamma_{std} + (-bcx - acy - abz) = 0; \begin{pmatrix} -bc \\ -ac \\ -ba \\ 1 \end{pmatrix}$$

Proposition 13.9.2. Centers of homothety with $\Gamma$ are in $=X(165)$, and $ex=X(1)$. Radical trace $X(1155)$, center axis $X(517)$, radical axis $X(513)$. Moses parametrization leads to $Q$ (bad looking) and $Q' = -(a + 2b + 2c)p + aq + ar$.

Remark 13.9.3. On ETC $n \leq 3587$, there are:

- 9 named points on the Bevan circle, namely: 1054, 1282, 1768, 2100, 2101, 2448, 2449, 2948, 3464
- 5 pairs of "true" inverses that both are named
- 2 pairs of named antipodal points

13.10 Spieker circle

Definition 13.10.1. Spieker circle is the incircle of the medial triangle. Perspector $X(2)$, center $X(10)$, radius $r/2$ (half the $ABC$ inradius), equation:

$$\Gamma_{std} + \frac{1}{16} \sum x \ (5c^2 + 5b^2 - 3a^2 + 2ac + 2ab - 6bc)$$
Proposition 13.10.2. Centers of homothety with $\Gamma$ are in-$X(958)$, and ex-$X(1376)$. Radical trace not named, center axis $X(515)$, radical axis $X(522)$. Moses parametrization leads to:

$$Q = (b + c - a) \left( (ab^2 + ac^2 + b^3 - b^2c - c^2b + c^3) p + a (q + r) \left( a^2 + ab + ac + 2bc \right) \right)$$

$$Q' = (ab^2 + c^2a + b^3c + c^3b - b^3) p + a (q + r) \left( +a^2 - ab - ac + 2bc \right), \text{ etc}$$

Centers of homothety with the incircle are in-$X(8)$ and ex-$X(2)$.

Remark 13.10.3. On ETC $n \leq 3587$, there are:

- 8 named points on the Spieker circle, namely: 3035, 3036, 3037, 3038, 3039, 3040, 3041, 3042
- no pairs of "true" inverses that both are named
- 2 pairs of named antipodal points [3035,3036], [3042,3042]

13.11 Alt-Spieker circle

Definition 13.11.1. The alt-Spieker circle is the common orthogonal cycle to the three excircles (see Subsection 14.10.4). Center $X(10)$, radius $\sqrt{r_0^2 + s^2} \div 2$, equation and column:

$$4 \Gamma_{std} - \sum x (a - b + c) (a + b - c) = 0; \begin{pmatrix} - (a - b + c) (a + b - c) \\ - (b - c + a) (b + c - a) \\ - (c - a + b) (c + a - b) \\ 4 \end{pmatrix}$$

13.12 Apollonius circle

Definition 13.12.1. The Apollonius circle is tangent to the three excircles and encloses them (see Subsection 14.10.4). Center $X(970)$, perspector not named, equation:

$$\Gamma_{std} = \frac{a + b + c}{4} \sum \frac{a^2 + ab + ac + 2bc}{a} x = 0$$

radius = $(abc + \sum a^2b) \div 8S = \frac{r_0^2 + s^2}{4r_0}$

Proposition 13.12.2. Centers of homothety with $\Gamma$ are in-$X(573)$, and ex-$X(386)$. Radical trace not named, center axis $X(511)$, radical axis $X(512)$. Moses parametrization leads to bad looking $Q$ and

$$Q' = -(b + c)^2 (a + c) (a + b) p + (ab + ac + bc + b^2 + c^2) a^2 q + r (ab + ac + bc + b^2 + c^2) a^2, \text{ etc}$$

Centers of homothety with the nine-points circle are in-$X(10)$ and ex-$X(2051)$.

Remark 13.12.3. On ETC $n \leq 3587$, there are:

- 8 named points on the Apollonius circle, namely: 2037, 2038, 3029, 3030, 3031, 3032, 3033, 3034
- no pairs of "true" inverses that both are named
- 1 pairs of named antipodal points [2037, 2038].
13. More about circles

13.13 First Lemoine circle

Definition 13.13.1. The first Lemoine circle of $ABC$ is obtained as follows. Draw parallels to the sidelines of $ABC$ through Lemoine point $X_6$. The six intersections of these lines with sidelines are concyclic on the required circle. The following surd is useful:

$$W_1 = \sqrt{a^2b^2 + a^2c^2 + b^2c^2} \quad (13.11)$$

Proposition 13.13.2. Center is $X(182)$ (i.e. $[O, K]$ midpoint), radius $R:\frac{2\cos \omega}{(a^2 + b^2 + c^2)}$, perspector:

$$\frac{a^2}{2a^2b^2 + 2a^2c^2 + b^2c^2}, \frac{b^2}{2a^2b^2 + 2a^2c^2 + b^2c^2}, \frac{c^2}{2a^2b^2 + 2a^2c^2 + b^2c^2}$$

is not named, equation:

$$\Gamma_{std} + \frac{1}{(a^2 + b^2 + c^2)^2} \sum x \left( b^2 + c^2 \right) b^2 c^2$$

This circle is concentric with and external to the first Brocard circle.

Proof. Difference of squared radiuses factors into $(\frac{abc}{(a^2 + b^2 + c^2)})^2$.

Proposition 13.13.3. Centers of homothety with $\Gamma$ are in-$X(1342)$, and ex-$X(1343)$. Radical trace $X(1691)$, center axis $X(511)$, radical axis $X(512)$. Moses parametrization leads to bad looking $Q$ and $Q'$. Poncelet centers of the pencil $X(1687)$ (inside) and $X(1688)$ outside.

Proof. In order to see that $X(1687)$ is inside, compute $\Omega (X(182)) \times \Omega (X(1687))$ and obtain a quantity that is clearly positive.

Remark 13.13.4. On ETC $n \leq 3587$, there are:

- 2 named points on the first Lemoine circle, namely : 1662, 1663 (intersection with the Brocard axis, $X(3)X(6)$).
- 7 pairs of "true" inverses that both are named:

$$\begin{bmatrix} 3 & 6 & 32 & 39 & 371 & 372 & 1687 \\ 2456 & 1691 & 1692 & 2458 & 2461 & 2462 & 1688 \end{bmatrix}$$

- 1 pairs of named antipodal points [1662, 1663].

13.14 Second Lemoine circle

Definition 13.14.1. The second Lemoine circle of $ABC$ is obtained as follows. Draw parallels to the sidelines of orthic triangle through Lemoine point $X_6$. The six intersections of these lines with sidelines are concyclic on the required circle.

Proposition 13.14.2. Center is $X(6)$ itself, radius $\frac{abc}{(a^2 + b^2 + c^2)}$, perspector $X(3527)$, equation:

$$\Gamma_{std} + \frac{4}{(a^2 + b^2 + c^2)^2} \sum x S_0 b^2 c^2 = 0$$

Centers of homothety with $\Gamma$ are in-$X(371)$, and ex-$X(372)$. Radical trace $X(1692)$, center axis $X(511)$, radical axis $X(512)$. Moses parametrization leads to bad looking $Q$ and $Q'$. Poncelet centers of the pencil are involving radical $\sqrt{6 \sum a^2b^2 - 5 \sum a^4}$ and are not named. Moreover, the second Lemoine circle is bitangent to the Brocard ellipse.

Remark 13.14.3. On ETC $n \leq 3587$, there are:

- 2 named points on the second Lemoine circle, namely : 1666, 1667 (intersection with the Brocard axis, $X(3)X(6)$).
- 5 pairs of "true" inverses that both are named:

$$\begin{bmatrix} 3 & 576 & 1316 & 1351 & 2452 \\ 1570 & 1691 & 2451 & 1692 & 3049 \end{bmatrix}$$

- 1 pairs of named antipodal points [1666, 1667].
13.15 Sine-triple-angle circle

Definition 13.15.1. Define inscribed triangles $T_1$ and $T_2$ by the properties:

$$\angle (AB, AC) = \angle (B_1 A, B_1 C_1) = \angle (C_2 B_2, C_2 A),$$

the idea being to obtain isosceles "remainders". Then all the six vertices are on the same circle.

Proof. Using the tangent formula, the six points are easily obtained. $T_1$ is a central triangle, and each vertex of $T_2$ is obtained by a transposition.

$$A_1 \simeq \begin{pmatrix}
0 \\
(a^2 - ac - b^2) (a^2 + ac - b^2) (a^2 + b^2 - c^2) \\
(a^2 - bc - c^2) (a^2 + bc - c^2) c^2
\end{pmatrix}$$

Proposition 13.15.2. Center $X(49)$, perspector not named, equation horrific, radius $R_{tsa} = R^3/\left(|OH|^2 - 2 R^2\right)$. Centers of homothety with $\Gamma$ are in=$X(1147)$, and ex=$X(184)$, direction of radical axis $X(924)$. Moses parametrization leads to bad looking $Q$ and:

$$Q' = a^2 (a^2 - c^2) (a^2 - b^2) p - a^4 (b^2 + c^2 - a^2) (q + r),$$

Remark 13.15.3. On ETC $n \leq 3587$, there are:

- 6 named points on the Sine Triple Angle circle, namely: 3043, 3044, 3045, 3046, 3047, 3048
- 0 pairs of "true" inverses that are both named
- 1 pairs of named antipodal points [3043, 3047].

13.16 Brocard 3-6 circle

Definition 13.16.1. The Brocard 3-6 circle has $[X_3, X_6]$ for diameter. Center $X(182)$. Radius $e R \div 2 \cos \omega = W_2 R/ (a^2 + b^2 + c^2)$ where:

$$W_2 = \sqrt{a^4 + b^4 + c^4 - (b^2 c^2 + a^2 b^2 + a^2 c^2)}$$

(13.12)

while the perspector:

$$\frac{a^2}{2a^4 + b^2 c^2} : \frac{b^2}{2b^4 + a^2 c^2} : \frac{c^2}{2c^4 + a^2 b^2}$$

is not named. Equation:

$$\Gamma_{std} + \frac{1}{(a^2 + b^2 + c^2)} \left( x b^2 c^2 + y c^2 a^2 + z a^2 b^2 \right) = 0$$
Proposition 13.16.2. Centers of homothety with $\Gamma$ are in $X(1340)$, and $ex=X(1341)$. Radical trace $X(187)$, center axis $X(511)$, radical axis $X(512)$. Moses parametrization leads to:

$$Q = \left( \pm W \left( (a^2 b^2 + a^2 c^2 + 2b^2 c^2 - b^4 - c^4) p - a^2 (b^2 + c^2 - a^2) (q + r) \right) \right),$$

Poncelet centers of the pencil are $X(15)$ and $X(16)$, the isodynamic points (see Section 14.9).

Remark 13.16.3. On ETC $n \leq 3587$, there are:

- 4 named points on the first Brocard circle, namely: 3, 6, 1083, 1316. Moreover, this circle goes through the Brocard points (cf 7.7.1).
- 47 pairs of "true" inverses that are both named
- 1 pairs of named antipodal points [3, 6].

13.17 Second Brocard circle

Definition 13.17.1. First anti-Brocard circle: circumcircle of the first anti-Brocard triangle. Equation:

$$\Gamma_{std} - \sum_3 a^2 \left( b^2 + bc + c^2 \right) \left( b^2 - bc + c^2 \right) \frac{W^2}{x} = 0$$

Definition 13.17.2. The Brocard second circle is centered on $X(3)$ and goes through the Brocard’s centers. Radius $eR = W_2 R/\sqrt{a^2 b^2 + b^2 c^2 + c^2 a^2}$, while the perspector:

$$\frac{a^2}{2a^4 - a^2 b^2 - a^2 c^2 + b^2 c^2},$$

is not named. Equation:

$$\Gamma_{std} + \frac{a^2 b^2 c^2}{a^2 b^2 + a^2 c^2 + b^2 c^2} = 0$$

Remark 13.17.3. On ETC $n \leq 3587$, there are:

- 6 named points on the second Brocard circle, namely: 1670, 1671, 2554, 2555, 2556, 2557. Moreover, this circle goes through the Brocard points (cf 7.7.1).
- 17 pairs of "true" inverses that are both named
- 3 pairs of named antipodal points [1670, 1671], [2554, 2555], [2556, 2557].

13.18 Orthocentroidal 2-4 circle

Definition 13.18.1. Orthocentroidal circle has $[X_2, X_4]$ for diameter. Center is $X(381)$ and radius $RW_3 = 3abc$ where:

$$W_3 = \sqrt{a^6 + b^6 + c^6 - \left( a^4 b^2 + a^4 c^2 + a^2 b^4 + a^2 c^4 + b^4 c^2 + b^2 c^4 \right)} + 3 a^2 b^2 c^2$$

while the perspector:

$$\frac{1}{b^2 c^2 + 2a^2 (b^2 + c^2 - a^2)}$$
is not named. Equation and column are:

\[
\Gamma_{std} + \frac{2}{3} (x S_a + y S_b + z S_c) = 0 ; \quad \begin{pmatrix} 2 S_a \\ 2 S_b \\ 2 S_c \\ 3 \end{pmatrix}
\]

**Proposition 13.18.2.** Centers of homothety with \( \Gamma \) are in-\( \times \)X(1344), and ex-\( \times \)X(1345). Radical trace \( X(468) \), center axis \( X(30) \), radical axis \( X(523) \). Moses parametrization leads to:

\[
Q = \pm W \left( \left( a^2 b^2 + a^2 c^2 + 2 b^2 c^2 - b^4 - c^4 \right) p - a^2 \left( b^2 + c^2 - a^2 \right) (q + r) \right),
\]

Poncelet points are real when triangle is acute.

**Remark 13.18.3.** On ETC \( n \leq 3587 \), there are:

- 2 named points on the orthocentroidal circle, namely (antipodal) 2, 4
- 45 pairs of "true" inverses that are both named

**Proposition 13.18.4.** The orthocentroidal circle goes through points \( A' = (A + 2A_H)/3 \), etc. where \( A_B A_H \) are the feet of the altitudes.

*Proof.* One has \( A_H \approx a^2 : 2 S_c : 2 S_b \). Then one has \( \text{Ver}(A_H) \cdot \mathcal{V} = 0. \)

**Proposition 13.18.5.** Among the four in/ex-centers, the in-center is inside the orthocentroidal circle, and the other three are outside.

*Proof.* One can check that

\[
|O I_0|^2 - 4 |N I_0|^2 = +2r_0(R - 2r_0) = |O I_0|^2 \times (+2r_0/R) \geq 0
\]

\[
|O I_A|^2 - 4 |N I_A|^2 = -2r_A(R + 2r_A) = |O I_A|^2 \times (-2r_A/R) \leq 0
\]

while the locus of \( |O M|^2 = 4 |N M|^2 \) is precisely the said circle (\( M = H \) and \( M = G \) are solutions).

**Proposition 13.18.6.** Let \( O, N, I \) be three points on the plane and let \( \psi(\Phi) \) be the image of the curve

\[
\Phi : 3 \overline{Z}^2 Z^2 + 14 \overline{Z} Z (Z + \overline{Z}) T + 27 \left( Z^2 + \frac{4}{3} \overline{Z} Z + \overline{Z}^2 \right) T^2 + 54 (Z + \overline{Z}) T^3 + 27 T^4 = 0
\]

by the similitude \( \psi \) which sends \( z = -1 \) to \( O \) and \( z = 0 \) to \( N \). When \( I \) is outside \( \psi(\Phi) \), it exists a triangle such that \( O \) is the circum-center, \( N \) is the Euler-center and \( I \) is one of the in/ex-centers. When \( I \) is inside this exclusion curve, no such triangle can be found.

**Sketch of the proof,** more details in Guinand, 1984. Consider the polynomial:

\[
P(X) = X^3 - X^2 \left( \frac{3}{2} - \frac{2\sigma}{\rho} \right) + X \left( \frac{2\sigma(\sigma - \kappa)}{\rho^2} - \frac{3\sigma}{4} + \frac{3}{4} \right) - \frac{1}{8} + \frac{2\sigma \kappa}{\rho^2}
\]

When \( \rho = |O I_0|^2 \), \( \sigma = |N I_0|^2 \), \( \kappa = |O N|^2 \), the roots of \( P \) are the three \( \cos A \), etc. When \( a \) is changed into \( -a \), \( I_0 \) becomes \( I_a \) and the roots of \( P \) become \( \cos A \), \( -\cos B \), \( -\cos C \). This is easily checked using the Al-Kashi formula. (How to obtain \( P \) is another story, see Guinand).

Since all roots have to be real, we must have

\[
0 \geq \Delta(P) = 16 \frac{\sigma^2 S^2 (\rho, \sigma, \kappa)}{\rho^6} \left( 32 \sigma \kappa - 27 \rho^2 + 40 \sigma \rho - 16 \sigma^2 \right)
\]

And then \( \Phi \) is obtained by substituting

\[
\rho = (Z + T)(Z + T)/T^2 ; \quad \sigma = ZZ/T^2 ; \quad \kappa = 1
\]

in the last factor. When \( \Delta = 0 \), the roots are \( X = -1 \) (simple) and \( X = 5/4 - \sigma/\rho \) (double).
Remark 13.18.7. Due to the relation $A + B + C = \pi$, the three cosines are related by

$$\cos^2 A + \cos^2 B + \cos^2 C = 1 - 2 \cos A \cos B \cos C.$$ 

In other words, we have $s_1^2 - 2 s_2 + 2 s_3 - 1 = 0$. And this can be checked for polynomial $P$. The Lemoine's transforms changes two signs, and the relation remains.

Exercise 13.18.8. Generate "a lot" of $\alpha, \beta, \gamma$ on the unit circle, and draw the corresponding

$$\psi^{-1}(I) = \frac{(\alpha + \beta + \gamma)^2}{\alpha^2 + \beta^2 + \gamma^2}$$

(the green points). Superpose the graph of $\Phi$, and draw the circle $[-1/3; +1]$.

### 13.19 Fuhrmann 4-8 circle

**Definition 13.19.1.** Fuhrmann circle has $[X_4, X_8]$ for diameter (see also ???). Center $X(355)$, radius $RW_4 \div \sqrt{abc}$ where:

$$W_4 = \sqrt{a^3 + b^3 + c^3 - (a^2 b + a^2 c + ab^2 + ac^2 + b^2 c + bc^2) + 3 abc} \tag{13.14}$$

perspector not named (and not handy). Equation and column are:

$$\Gamma_{std} + \frac{2}{a + b + c} (x a S_a + y b S_b + z c S_c) = 0 ; \begin{pmatrix} 2 S_a a \\ 2 S_b b \\ 2 S_c c \\ b + a + c \end{pmatrix}$$

**Remark 13.19.2.** The only named points of this circle are $X(2)$ and $X(4)$. Five inverse pairs are:

<table>
<thead>
<tr>
<th>1</th>
<th>11</th>
<th>72</th>
<th>2475</th>
<th>3434</th>
</tr>
</thead>
<tbody>
<tr>
<td>80</td>
<td>1837</td>
<td>3419</td>
<td>3448</td>
<td>3436</td>
</tr>
</tbody>
</table>

### 13.20 Taylor circle

**Definition 13.20.1.** Project the foot of any altitude onto the two other sidelines. The six points obtained are concyclic, defining the Taylor circle. Center $X(389)$ with barycentrics

$$a^2 \left(4 S^2 + S_b S_c\right) \left(2 S_a^2 - b^2 c^2\right) - a^4 b^2 c^2 S_a, \text{ etc}$$

—— pldx : Translation of the Kimberling's Glossary into barycentrics ——
squared radius

\[ \frac{4S^4}{a^2b^2c^2} + \frac{S^2S_a^2 S_b^2}{16a^2b^2c^2S^2} = \frac{\text{rad}H + 4S^2}{16R^2} \]

where \( \text{rad}H \) is the radius of the polar circle (so that \( \text{rad}H \geq 0 \)), perspector not so simple (and not named). No named point on it. Equation and column are:

\[ \Gamma_{\text{std}} + \frac{4S^2}{a^2b^2c^2} (xS_a^2 + yS_b^2 + zS_c^2) = 0 ; \begin{bmatrix} S_a^2 \\ S_b^2 \\ S_c^2 \\ a^2b^2c^2 \end{bmatrix} \simeq \begin{bmatrix} S_a^2 \\ S_b^2 \\ S_c^2 \\ 4R^2 \end{bmatrix} \]

### 13.21 Kiepert RH and isosceles adjunctions

**Proposition 13.21.1. Kiepert RH.** Chose angle \( \phi \) and construct isosceles triangles \( BA'C, \ CB'A, \ AC'B \) with basis angle \( \angle (BC, BA') = \phi \) (\( \phi < 0 \) when \( A' \) is outside). Then triangles \( ABC \) and \( A'B'C' \) are perspective wrt a point \( N (\phi) \):

\[ N (\phi) \simeq \frac{1}{2S \cot \phi - S_a}, \text{ etc} \simeq \frac{a}{\sin (A - \phi)}, \text{ etc} \quad (13.15) \]

and Kiepert RH is the locus of such points. Perspector of this conic is \( X(523) \), DeLongchamps point at infinity, and center is \( X(115) \).

**Proof.** One has : \( A' = a^2 \tan \phi : 2S - S_c \tan \phi : 2S - S_b \tan \phi \).

**Remark 13.21.2.** Triangle \( A'B'C' \) is perspective from \( X(3) \) to the tangential triangle of \( ABC \).

**Proposition 13.21.3.** For a given \( K \), the points \( A'B'C' \) are on the same cubic as the vertices, the incenter, orthocenter, circumcenter and the points \( A'B'C' \) relative to the opposite value of \( K \). This cubic can be written as \((K^2 + 1)K001 + (3 - K^2)K003\) where \( K001 \) and \( K003 \) are the standardized equations of, respectively, the Neuberg and the McKay cubics, as given in Proposition 20.4.25 and Proposition 20.4.24. The pole is \( X(6) \), while pivot is \( 3vX(2) - K^2vX(20) \) i.e. : \((1 + K^2) s_1 s_3 : (3 - K^2) s_3 : (1 + K^2) s_2 \).

**Proof.** Details are given in Proposition 20.4.22.

**Remark 13.21.4.** Points at infinity of Kiepert RH are parametrized by:

\[
\cot \phi = \frac{-1}{3} \left( 1 + \frac{2W_2}{a^2 + b^2 + c^2} \right) \cot (\omega) = \frac{-1}{12S} \left( a^2 + b^2 + c^2 + 2W_2 \right) \\
W_2 = \sqrt{a^4 - a^2b^2 - a^2c^2 - b^4 - b^2c^2 + c^4} \quad \text{is the Brocard radical}
\]

**Remark 13.21.5.** Fixed values of angle \( \phi \) can be obtained by adding some regular shape to each side of the reference triangle. This created a race for the most inventive adjunction. Among them is the Pelle à Tarte, whose name was coined from a charade whose solution was "neon lamp, pie shovel" (i.e. lampe au néon, pelle à tarte, an approximation for Napoleon Bonaparte)

**Example 13.21.6.** In Figure 13.3, a Pelle à Tarte (pie shovel) like \( AFUGC \) is made of a square \( AFGC \) and an equilateral triangle \( FUG \). It defines the angle \( \phi = \left( \frac{AU}{AC} \right) = -75^\circ \). When the triangle is inside the square, we obtain a Pelle Pliée (folded shovel) like \( AFWGC \) that defines angle \( \phi = \left( \frac{AU}{AC} \right) = -15^\circ \).

**Example 13.21.7.** Arbelos. Another idea to obtain some adding object is as follows. Divide \([AB]\) in a given ratio, obtaining \( D \). Use the same ratio to obtain \( E, F \). Draw the half circles and perpendicular \( DM \), then the tangent circles. The triangle of the centers is perspective with \( ABC \) if and only if the ratio is \((\sqrt{3} - 1)/2\). Perspector is \( X(2672) \). And belongs to Kiepert RH. Therefore, we have isosceles triangle. Value is \( \pm \tan \phi = 3 - \sqrt{5} \) (plus is inward, minus is outward).
Example 13.21.8. When using the center of a regular triangle, square, pentagon, we have $\phi = 30^\circ$, $\phi = 45^\circ$, $\phi = 54^\circ$; when using the farthest vertex (or the midpoint of the farthest side), $\phi = 60^\circ$, $\phi = \arctan 2 = 2\phi = 72^\circ$. We even have points relative to $\arctan 3 = \phi - 90^\circ$, $\phi = 2\arctan 2 = \arctan 3$.

Example 13.21.9. Brocard angle is often involved since:

$$P_\phi \simeq \frac{1}{(b^2 + a^2 + c^2) \cot (\phi) - S_a \cot (\omega)},$$

etc.

Proposition 13.21.10. Let $\sigma$ be a constant value. Then all lines $N(\phi) N(\sigma - \phi)$ are passing through point $T(\sigma)$ where:

$$T(\sigma) \simeq \frac{a^2 S_a}{b^2 S_b} + 2S \cot \sigma \left( \begin{array}{c} a^2 \\ b^2 \\ c^2 \end{array} \right)$$

Therefore all the $N(\phi) N(-\phi)$ lines (columns in a block) are passing through $X(6)$, the Lemoine point. And all the $N(\phi) N\left(\frac{\pi}{2} - \phi\right)$ lines (rows in a block) are passing through $X(3)$ the circum-center. All the $T(\sigma)$ are on the Brocard axis $X(3)X(6)$.

Proof. Obvious from (13.15).

Proposition 13.21.11. Let $\delta$ be a constant value. Then all lines $N(\phi) N(\phi + \delta)$ are tangent to a conic $D(\delta)$. When $\delta = 0$ this conic is the Kiepert RH itself. When $\delta = \pi/2$ (diagonals in a block), the conic reduces to a real point: the Euler center $X(5)$. In the general case, $D(\delta) = 16S^2 \left( \prod (a^2 - b^2) \right) \cot^2 (\delta) D(0) + D(\pi/2)$.

Proof. This can be proved using the usual techniques: differentiate and wedge. When $\delta$ is rational wrt $\pi$, conics $D(\delta)$ and $D(0)$ are in a Poncelet configuration.

Lemma 13.21.12. Consider circle $C_0 : x^2 + y^2 = 1$ and points $U = (u, 0), V = (v, 0)$ in the Cartesian plane. When $u + v \neq 0$, these points are the centers of homothety of circles $C_0$ and $C(P, \rho)$ where

$$P = \left( \frac{2uv}{u+v}, 0 \right), \quad \rho = \left| \frac{u-v}{u+v} \right|$$

--- pdlx : Translation of the Kimberling's Glossary into barycentrics ---
Proposition 13.21.13. A circle $C(P, \rho)$ can be found such that points $U = N(\phi)$, $V = N(\phi + \pi/2)$ are the centers of homothety between $C$ and the nine-points circle. We have:

\[
\cot (2 \phi) \sim \frac{a^2 S_a}{b^2 S_b} - 2 S \left( \frac{a^2}{b^2} \right) \approx \cot (2 \phi) \quad \text{for} \quad P = (U + V) / 2.
\]

Proof. These two points are aligned with $E=X(5)$. Define $W = (U + V) / 2$. From the above lemma, we have:

\[
P = E + k (U - E) ; \quad \rho = (1 - k) \frac{R}{2} \quad \text{where} \quad k = \frac{V - E}{W - E}.
\]

Computations are greatly simplified when remarking that the results depends not really from $\cot \phi$ itself, but rather from $\cot \phi - 1 / \cot \phi$. That is the reason why all these formulas are involving $\cot (2\phi)$.

### 13.22 Cyclocevian conjugate

**Definition 13.22.1.** Two points are called cyclocevian conjugates when their cevian triangles share the same circumcircle. This definition has to be compared with cyclopedal conjugacy, see Section 9.3.

**Proposition 13.22.2 (Terquem).** Each point not on the sidelines has a cyclocevian conjugate. Its barycentrics are given by (Grinberg, 2003a):

\[
U = \text{cyclocevian} (P) = (\text{isot} \circ \text{anticomplement} \circ \text{isog} \circ \text{complement} \circ \text{isot}) (P)
\]

Proof. This proposition asserts that the other intersections of the circumcircle of $ABPC_P$ with the sidelines are the vertices of another cevian triangle. As it should be, the formula is involutory.
<table>
<thead>
<tr>
<th>$\phi$</th>
<th>$U$</th>
<th>$V$</th>
<th>$P$</th>
<th>$\rho$</th>
<th>name</th>
<th>circum</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>2</td>
<td>4</td>
<td>3</td>
<td>$\frac{2 \sqrt{3}abc}{12 S - \sqrt{3} (a^2 + b^2 + c^2)}$</td>
<td></td>
<td></td>
</tr>
<tr>
<td>$\pi/12$</td>
<td>3392</td>
<td>3391</td>
<td>16</td>
<td>$\frac{2 \sqrt{3}abc}{12 S + \sqrt{3} (a^2 + b^2 + c^2)}$</td>
<td></td>
<td></td>
</tr>
<tr>
<td>$-\pi/12$</td>
<td>3366</td>
<td>3367</td>
<td>15</td>
<td>$\frac{2 \sqrt{3}abc}{12 S - \sqrt{3} (a^2 + b^2 + c^2)}$</td>
<td></td>
<td></td>
</tr>
<tr>
<td>$\pi/10$</td>
<td>3397</td>
<td>1139</td>
<td>3393</td>
<td>$\frac{5 abc}{16 S + 3 (a^2 + b^2 + c^2)}$</td>
<td></td>
<td></td>
</tr>
<tr>
<td>$-\pi/10$</td>
<td>3370</td>
<td>1140</td>
<td>3379</td>
<td>$\frac{5 abc}{16 S - 3 (a^2 + b^2 + c^2)}$</td>
<td></td>
<td></td>
</tr>
<tr>
<td>$\arctan(3)$</td>
<td>1328</td>
<td>?</td>
<td>?</td>
<td>$\frac{5 abc}{12 S + 4 (a^2 + b^2 + c^2)}$</td>
<td></td>
<td></td>
</tr>
<tr>
<td>$-\arctan(3)$</td>
<td>1327</td>
<td>?</td>
<td>?</td>
<td>$\frac{5 abc}{12 S - 4 (a^2 + b^2 + c^2)}$</td>
<td></td>
<td></td>
</tr>
<tr>
<td>$\pi/8$</td>
<td>3388</td>
<td>3387</td>
<td>372</td>
<td>$\frac{\sqrt{2}abc}{4 S - (a^2 + b^2 + c^2)}$</td>
<td></td>
<td></td>
</tr>
<tr>
<td>$-\pi/8$</td>
<td>3373</td>
<td>3374</td>
<td>371</td>
<td>$\frac{\sqrt{2}abc}{4 S + (a^2 + b^2 + c^2)}$</td>
<td></td>
<td></td>
</tr>
<tr>
<td>$\arctan(2)$</td>
<td>1132</td>
<td>3316</td>
<td>?</td>
<td>$\frac{5 abc}{12 S + 4 (a^2 + b^2 + c^2)}$</td>
<td></td>
<td></td>
</tr>
<tr>
<td>$-\arctan(2)$</td>
<td>1131</td>
<td>3317</td>
<td>?</td>
<td>$\frac{5 abc}{12 S - 4 (a^2 + b^2 + c^2)}$</td>
<td></td>
<td></td>
</tr>
<tr>
<td>$\pi/6$</td>
<td>18</td>
<td>13</td>
<td>62</td>
<td>$\frac{2 abc}{4 S - \sqrt{3} (a^2 + b^2 + c^2)}$</td>
<td></td>
<td></td>
</tr>
<tr>
<td>$-\pi/6$</td>
<td>17</td>
<td>14</td>
<td>61</td>
<td>$\frac{2 abc}{4 S + \sqrt{3} (a^2 + b^2 + c^2)}$</td>
<td></td>
<td></td>
</tr>
<tr>
<td>$\pi/5$</td>
<td>3382</td>
<td>3381</td>
<td>3395</td>
<td>$\frac{\sqrt{2}abc}{10 \omega}$</td>
<td>2° Lemoine</td>
<td>Apollonius</td>
</tr>
<tr>
<td>$-\pi/5$</td>
<td>?</td>
<td>?</td>
<td>3368</td>
<td>$\frac{\sqrt{2}abc}{10 \omega}$</td>
<td>2° Lemoine</td>
<td>Apollonius</td>
</tr>
<tr>
<td>$\arctan(3-\sqrt{5})$</td>
<td>2672</td>
<td>?</td>
<td>?</td>
<td>$\frac{3 (4 \sqrt{5} + 5) abc}{44 S - 2 (9 + 5 \sqrt{5}) (a^2 + b^2 + c^2)}$</td>
<td></td>
<td></td>
</tr>
<tr>
<td>$-\arctan(3-\sqrt{5})$</td>
<td>2671</td>
<td>?</td>
<td>?</td>
<td>$\frac{3 (4 \sqrt{5} + 5) abc}{44 S + 2 (9 + 5 \sqrt{5}) (a^2 + b^2 + c^2)}$</td>
<td></td>
<td></td>
</tr>
<tr>
<td>$\arctan((a+b+c)^2)$</td>
<td>486</td>
<td>485</td>
<td>6</td>
<td>$\frac{abc}{a^2 + b^2 + c^2}$</td>
<td>2° Lemoine</td>
<td>Apollonius</td>
</tr>
<tr>
<td>$-\arctan((a+b+c)^2)$</td>
<td>?</td>
<td>?</td>
<td>?</td>
<td>$\frac{abc}{a^2 + b^2 + c^2}$</td>
<td>2° Lemoine</td>
<td>Apollonius</td>
</tr>
<tr>
<td>$2 \omega$</td>
<td>1916</td>
<td>3399</td>
<td>?</td>
<td>$\frac{(a^2 b^2 + a^2 c^2 + b^2 c^2) R}{a^4 + b^4 + c^4 + a^2 c^2 + a^2 b^2 + b^2 c^2}$</td>
<td></td>
<td></td>
</tr>
<tr>
<td>$-2 \omega$</td>
<td>3407</td>
<td>3406</td>
<td>?</td>
<td>$\frac{(a^2 b^2 + a^2 c^2 + b^2 c^2) R}{a^4 + b^4 + c^4 + a^2 c^2 + a^2 b^2 + b^2 c^2}$</td>
<td></td>
<td></td>
</tr>
<tr>
<td>$\omega$</td>
<td>76</td>
<td>262</td>
<td>3095</td>
<td>$\frac{R}{a^4 + b^4 + c^4 + a^2 b^2 + b^2 c^2}$</td>
<td></td>
<td></td>
</tr>
<tr>
<td>$-\omega$</td>
<td>83</td>
<td>98</td>
<td>3398</td>
<td>$\frac{R}{a^4 + b^4 + c^4 + a^2 c^2 + a^2 b^2 + b^2 c^2}$</td>
<td></td>
<td></td>
</tr>
<tr>
<td>$\omega/2$</td>
<td>?</td>
<td>?</td>
<td>511</td>
<td>$\frac{R}{2 \cos(\omega)}$</td>
<td>1° Lemoine</td>
<td>Gallatly</td>
</tr>
<tr>
<td>$-\omega/2$</td>
<td>1676</td>
<td>1677</td>
<td>182</td>
<td>$\frac{R}{2 \cos(\omega)}$</td>
<td>1° Lemoine</td>
<td>Gallatly</td>
</tr>
<tr>
<td>$\pi/4 + \omega/2$</td>
<td>2010</td>
<td>2009</td>
<td>39</td>
<td>$\frac{abc (b^2 + a^2 + c^2)}{2 (b^4 + c^4 + a^4) \cos(\omega)}$</td>
<td></td>
<td></td>
</tr>
<tr>
<td>$-\pi/4 - \omega/2$</td>
<td>?</td>
<td>?</td>
<td>32</td>
<td>$\frac{abc (b^2 + a^2 + c^2)}{2 (b^4 + c^4 + a^4) \cos(\omega)}$</td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

Table 13.5: Similicenters on Kiepert RH

— pldx : Translation of the Kimberling’s Glossary into barycentrics —
Example 13.22.3. Some examples:

<table>
<thead>
<tr>
<th>point</th>
<th>code</th>
<th>bary</th>
<th>cycc</th>
<th>circumcenter</th>
</tr>
</thead>
<tbody>
<tr>
<td>Gergonne</td>
<td>X (7)</td>
<td>(1/(−a + b + c))</td>
<td>X (7)</td>
<td>X (1)</td>
</tr>
<tr>
<td>centroid</td>
<td>X (2)</td>
<td>1</td>
<td>X (4)</td>
<td>X (5)</td>
</tr>
<tr>
<td>orthocenter</td>
<td>X (4)</td>
<td>(1/(−a^2 + b^2 + c^2))</td>
<td>X (2)</td>
<td>X (5)</td>
</tr>
<tr>
<td>Nagel</td>
<td>X (8)</td>
<td>(-a + b + c)</td>
<td>X (189)</td>
<td>X (1158)</td>
</tr>
</tbody>
</table>

Point \(X_7\) is the only center that is invariant by cyclocevian. Three other points share this property, obtained by changing one of the \(a, b, c\) into its opposite in the barycentrics of \(X_7\).

13.23 Mixtilinear circles

Definition 13.23.1. In triangle \(ABC\), the circle \(\gamma\) tangent to the sidelines \(AB, AC\) and the circumcircle is called the A mixtilinear circle when \(\gamma_A\) is inside the circumcircle. The other circle tangent to these three lines is called the external mixtilinear circle and noted \(\hat{\gamma}_A\).

Proposition 13.23.2. Using barycentrics or Lubin-2 lead to:

\[
\gamma_A \simeq \left[ \begin{array}{c} \frac{4b^2c^2}{a^2(b+c)} \\ \frac{c^2(a-b+c)^2}{b^2(a+b-c)^2} \\ \frac{(b+a+c)^2}{(b+a+c)^2} \end{array} \right]; \quad \gamma_A \simeq \left[ \begin{array}{c} \frac{(as_2 + s_3)^2}{\alpha^2(\beta - \gamma)^2} \\ \frac{(\alpha + s_1)^2}{(\beta - \gamma)^2\alpha} \end{array} \right]; \quad \rho = \frac{2(s_1s_2 - s_3)}{(\beta - \gamma)^2\alpha}
\]

The external mixtilinear circle is obtained using the A-Lemoine transform.

Proof. Formulas are easy to verify... and then non-displayed ones are rather huge! \(\square\)

Construction 13.23.3. The second intersection of line \(A-X(56)\) cuts again the circumcircle at \(A_1 \in \gamma_A\). Line \(OA_1\) cuts line \(AI_0\) at \(Ka\), the center of \(\gamma_A\). Contacts \(A_b, A_c\) with the sidelines are aligned with \(I\). Moreover \(A_bA_c \perp AI_0\).
Chapter 14

Pencils of Cycles in the Triangle Plane

14.1 Introductory remarks

14.1.1 How many points at infinity should be used?

In the context of the (barycentric) Triangle Plane, points are described by projective columns living in \( P_\mathbb{R}(\mathbb{R}^3) \). In the same vein, the present chapter will describe the circles and their pencils by projective columns \( V \) living in \( P_\mathbb{R}(\mathbb{R}^4) \). Later, the Morley plane will be introduced, where points are described by columns living in \( P_\mathbb{C}(\mathbb{C}^3) \) while, in Chapter 18, circles will be described by projective columns \( V \) living in \( P_\mathbb{C}(\mathbb{C}^4) \).

Efficient notations are powerful, but poor notations can be confusing. Thus we will use \( V_b, V_z, V_p, V_s \) to distinguish between barycentric, general Morley, Pedoe and spherical objects. When a formula is valid in all the four contexts, indices will be omitted.

Using such 4D spaces is not the most frequent method to describe the circles and their pencils. The tradition (Poncelet, 1822, 1865) is rather to use the Riemann sphere \( P_\mathbb{C}(\mathbb{C}^2) \) where \( (z_1, z_2) \simeq (\lambda z_1, \lambda z_2) \) for any non-zero \( \lambda \in \mathbb{C} \). But Triangle Geometry deals with points and lines living in projective 3D-spaces, i.e. described as \( x : y : z \) where \( x : y : z \simeq kx : ky : kz \) for any non-zero multiplier \( k \).

Obviously, the 2D and 4D points of view are reducing to the same elementary Cartesian co-ordinates when restricted to the finite domain. But they are conflicting where they are the most useful, i.e. where they are implementing the Poncelet’s continuity principle for objects at infinity. This is even more true concerning the "circular points at infinity", the so-called umbilics of the plane.

An ordinary line must be completed in a way or another to become a "circle with infinite radius" and having a clear definition of this completion is required in order to unify the three concepts of circle \( (0 < \rho < \infty) \), point \( (\rho = 0) \) and line \( (\rho = \infty) \) into a single concept of cycle.

In the Riemann sphere \( P_\mathbb{C}(\mathbb{C}^2) \), there exists only one point at infinity (noted \( \infty \)). In this context, a "circle with infinite radius" is an ordinary line \( \Delta \) completed with the point noted \( \infty \), i.e. \( \Delta = \Delta \cup \{\infty\} \), while point-circles are either the set \( \{\infty\} \) or a circle with radius \( 0 \) around an ordinary point.

In the Triangle Plane \( P_\mathbb{R}(\mathbb{R}^3) \), there exists a whole line \( L_\infty \) of points at infinity, verifying \( x + y + z = 0 \). In this document, barycentrics are used. Using trilinears would only change some formulas, but not the very nature of the underlying space. In this context, the barycentric equation of an ordinary circle leads to define a cycle as a second degree curve (a conic) that goes through the so-called umbilics \( \Omega^\pm \). Therefore, the equation of a completed line becomes \((x + y + z)(ux + vy + wz) = 0\) so that we must define \( \Delta \) as \( \Delta \cup L_\infty \), while the role of \( \{\infty\} \) in \( P_\mathbb{C}(\mathbb{C}^2) \) is played now by the horizon circle \( C_\infty \) defined by \((x + y + z)^2 = 0\), i.e. defined as the object having the same points as \( L_\infty \) but each of them counted twice.

In Chapter 7, orthogonality of lines has been reduced to a polarity wrt operator \([M]\). Here, the same treatment will be applied to cycles, i.e. the family of all curves that are either a circle or a line. This leads to a fundamental quadric \( Q \) in a 4-D projective space \( P_\mathbb{R}(\mathbb{R}^4) \) [here] or \( P_\mathbb{C}(\mathbb{C}^4) \) [some chapters later]. Finite points on the quadric can be interpreted as representatives of point-circles, quite the same thing as an ordinary point in the triangle plane. Points outside this quadric
are representatives of cycles, while points inside are assigned to the later defined virtual circles.

When dealing with tangency of cycles, a better description is secured by using oriented cycles, living in a Lie sphere, embedded into a 5D space obtained by a double coating of the ordinary 4D space of cycles. We haven’t developed this concept here.

14.1.2 Umbilics

Lemma 14.1.1. We have \( \mathbf{OrtO} \cdot \mathbf{OrtO} \cdot \mathbf{W} = -\mathbf{W} \). Therefore, restricted to \( \mathcal{V} \), \( \mathbf{OrtO}^2 \) is nothing but an half turn. Multiplied by \( \mathbf{Pyth} \) this leads to \( \mathbf{OrtO}^2 + \mathbf{OrtO} = 0 \), so that eigenvalues of \( \mathbf{OrtO} \) are 0, \( +i \), \( -i \).

Definition 14.1.2. Umbilics. Let \( U \in \mathcal{L}_\infty \) be a point at infinity, i.e a point such that \( u + v + w = 0 \). The associated umbilics are the complex points:

\[
\Omega^+ \simeq \left(1 - i \mathbf{OrtO}\right) \cdot U \quad ; \quad \Omega^- \simeq \left(1 + i \mathbf{OrtO}\right) \cdot U
\]

Caveat : signs are crossed ! A possible choice is:

\[
\Omega^+ \simeq 4S X_{512} - i X_{511}
\]

\[
\simeq 4S \left( a^2 (b^2 - c^2) \right) \pm i \left( a^2 (b^2 + c^2) - b^4 - c^4 \right)
\]

\[
= \left( b^2 (c^2 + a^2) - c^4 - a^4 \right) \quad (14.1)
\]

where \( i \) is the imaginary unit and \( S \) the area of the triangle. Another choice is:

\[
\Omega^+ = \Omega_y \simeq \left( S_b + 2iS \right) \quad ; \quad \Omega^- = \Omega_x \simeq \left( S_a + 2iS \right)
\]

These expressions are no more symmetric, but computations become easier.

Remark 14.1.3. Spoiler. Using Morley representation \( z_A = \alpha \), etc, we obtain:

\[
\Omega_y \simeq \left( \begin{array}{cccc}
1 & 1 & 1 & \gamma \\
\gamma & 1 & 1 & 1 \\
1 & \gamma & 1 & 1 \\
\gamma & 1 & \gamma & 1 \\
\end{array} \right) \simeq \left( \begin{array}{cccc}
1 & 0 & 0 & 0 \\
0 & \gamma - \alpha & 0 & 0 \\
0 & 0 & \alpha - \beta & 0 \\
0 & 0 & 0 & 1 \\
\end{array} \right) \quad \Omega_x \simeq \left( \begin{array}{cccc}
1 & 0 & 0 & 0 \\
0 & \gamma - \alpha & 0 & 0 \\
0 & 0 & \alpha - \beta & 0 \\
0 & 0 & 0 & 1 \\
\end{array} \right)
\]

Proposition 14.1.4. When seen as elements of \( \mathcal{V}_C \), all \( \Omega^\pm \) are eigenvectors of operator \( \mathbf{OrtO} \), with eigenvalues (respectively) \( \pm i \) and therefore belong to the light cone. When seen as elements of \( \mathcal{P}_C (\mathbb{C}^3) \), points \( \Omega^+ \) and \( \Omega^- \) are now independent of the choice of \( U \), and are the fixed points of the orthopoint transform. They both belong to the circumcircle, and are isogonal conjugates to each other.

Proof. See Postnikov (1982, 1986) for better insights on real-complex spaces. Property \( \mathcal{L}_\infty \cdot \Omega = 0 \) is obvious. Umbilics are eigenvectors because of

\[
\mathbf{OrtO} \cdot \Omega^+ = \mathbf{OrtO} \cdot \left(1 - i \mathbf{OrtO}\right) \cdot U = \left(\mathbf{OrtO} + i\right) \cdot U = +i\Omega^+
\]

Since eigenvalues of \( \mathbf{OrtO} \) are simple, the C-dimension of eigenspaces is one, and uniqueness in \( \mathcal{P}_C (\mathbb{C}^3) \) follows. For this reason, points \( \Omega^\pm \) are also called the "circular points at infinity". Finally, intersection of circumcircle and the infinity line must be invariant by isogonal conjugacy, while \( \Omega^+_b \cdot \Omega^+ \) cannot be real, even up to a complex proportionality factor.

Proposition 14.1.5. A circle is a conic that goes through the umbilics. Above all, choosing the umbilics is deciding which of the circum-ellipses is the circumcircle.
Proof. By definition, umbilics are the (non real) points where line at infinity intersects the circumcircle. By (13.3), these points belong to any circle. For the converse, consider the values taken by $x^2$, $y^2$, $z^2$, $xy$, $yz$, $zx$ at $A, B, C$ together with both umbilics. This gives a $5 \times 6$ matrix whose rank is 5: the first three lines are 1, 0, 0, 0, 0 etc. and it remains only to show that rank of submatrix 4..5, 4..6 is two. A direct inspection shows that critical factors are $a^2 + b^2 - c^2$ (straight angle, that can occur only once) and $a^4 + b^4 + c^4 - b^2c^2 - a^2b^2 - a^2c^2$ (condition of equilaterality). In such a case, the property remains when umbilics are written as $1 : j : j^2$ and $1 : j^2 : j$.

Remark 14.1.6. When the umbilics are given, the euclidian structure of the Triangle Plane is known. From $\Omega^+ \ast \Omega^- = a^2 : b^2 : c^2$, the Pyth matrix is known (up to the value of $R$), while the orthopoint transform, and its matrix $OrtO$ is characterized by its diagonal shape, namely $(0, +i, -i)$, when using the triple $X(3)$, $\Omega^+ \ast \Omega^-$ as barycentric basis.

14.1.3 Notations

We have done our best effort to use unified notations. In this whole chapter,

Notation 14.1.7. Conventions about letters.

$P, Q$ denote some flat true points in the Triangle Plane and $X_n$ a Kimberling-named triangle center, all of them being 3-columns.

$\Gamma, \Omega$ denote the circumcircle of the fundamental triangle $ABC$ (and nothing else) while $\Omega$ denotes a cycle, both of them being curves, i.e. a set of points. The $3 \times 3$-matrices describing their equations as a quadratic form are noted using a box.

$U, V$ denote the representative of a circle-point, while $V$ denotes the representative of any cycle, all of them being columns in $P_R(R^3)$.

$Y$ denotes the representative of an oriented cycle in $P_R(R^5)$

$G$ denotes a Gram matrix. The elements of this matrix are noted $w^2$ along the diagonal and $W$ for the non-diagonal elements.

Notation 14.1.8. To design a circle known by a pair center/radius, parentheses will be used, exemplified by $\Gamma = (X_3, R)$. Using parentheses around a single Roman letter —e.g. $(P)$— will be reserved to denote the circle $(P, 0)$ i.e. the circle whose unique real point is $P$ itself.

14.2 Cycles and representatives

Definition 14.2.1. The (barycentric) Veronese map is the correspondence that maps a $P_R(R^3)$ column into a $P_R(R^4)$ row proportional to:

$$t \begin{pmatrix} x (x + y + z) \\ y (x + y + z) \\ z (x + y + z) \\ -a^2yz - b^2xz - c^2xy \end{pmatrix}$$

 Proposition 14.2.2. The Veronese map is obviously homogeneous. Both umbilics are send to $[0, 0, 0, 0]$ (points of indeterminacy). The other points at infinity are send to $[0, 0, 0, 1]$. Otherwise, the map is injective.

Definition 14.2.3. Veronese map. For points at finite distance, we will use (7.13) and define $Ver(x, y, z)$ by the simpler formula:

$$Ver(x : y : z) = [x, y, z, \Gamma_{std}(x, y, z)]$$

(14.4)
Remark 14.2.4. Requiring that four points are on the same circle leads to Proposition 13.1.4, i.e.
to equation:
\[ \det_{i=1}^{i=4} [p_i, q_i, r_i, \Gamma_{std}(p_i, q_i, r_i)] = 0 \]
But, conversely, this equation only implies that our four points are on the same circle or on the
same straight line. To summarize both situations under a single concept, we introduce :
Definition 14.2.5. The cycle \( \Omega \) associated with the representative \( \mathcal{V}_b \simeq u : v : w : t \in \mathbb{P}_R(\mathbb{R}^4) \)
is the locus of the points \( X \in \mathbb{P}_R(\mathbb{R}^3) \) that satisfies equation:
\[ \text{Ver}_b(X) \cdot (u : v : w : t) = 0 \] (14.5)
For example, the representative of circumcircle \( \Gamma \) is \( \mathcal{V}_b \simeq 0 : 0 : 0 : 1 \).
Remark 14.2.6. When \( t \neq 0 \), cycle \( \Omega \) is the (ordinary) circle whose standardized equation is:
\[ \frac{u}{t}x + \frac{v}{t}y + \frac{w}{t}z + \Gamma_{std}(x, y, z) \] (14.6)
Remark 14.2.7. The representative of a circle is seen as a 3D point (described by a column). The
Veronese of a 2D point is an action over the 3D points: i.e. a plane, described by a row.
Definition 14.2.8. Cycle \( \mathcal{L}_\infty \) represented by \( \mathcal{V}_\infty \simeq 1 : 1 : 1 : 0 \) has to be understood as the line at
infinity \( \mathcal{L}_\infty \) described twice, and will be called the horizon circle. This object has to be perceived
as a circle "whose center is everywhere and circumference nowhere" (Empedocles).
The representative itself, i.e. the point \( \mathcal{V}_\infty \simeq 1 : 1 : 1 : 0 \), will be called Sirius, following
Kimberling (1998-2021) in using stars to coin the name given to a point. While using that specific
star for a very distant point is from (Voltaire, 1752).
Definition 14.2.9. Otherwise, the cycle represented by \( u : v : w : 0 \) is the union of an ordinary
line and \( \mathcal{L}_\infty \), and will be called a completed line.
Theorem 14.2.10. The representative of the point-circle \( (P) \) associated with a point at finite
distance \( P \simeq p : q : r \) is the column given by:
\[ U_P \simeq \mathcal{Q}^{-1}_b \cdot \text{Ver}(P) \quad \text{where} \quad \mathcal{Q}^{-1}_b = \begin{pmatrix} 0 & c^2 & b^2 & 1 \\ c^2 & 0 & a^2 & 1 \\ b^2 & a^2 & 0 & 1 \\ 1 & 1 & 1 & 0 \end{pmatrix} \] (14.7)
\[ U_P = c^2 q^2 + b^2 r^2 + 2 S_a q r : c^2 p^2 + a^2 r^2 + 2 S_b p r : b^2 p^2 + a^2 q^2 + 2 S_c p q : (p + q + r)^2 \] (14.8)
Conversely, we have:
\[ \text{Ver}(P) \simeq U_P \cdot \mathcal{Q}_b \quad \text{where} \quad \mathcal{Q} = \frac{-1}{8S^2} \begin{bmatrix} a^2 & -S_c & -S_b & -a^2 S_a \\ -S_c & b^2 & -S_a & -b^2 S_b \\ -S_b & -S_a & c^2 & -c^2 S_c \\ -a^2 S_a & -b^2 S_b & -c^2 S_c & a^2 b^2 c^2 \end{bmatrix} \] (14.9)
Proof. Direct computation. Mind the fact that both formulas are hard equalities, and that, when
both matrices are written exactly that way, the obvious result \( \mathcal{Q}_b \mathcal{Q}^{-1}_b = +1 \) is enforced !
Remark 14.2.11. It must be taken into account that radiiuses that are not "up to a proportionality
factor". The values of matrix \( \mathcal{Q}_b \) and \( \mathcal{Q}^{-1}_b \) were chosen to obtain the best looking formula at
(14.10) and the normalized Minkowski formula (14.12)
\[ \frac{1}{b} \mathcal{V}_1 \cdot \mathcal{Q} \cdot \mathcal{V}_2 = d^2 - r_1^2 - r_2^2 \]
The price to pay is the appearence of a \( -2 \) factor when computing radiiuses, at the general
formula (14.15), and at the orthogonal formula (14.18). Be prudent, don't over-simplify !
Remark 14.2.12. Sometimes, "the representative of the point-circle \( P^n \) will be shortened into "the representative of \( P \)." This object (a column) is not to be confused with the Veronese image of \( P \) (a row)!

Example 14.2.13. Here are some point representatives:

<table>
<thead>
<tr>
<th>( P \setminus x )</th>
<th>( u )</th>
<th>( v )</th>
<th>( w )</th>
<th>( t )</th>
</tr>
</thead>
<tbody>
<tr>
<td>1 : 0 : 0</td>
<td>0</td>
<td>( c^2 )</td>
<td>( b^2 )</td>
<td>1</td>
</tr>
<tr>
<td>0 : 1 : 1</td>
<td>( 2b^2 + 2c^2 - a^2 )</td>
<td>( a^2 )</td>
<td>( a^2 )</td>
<td>4</td>
</tr>
<tr>
<td>( X(1) )</td>
<td>( bc ) (( b + c - a ))</td>
<td>( ac ) (( c + a - b ))</td>
<td>( ab ) (( b + a - c ))</td>
<td>( a + b + c )</td>
</tr>
<tr>
<td>( X(2) )</td>
<td>( 2b^2 + 2c^2 - a^2 )</td>
<td>( 2a^2 + 2c^2 - b^2 )</td>
<td>( 2b^2 + 2a^2 - c^2 )</td>
<td>9</td>
</tr>
<tr>
<td>( X(3) )</td>
<td>( R^2 )</td>
<td>( R^2 )</td>
<td>( R^2 )</td>
<td>1</td>
</tr>
<tr>
<td>( X(4) )</td>
<td>( R^2 a^2 (b^2 + c^2 - a^2)^2 )</td>
<td>( R^2 b^2 (c^2 + a^2 - b^2)^2 )</td>
<td>( R^2 c^2 (a^2 + b^2 - c^2)^2 )</td>
<td>( a^2 b^2 c^2 )</td>
</tr>
<tr>
<td>( X(6) )</td>
<td>( b^2 c^2 (2b^2 + 2c^2 - a^2) )</td>
<td>( a^2 c^2 (2c^2 + 2a^2 - b^2) )</td>
<td>( a^2 b^2 (2a^2 + 2b^2 - c^2) )</td>
<td>( (a^2 + b^2 + c^2)^2 )</td>
</tr>
<tr>
<td>( \infty )</td>
<td>1</td>
<td>0</td>
<td>1</td>
<td>0</td>
</tr>
<tr>
<td>( \emptyset )</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
</tr>
</tbody>
</table>

The fact that formula (14.8) *would* give \( \text{Sirius} \simeq 1 : 1 : 1 : 0 \) for each point on \( L_\infty \) is the reason of their exclusion from the definition of the point representatives.

Corollary 14.2.14. The representative of the circle \( (P, \omega) \) where \( P \simeq p : q : r \) is a point at finite distance is obtained by:

\[
\begin{align*}
\frac{P}{p + q + r} & \text{Ver} \left( \frac{P}{p + q + r} \right) - \omega^2 \text{Sirius} \\
\end{align*}
\]

(14.10)

where \( \text{Sirius} \) is exactly \( [1, 1, 1, 0] \) and \( \frac{Q}{b} \) is exactly as in (14.7).

Proof. Obvious from the above theorem. \( \square \)

Example 14.2.15. Compute the representative of the incircle. Two equivalent methods are:

\[
\begin{align*}
\text{mQCI}(\text{Tr}@\text{Factor}@\text{Ver}@\text{norb}@vX)(1)-\text{ri}^2*\text{Sirius}:
\text{method1} & := (\text{Factor}@\text{subs})(\text{kitcircleS}, \text{valS6}, \%); \\
\text{method2} & := (\text{nor4}@\text{wedge3})(\text{seq}(\text{Ver}(j), j=\text{Column(\text{matcev}(vX(7)), 1..3))});
\end{align*}
\]

Remark 14.2.16. Assuming that representatives are living in \( \mathbb{P}_R(\mathbb{R}^4) \) has many advantages. The top one could be to enforce the fact that a representative is not a point in the Triangle Plane. It is a key fact that the triple \( [\hat{u}, \hat{v}, \hat{w}] \) appearing in the standardized equation (14.6) is not defined up to a proportionality factor. The same remark applies to the so-called "circle function" \( [\hat{u} \div bc, \hat{v} \div ca, \hat{w} \div ab] \in \mathbb{R}^3 \) that appears when using trilinears as in Weisstein (1999-2009).

14.3 Fundamental quadric and orthogonality

Theorem 14.3.1. Any point representative \( U = u : v : w : t \) belongs to the quadric \( Q \) :

\[
^t U \cdot \frac{Q}{b} U = 0 = 0
\]

where \( \frac{Q}{b} \) is as given in (14.9).

Proof. One has \( \text{Ver}(P) \cdot \frac{Q}{b} \) is as given in (14.9).

\[
\begin{align*}
\text{Ver}(P) & = 0 \text{ by the very definition of } \text{Ver}(P). \quad \square
\end{align*}
\]

--- pldx : Translation of the Kimberling's Glossary into barycentrics ---
Proposition 14.3.2. Object Sirius \(1 : 1 : 1 : 0\) is the only (real) point at infinity of the quadric \(Q\). Therefore, \(Q\) is a paraboloid.

Proof. Substitute \(t = 0\), then compute the discriminant with respect to \(w\) and obtain \(-(u - v)^2 a^2 b^2 c^2 / R^2\). This requires \(u = v\), etc. \(\square\)

Remark 14.3.3. In the usual \(\mathbb{P}_C(\mathbb{C}^2)\) model, \(\mathcal{L}_\infty\) is "in the South plane" while the horizon circle \(\mathcal{C}_\infty\) is nothing but the point-circle \(\{\infty\}\).

Proposition 14.3.4. An element \(\mathbb{V} = u : v : w : t\) of \(\mathbb{P}_R(\mathbb{R}^4)\) is the representative of a (real) cycle if only if \(\mathbb{V}\) is outside of \(Q\) (i.e. on the same side as \(0 : 0 : 0 : 1\) characterized by \(t(u : v : w : t) \cdot Q = (u : v : w : t) \geq 0\) when (14.11) is used.

Proof. Obvious from (14.10), that states that representative of \((P, \omega)\) is "below" the representative of \((P)\), while representatives of completed lines are at infinity in \(\mathbb{P}_R(\mathbb{R}^4)\) and therefore outside of paraboloid \(Q\). \(\square\)

Theorem 14.3.5. Orthogonal cycles. Consider two cycles \(\Omega_1, \Omega_2\) with representatives \(\mathbb{V}_1, \mathbb{V}_2\).

When \(\mathbb{V}_2\) belongs to the polar plane of point \(\mathbb{V}_1\) wrt the fundamental quadric then cycles \(\Omega_1\) and \(\Omega_2\) are orthogonal –and conversely.

Computed Proof. Begin with two circles. Write representative \(\mathbb{V}_i\), as in (14.10) from representative \(U_j\) of point-circle \((P_j)\). This implies that \(\mathbb{V}_i[4] = 1\). Compute \(\mathbb{V}_1 : Q \cdot \mathbb{V}_2\) and –using (7.11)– obtain :

\[
\mathbb{V}_1 \cdot \mathbb{Q} \cdot \mathbb{V}_2 = (|P_1P_2|^2 - \nu_1^2 - \nu_2^2)
\] (14.12)

Compute now \(\mathbb{V}_1 : Q \cdot \mathbb{V}_3\) where \(\mathbb{V}_3 = u_3 : v_3 : w_3 : 0\) and obtain :

\[
\mathbb{V}_1 \cdot \mathbb{Q} \cdot \mathbb{V}_3 = (p_1u_3 + q_1v_3 + r_1w_3) \div (p_1 + q_1 + r_1)
\] (14.13)

In both cases, the result is the orthogonality condition times a non vanishing factor. Finally, when the representatives of two lines are involved, the conclusion follows directly from the properties of the orthopoint transform. \(\square\)

Remark 14.3.6. The elementary formula \(\mathbb{V}_1 \cdot Q \cdot \mathbb{V}_2 = d^2 - r_1^2 - r_2^2\) has to be enforced at any cost. But this implies that \(\mathbb{V} : Q \cdot \mathbb{V} = -2v^2\), and we have to live with this disgracious \(-2\) factor when computing the radius of a circle.

Corollary 14.3.7. The locus of the representatives of the points of a given cycle \(\Omega\) is the intersection between \(Q\) and the polar plane –wrt \(Q– of the representative of \(\Omega\).

Proof. By definition, point \(P\) belongs to cycle \(\Omega\) if and only if \(\Omega\) and \((P)\) are orthogonal. \(\square\)

Theorem 14.3.8. Back to barycentrics. Let \(\mathbb{V} = u : v : w : t\in \mathbb{P}_R(\mathbb{R}^4)\) be a representative.

Then either

(1) \(\mathbb{V} \simeq 1 : 1 : 1 : 0\). Then \(\mathbb{V}\) is Sirius, i.e. represents the horizon circle
(2) \(t = 0\), but \(\mathbb{V} \neq \text{Sirius}\). Then \(\mathbb{V}\) represents a line.
(3) \(t \neq 0\). Then \(\mathbb{V}\) represents a circle (may be reduced to a point). The associated center and squared radius are given by :

\[
(p : q : r) \simeq \left(\mathbb{Q} \cdot \mathbb{V}\right)_{1.3}
\] (14.14)

\[
\omega^2 = \left(-\frac{1}{2}\right) \mathbb{t} \cdot \mathbb{V} / t^2
\] (14.15)

Moreover, the representative of the center (as a point-circle) is \(U = \mathbb{V} / t + \omega^2\) Sirius \(\in Q\).
Proof. The radius formula is a corollary of the preceding theorem. Let \( \mathcal{V}_b = x : y : z : \tau \) be any cycle representative and \( U \in \mathcal{Q} \) be the representative of point \( P = p : q : r \). Then \( {}^t W \cdot \mathcal{Q} \cdot U = 0 \) implies
\[
xp + yq + zr + \tau \Gamma_{std}(p, q, r) = 0
\]
where \( \Gamma_{std}(p, q, r) = -(a^2 qr + b^2 rp + c^2 pq) \div (p + q + r) \), so that equation (14.14) must hold for rank reason (and can be checked directly). Conversely, starting from any \( \mathcal{V}_b \) and applying (14.14) and then (14.8) leads back to \( \mathcal{V}_b \).

Example 14.3.9. Reobtain the radius of the incircle.

\[
\text{factor(Tr(incircle).mQQ.incircle); subs(kitcircleS, valS6, %/ri^2); } \mapsto 1
\]

Proposition 14.3.10. Power of a point wrt a circle. When \( W \) is the Veronese of a point \( M \) and \( \mathcal{V}_b \) is the representative of circle \( (P, \omega) \), then
\[
\text{power}(M, (P, \omega)) = |PM|^2 - \omega^2 = \frac{W \cdot \mathcal{V}_b}{W_1 + W_2 + W_3} \cdot \mathcal{V}_b
\]
This requires that \( W_1 + W_2 + W_3 \neq 0 \), i.e. \( M \notin L_\infty \) and that \( \mathcal{V}_b \neq 0 \), i.e. that \( (P, \omega) \) is a true circle (and not a completed line).

Proof. Obvious from definitions. Mind the normalizations!

Exercise 14.3.11. Compute the power of \( X(3) \) wrt the circumcircle (and obtain \( -R^2 \)). Compute the power of \( X(4) \) wrt the same circle. This formula enforces the fact that \( X(4) \) is inside the circle when triangle is acute.

Proposition 14.3.12. The angle of two circles is defined as:
\[
\cos(\Omega_1, \Omega_2) = \frac{-{}^t \mathcal{V}_b \cdot \mathcal{Q}_b \cdot \mathcal{V}_2}{\sqrt{{}^t \mathcal{V}_b \cdot \mathcal{Q}_b \cdot \mathcal{V}_b} \sqrt{{}^t \mathcal{V}_b \cdot \mathcal{Q}_b \cdot \mathcal{V}_b}}
\]

Proof. When the circles intersect at a visible point \( M \), this value is nothing but the Al-Kashi formula applied to triangle \( M\omega_1\omega_2 \). In any case, this formula is homogeneous wrt each of \( \mathcal{Q}_b \) and \( \mathcal{V}_b \).

Theorem 14.3.13. Common orthogonal cycle. Let be given three cycles \( \Omega_1, \Omega_2, \Omega_3 \). If they don't belong to the same pencil, the bundle they generate is exactly the set of all the cycles orthogonal to a fixed cycle \( \Omega_\perp \). We have the formulas (see (14.7), (14.9) for the precise values of the matrices):
\[
W = \bigwedge_3 \left( \mathcal{V}_b, \mathcal{V}_b, \mathcal{V}_b \right) \quad \text{(a 4-sized row)}
\]
\[
\mathcal{V}_b = \mathcal{Q}_b^{-1} \cdot \mathcal{W}
\]
\[
\text{center} = W_1 : W_2 : W_3
\]
\[
\text{squared radius} = \left( -\frac{1}{2} \right) \left( \mathcal{V}_b \cdot \mathcal{Q}_b \cdot \mathcal{V}_b \right) = \left( -\frac{1}{2} \right) \left( \mathcal{W} \cdot \mathcal{V}_b \right)
\]

Proof. Let \( j = 1, 2, 3 \). By definition, \( W \cdot \mathcal{V}_j = 0 \), so that \( {}^t \mathcal{V}_j \cdot \mathcal{Q}_b \cdot \left( \mathcal{Q}_b^{-1} \cdot {}^t \mathcal{W} \right) \) vanishes. Then the center follows by (14.14) and the radius as well.
Remark 14.3.14. Mind the normalizing factor in the formula just above (see Rem. Remark 14.2.11). Don’t simplify anything in the formula giving the radius! Remember that

\[ t \psi_1 - \psi_2 = (d_1^2 - r_1^2 - r_2^2) \times \left( \frac{\psi_1}{d_1} \frac{\psi_2}{d_2} \right) \]

Example 14.3.15. Compute the radius of the Euler circle (useless factor K, means whatever).

seq(mQQI.Tr(Ver(j)), j= Column(matcev(vX(2)),1..3) );
  tmp1:= K*(wedge3)(%);
  tmp2:= FActor(mQQI.Tr(tmp1)); # (Tr@reduce)(tmp1[1..3]); ency(%);
  methode1:= (-1/2)* tmp1.tmp2/tmp2[4]^2;
  subs(rapbpc, factor(%));
  methode2:= (-1/2)* Tr(tmp2).mQQ.tmp2/tmp2[4]^2;
  subs(kitRcircle, factor(%));
  And obtain \( R^2 = \frac{b^2c + a^2b + bc^2 + 2ab^2 + 2ac^2 + 2abc}{4(a + b + c)} \) by each method.

Definition 14.3.16. Radical center. The ever visible point \( W_1 : W_2 : W_3 \) in the former theorem is called the radical center of the three cycles. Having the same power w.r.t all the cycles of the bundle is a characteristic property of this point.

Remark 14.3.17. As emphasized later, the nature of the radius of \( \Omega_4 \), i.e. real, zero or imaginary fixes the nature of the bundle defined by \( \Omega_1, \Omega_2, \Omega_3 \).

Example 14.3.18. The example of the three excircles is examined in Subsection 14.10.4. Center is X(10) and radius is

\[ \omega_4 = \sqrt{\frac{b^2c + a^2b + bc^2 + 2ab^2 + 2ac^2 + 2abc}{4(a + b + c)}} \]

14.4 Pencils of cycles

Definition 14.4.1. Pencil. When \( \Omega_1, \Omega_2 \) are distinct cycles (with non proportional representatives), all curves \( \lambda_1 \Omega_1 + \lambda_2 \Omega_2 = 0 \), where \( (\lambda_1, \lambda_2) \neq (0, 0) \), are cycles and the set of all these cycles is called the pencil generated by \( \Omega_1, \Omega_2 \). It is clear that representatives of the cycles of a given pencil are on the same projective line in \( \mathbb{P}^2(\mathbb{R}^4) \) –called the representative of the pencil.

Example 14.4.2. Formula (14.6) describes circle \( \Omega \) as a member of the pencil generated by the circumcircle and a completed line. Therefore, the ordinary line \( ux + vy + wz = 0 \) is the radical axis \( \Delta \) of both circles \( \Omega \) and \( \Gamma \). That’s another way to see that knowing \( u:v:w \) is not enough to determine a circle.

Example 14.4.3. Formula (14.10), i.e. \( \Omega (P, \omega) = Q^{-1} \cdot \text{Ver} (P) = \omega^2 \mathcal{C}_\infty \), describes the circle \( (P, \omega) \) as a member of the pencil generated by the point-circle \( (P, 0) \) and the horizon circle, i.e. the pencil of all circles concentric with \( (P, 0) \).

Remark 14.4.4. Here again, the triple \( u:v:w \) is not sufficient to specify \( P \), and \([u:v:w:t]\) must be used. It can be checked that representative is well specified, i.e. doesn’t depends on whichever triple \((kp,kq,kr)\) is chosen as barycentrics of point \( P \).

14.5 Classification of pencils

Theorem 14.5.1 (Classification). Pencils of cycles fall in three classes, depending on the way their representative line \( \mathcal{P} \) intersects –in \( \mathbb{P}^2(\mathbb{R}^4) \)– the fundamental quadric \( Q \).

\( Q, \mathcal{P} \) tangent : \( \mathcal{P} \) is the tangent pencil of all the cycles containing a given point \( \omega_0 \) and tangent at \( \omega_0 \) to a line \( \Delta_1 \) containing \( \omega_0 \). Archetype : \( \omega_0 = \infty \) and \( \mathcal{P} \) is "all the lines parallel to a given line \( \Delta_1 \)."
Q, \( \mathcal{P} \) secant : \( \mathcal{P} \) is the isoptic pencil generated by two different point-circles \( \{ \omega_1 \} \) and \( \{ \omega_2 \} \) (\( \omega_1 \) are the limit points of \( \mathcal{P} \)). Archetype : \( \omega_2 = \infty \) and \( \mathcal{P} \) is, apart from \( \{ \infty \} \), "all the circles centered at a finite point \( \omega_1 \)".

Q, \( \mathcal{P} \) disjoint : \( \mathcal{P} \) is the isoptic pencil of all the cycles going through two different points \( \omega_1 \) and \( \omega_2 \) (the base points). Archetype : \( \omega_2 = \infty \) and \( \mathcal{P} \) is "all the lines through a finite point \( \omega_1 \)".

When \( \mathcal{P} \) is a tangent pencil, so is \( \mathcal{P} \perp \) (using \( \omega_0 \) and \( \Delta \perp \) orthogonal to \( \Delta_1 \) at \( \omega_0 \)). When \( \mathcal{P} \) is isoptic (\( \omega_1, \omega_2 \)) then \( \mathcal{P} \perp \) is isotomic (\( \omega_1, \omega_2 \)) and conversely. In all cases, representative of \( \mathcal{P} \) and \( \mathcal{P} \perp \) are conjugate lines wrt \( Q \).

Proof. Everything goes as in (Pedoe, 1970) –using another paraboloid– or (Douillet, 2009) –using a sphere. The only striking thing is that the usual point at infinity of the complex plane, namely \( \infty \in \mathbb{P}_C(\mathbb{C}^2) \), has to be replaced by the horizon circle \( C_\infty : (x + y + z)^2 = 0 \).

**Proposition 14.5.2.** A pencil of cycles that contains two lines is a pencil of lines. A concentric pencil contains the horizon cycle. All other pencils (i.e. all the non archetypal pencils) contain exactly one straight line (the radical axis of the pencil).

Proof. The representative of \( \mathcal{P} \) ever intersects the plane at infinity of \( \mathbb{P}_R(\mathbb{R}^4) \).

**Proposition 14.5.3.** Let \( P = p : q : r \) be a point in the Triangle Plane. Define its shadow in the Triangle Plane as point \( S = u : v : w \) where \( u, v, w \) are defined in (14.8). Then \( S \) is not outside the inconic \( IC(X_{76}) \). Any point on the border of \( IC(X_{76}) \) is the shadow of exactly one point on the circumcircle, while a point inside \( IC(X_{76}) \) –except from \( X_2 \)– is the shadow of exactly two points. Moreover, these points are inverse in the circumcircle.

**Remark 14.5.4.** Figure 14.1 shows the shadows of all the named points in ETC (Kimberling, 1998-2021), using the standard values \( a = 6, b = 9, c = 13 \). One can see two lines of points : \( L(X_2, X_6) \)–Euler line– and \( L(X_3, X_6) \)–Brocard axis– respectively.

**Figure 14.1:** No point-shadow fall outside of the IC(X76) inconic

**Proof of Proposition 14.5.3.** The locus of representatives of the points \( P_0 \) that belongs to \( \Gamma \) is the intersection of quadric \( Q \) and the polar plane \( \Pi \) of \( 0 : 0 : 0 : 1 \), namely :

\[
ua^2 (b^2 + c^2 - a^2) + vb^2 (c^2 + a^2 - b^2) + wc^2 (a^2 + b^2 - c^2) - 2ta^2b^2c^2 = 0
\]

Extracting \( t \) and substituting in \( Q \) leads (apart from a constant non-zero factor) to :

\[
u^2a^4 + b^4v^2 + c^4w^2 - 2uva^2b^2 - 2vwb^2c^2 - 2wuc^2a^2 = 0
\]

i.e. the equation of \( IC(X_{76}) \).

—– pldx : Translation of the Kimberling’s Glossary into barycentrics —–
When two points $P_1$, $P_2$ in the Triangle Plane share the same shadow, then points $U_1$, $U_2$ and $0 : 0 : 0 : 1$ are collinear in $P_b(\mathbb{R}^4)$ so that cycles $(P_1)$, $(P_2)$ and $\Gamma$ belongs to the same pencil. Therefore $P_1$, $P_2$ are inverse in the circumcircle. Moreover $U_0 \triangleq U_1U_2 \cap \prod$ is inside $IC(X_{76})$—otherwise $U_0$ would be the representative of a real circle belonging to pencil $(P_1),(P_2)$ and orthogonal to $\Gamma$.

**Proposition 14.5.5.** Points inside of $Q$ are representative of virtual circles (real center, imaginary radius). The reason to imagine such circles is that inversion in such a circle is a real transform. Moreover a real cycle $\Omega$ is orthogonal to $(X, i\omega)$ when $\Omega$ cuts $(X, \omega)$ along a diameter.

**Proof.** Straightforward computation.

**Proposition 14.5.6.** Formally, the isoptic $(\omega_1, \omega_2)$ pencil is also the isotomic $(\omega_3, \omega_4)$ pencil where $\omega_3 + \omega_4 = \omega_1 + \omega_2$ and $\omega_3 - \omega_4 = i(\omega_1 - \omega_2)$ (the $i$ are supposed to be at finite distance and normalized, while orthopoint is obtained using $\underline{OrtO}$).

**Proof.** Denote the common middle by $\omega_0$ and use Pythagoras theorem to compute $|\omega_1\omega_2|^2$. One has $|\omega_1\omega_2|^2 = |\omega_0\omega_1|^2 + |\omega_0\omega_2|^2 = 0$ and thus both $(\omega_3,\omega_4)$ are orthogonal to $(\omega_1,\omega_2)$. Using $\underline{OrtO}$ i.e. a rotation acting over $\mathcal{V}$.

### 14.6 Apexes

**Definition 14.6.1.** We define the (barycentric) apex of a point, or of a cycle, as the columns:

\[
\begin{align*}
\mathcal{A}(M) & = \begin{bmatrix} 1 & \text{Ver}(M) \end{bmatrix} \\
\mathcal{A}(C) & = \begin{bmatrix} Q \cdot \text{Ver}(C) \end{bmatrix}
\end{align*}
\]

With this definition, the apex of a point is the same as the apex of the null-radius circle centered at this point.

**Maple 14.6.2.** One can check these assertions by:

- mQQ, bar2colu(vp,0), (Tr@Ver)(vp): subs(sapbpc, zipd( 

**Proposition 14.6.3.** The apex of a line lies on the "South plane" $[1 : 1 : 1 : 0]$.

**Proof.** Obvious from $\mathcal{V}_b([p,q,r]) = p : q : r : 0$ and $[1,1,1,0][\underline{Q}^\top] = [0,0,0,1]$.

**Proposition 14.6.4.** Orthogonality formula (Spoiler). Let $\mathcal{V}_b, j = 1..4$ be the representatives of four cycles $C_j$. Describe the pencil generated by $C_1, C_2$ using the matrix $\varDelta_{12} = \begin{bmatrix} V_1 \land V_2 \end{bmatrix}$, and the pencil generated by $C_3, C_4$ using the matrix $\varDelta_{34} = \begin{bmatrix} V_3 \land V_4 \end{bmatrix}$. When each pencil is orthogonal to the other, then

\[
\varDelta_{12} \cdot \underline{Q}^\top \cdot \varDelta_{34} = \begin{bmatrix} Q \cdot \underline{Q}^\top \cdot \varDelta_{12} \cdot \underline{Q}^\top \cdot \varDelta_{34} \end{bmatrix}
\]

**Proof.** Cut $\Delta_{12}$ by the four base hyperplanes. Among the four expressions obtained, at most two are $0 : 0 : 0 : 0$. Do the same with $\Delta_{34}$ and assert that all the 16 orthogonality relations are fulfilled. Use eliminate to solve in $C_1, C_2$. Use eliminate onto the three shortest remaining equations and solve in $F_3, F_5, F_7$ (up to a common multiplier). Use the Klein’s relation $\Delta_{12} \cdot \underline{Q}^\top \cdot \Delta_{34} = E_3B_z + E_5B_y + E_7B_z = 0$ to simplify the results, obtained first degree formulas and check the compliance with our claim. This is for the necessity. Sufficiency is easily checked by direct examination (if you don’t trust the Maple’s eliminate ). One can also check that the process is involutive, as it should be.

**Exercise 14.6.5.** Check this formula using the following circles

$$C_1 \triangleq \mathcal{C}(A ; 0 : b : +c ; 0 : b : -c) ; C_2 \triangleq \mathcal{C}(B ; a : 0 : +c ; a : 0 : -c)$$

The second pencil is supposed to contain both the circumcircle and the 3-6 Brocard circle. See Section 14.9 for more details.
14.7 Inversion

14.7.1 One cycle

Definition 14.7.1. Two points $X_1$, $X_2$ are inverses in a given circle with center $P$ and radius $\omega$ when $P$, $X_1$, $X_2$ are in straight line and $(PU_1 \mid PU_2) = \omega^2$.

Proposition 14.7.2. The inverse of a point $X = x : y : z$ in the circle $\Omega$ having center $P = p : q : r$ and radius $\omega$ is given by:

$$\text{nor} (\text{inv} (X)) = \text{nor} (P) + (\text{nor} (X) - \text{nor} (P)) \frac{\omega^2}{\text{pytha} (X, P)}$$

(14.20)

This can be rewritten as

$$\text{inv} (X) = \left( \frac{\Gamma_{\text{std}} (X)}{x + y + z} + \frac{\Gamma_{\text{std}} (P)}{p + q + r} - \omega^2 \right) \text{nor} (P) + \left( \omega^2 - \frac{2}{(p + q + r)^2} \right) P \cdot \text{pytha} \cdot \text{nor} (X)$$

This formula is to be compared with formula (14.21) given in Theorem 14.7.4.

Proof. The first formula is nothing but the definition. Multiplying by $\text{pytha} (X, P)$, we obtain $\text{inv} (X) \simeq (\text{pytha} (X, P) - \omega^2) \text{nor} (P) + \omega^2 \text{nor} (X)$. Then $\text{pytha}$ is expanded using its definition, and we conclude using $(P \cdot X) = (P \cdot tP) \cdot X$.

Remark 14.7.3. When circle $\Omega$ is given by its equation (13.1) and $(P, \omega)$ are obtained from (13.6) and (13.8), the following identity can be useful:

$$\frac{\Gamma_{\text{std}} (P)}{(p + q + r)^2} - \omega^2 = \frac{U \cdot (vX \cdot (3) / 2) - a^2 b^2 c^2}{8 S^2}$$

(remember: $U$ and $vX$ (3) are "as is" and not "up to a proportionality factor", while $P$ is projective and $\omega$ is a number).

Theorem 14.7.4. Inversion of cycles in a cycle. Let $\Omega_0$ be a fixed cycle with representative $\mathcal{V}_0$ and $\Omega_1$ another cycle with representative $\mathcal{V}_1$. Assume that $\Omega_0$ is not a point-circle and call $\mathcal{V}_1$ the intersection of line $\mathcal{V}_0 \mathcal{V}_1$ with the polar plane of $\mathcal{V}_0$. Define $\sigma$ as the transform $\Omega_1 \mapsto \Omega_3$ where $\mathcal{V}_3$, the representative of $\Omega_3$, is such that division $\mathcal{V}_0 \mathcal{V}_1 \mathcal{V}_1 \mathcal{V}_3$ is harmonic. Then the cycle $\Omega_3$ is the inverse of $\Omega_1$ in cycle $\Omega_0$ (inversion in a straight line is the ordinary reflection in this line) while the matrix of the transform $\mathcal{V}_1 \mapsto \mathcal{V}_3$ is given by:

$$\sigma = \text{Id} - 2 \mathcal{V}_1 \mathcal{V}_0 \cdot \mathcal{Q} \mathcal{V}_0$$

(14.21)

Moreover, when $\mathcal{V}_2$ is yet another cycle representative, we have the conservation law:

$$\sigma \left( \mathcal{V}_1 \right) \cdot \mathcal{Q} \cdot \mathcal{V}_2 = \sigma \left( \mathcal{V}_1 \right) \cdot \mathcal{Q} \cdot \mathcal{V}_2$$

(14.22)

Proof. Write $\mathcal{V}_1$ as $\alpha_1 \mathcal{V}_0 + \mathcal{V}_1$ in $\mathcal{V}_0 \mathcal{Q} \mathcal{V}_1 = 0$ and then obtain $\mathcal{V}_3$ as $2 \alpha_1 \mathcal{V}_0 + \mathcal{V}_1$ since division $(\infty, 1, 0, 2)$ is harmonic. Equation (14.22) is obvious from (14.20), and shows that $\sigma$ preserves orthogonality. Moreover, (14.20) shows that cycles orthogonal to $\Omega_0$ are invariant while cycles concentric with $\Omega_0$ are transformed into cycles concentric with $\Omega_0$: all together, this proves that $\sigma$ is the inversion in cycle $\Omega_0$.
14.7.2 Two cycles

Here, all barycentrics are supposed to be in their normalized form.

**Proposition 14.7.5. Centers of homothety.** Let \( C_j (O_j, r_j) \), etc be two circles. Points \( U, V \) defined by:

\[
U = \frac{r_1 O_2 + r_2 O_1}{r_1 + r_2}, \quad V = \frac{r_1 O_2 - r_2 O_1}{r_1 - r_2}
\]

are respectively the internal and the external centers of homothety of these two circles. At Kimberling, ETC, they are called insimilicenter and exsimilicenter. When \( X_1 \in C_1 \) then:

\[
(r_1 + r_2) U - r_2 X_1 \in C_2 \quad \text{and} \quad (r_1 - r_2) V + r_2 X_1 \in C_2
\]

**Proof.** When \( r_1 = r_2 \), point \( V \) defines a translation, not an homothety. Otherwise, all steps are obvious.

**Proposition 14.7.6.** Let \( C_j (O_j, r_j) \), etc be two circles, \( X_1 \) the generic point of \( C_1 \) and \( U = (r_1 O_2 + r_2 O_1) / (r_1 + r_2) \) as above. Then the line \( UX_1 \) cuts \( C_2 \) in two points. The first one is \( X_2 = ((r_1 + r_2) U + r_2 X_1) / r_1 \), obtained by homothety. The second one is \( Y_2 \), obtained by inversion into the circle centered at \( U \) with power:

\[
\rho^2 = \left( 1 - \frac{|O_1 O_2|^2}{(r_1 + r_2)^2} \right) r_1 r_2
\]

Changing one of the radiuses into its opposite give the results relative to \( V \) (assuming \( r_1 \neq r_2 \))

14.7.3 Three circles

**Notation 14.7.7.** We start with three generic circles \( C_j (z_j, r_j) \), i.e. the centers are not aligned and all the radiuses are different. And we note \( \gamma_j (U_j, \rho_j) \) and \( \gamma'_j (V_j, \rho'_j) \) the six circles

\[
\gamma_1 \pm \frac{r_3 C_2 + r_2 C_3}{r_3 + r_2}, \quad \text{etc} ; \quad \gamma'_1 \pm \frac{r_3 C_2 - r_2 C_3}{r_3 - r_2}, \quad \text{etc}
\]

\[\begin{align*}
C_1 (z_1), C_2 (z_2), C_3 (z_3): & \quad \text{black} ; \\
C_4 (cenrad): & \quad \text{blue} ; \\
\gamma'_j (v_j): & \quad \text{green} ; \\
\gamma_j (u_j): & \quad \text{dot-red}
\end{align*}\]

**Figure 14.2:** Three circles, six inversions
so that \( \gamma_i \) is the internal circle of homothety of \( \mathcal{C}_i, \mathcal{C}_k \), and \( \gamma_j' \) is the external one. As stated in the previous subsection, we have:

\[
U_1, V_1 = r_3 O_2 \pm r_2 O_3 \quad \Rightarrow \quad (\rho_1)^2 = r_2 r_3 \left(1 - \frac{|z_3 - z_2|^2}{(r_3 + r_2)^2}\right) ; \quad (\rho_2^j)^2 = r_2 r_3 \left(1 - \frac{|z_3 - z_2|^2}{(r_3 - r_2)^2}\right)
\]

Finally, \( \mathcal{C}_4 \) is the common orthogonal cycle to all these circles.

**Lemma 14.7.8.** The product of three inversions \( \alpha, \beta, \gamma \) wrt circles of a same pencil is another inversion wrt a circle of the pencil. Thus the chain \( M_0 \to \alpha \to M_1 \to \beta \to M_2 \to \gamma \to M_3 \to \alpha \to \beta \to M_4 \to \gamma \to M_5 \to M_6 \) closes with \( M_6 = M_0 \). Moreover, these 3 points are on a same circle, which belongs to the orthogonal pencil.

**Proof.** Use Morley affixes and consider the circles \(-p : 0 : -p : 1\) (where \( p, q, r \in \mathbb{R} \)). Then

\[
\alpha \left( \begin{array}{c} Z \\ T \\ \overline{Z} \end{array} \right) \simeq \frac{Z + T}{|\overline{Z} - p T|} \quad \Rightarrow \quad \gamma_{\alpha} \left( \begin{array}{c} Z \\ T \\ \overline{Z} \end{array} \right) \simeq \frac{Z + T}{|\overline{Z} - p T|}
\]

Moreover the six points are on the circle \([-2(T^2 + Z \overline{Z}), 2T(Z - \overline{Z}), T^2 + Z \overline{Z}, T(Z - \overline{Z})]\). □

**Proposition 14.7.9 (Monge).** Centers \( V_1 V_2 V_3 \) are aligned, and also centers \( V_j U_j U_k \) (even number of internal centers).

**Proof.** Alignment of the \( V_j \) comes from:

\[
r_1 \left( r_2 - r_3 \right) V_1 + r_2 \left( r_3 - r_1 \right) V_2 + r_3 \left( r_1 - r_2 \right) V_3 = \overrightarrow{0}
\]

When changing \( r_1 \) into \(-r_1\), two circles are impacted, inducing the parity requirement. □

**Proposition 14.7.10.** Starting by \( M_0 \in \mathcal{C}_2 \), the inversions \( \gamma_1', \gamma_2', \gamma_3', \gamma_1, \gamma_2, \gamma_3 \) are leading to a set of six concyclic points where \( M_j \) belongs to \( \mathcal{C}_{j+2} \) (indexes taken modulo 3) and \( M_6 = M_0 \). All circles \( (M_k) \) are centered on the perpendicular to line \( V_1 V_2 V_3 \) issued from \( z_4 \) (radical axis of circles \( \gamma_1', \gamma_2', \gamma_3' \)).

**Proof.** Circles \( \mathcal{C}_j \) are orthogonal to \( \mathcal{C}_4 \) and circles \( \gamma_j \) inherit of this property. By Monge proposition, they are orthogonal to line \( V_1 V_2 V_3 \). Therefore circles \( \gamma_j \) belong to a same pencil, the lemma applies and the conclusion follows. When \( M_0 \in \mathcal{C}_2 \cap \mathcal{C}_4 \), all the \( M_j \) are concyclic on \( \mathcal{C}_4 \). See Figure 14.2 where the \( M_j \) are obtained using \( V_1 V_2 V_3 \). The inverse of this circle in \( \mathcal{C}_4 \) is also given (both in magenta). □

**Fact 14.7.11.** \( N_0 \) can be chosen on \( \mathcal{C}_2 \) so that \( N_3 = N_0 \). And then circle \( N_0 N_1 N_2 \) is tangent to the \( \mathcal{C}_3 \) circles. See Figure 14.2 where the \( N_j \) are obtained using \( V_1 V_2 V_3 \). The inverse of this circle in \( \mathcal{C}_4 \) is also given (both in violet). And then line \( V_1 V_2 V_3 \) is the radical axis of these two circles. See Section 14.10

### 14.8 Euler pencil and incircle

Consider \( \mathcal{C}_1 = (X_1, r), \mathcal{C}_2 = (X_3, R), \mathcal{C}_3 = (X_5, R/2) \) and \( \mathcal{C}_z = (X_z = X_{381}, |GH|/2 \) i.e., respectively, the in-, circum- nine points and orthocentroidal circles (Figure 14.3a). Let \( U_j, V_j, x_j, c_j \) be the respective representatives of centers and circles, together with their respective shadows (Figure 14.3b). Then:

1. Circles \( (X_j), \mathcal{C}_1, \mathcal{C}_b \) are concentric so that \( U_1, V_1, Sirius \) are aligned and therefore \( x_1, c_1, G \) are aligned too. The same happens for \( j = 5 \) and \( j = z \).

2. Cycles \( \mathcal{C}_4, \mathcal{C}_5, \mathcal{C}_2 \) belong to the same (Euler) pencil, together with their radical axis \( AR_{3,5} \), so that representatives \( V_{b_1}, V_{b_2}, V_{b_3}, V_{b_4}, V_{b_5} \) are aligned and therefore \( c_3, c_5, c_z, ar_{3,5} \) are aligned too. Since \( c_4 \) is "far below the paper sheet", we have \( c_5 = c_z = ar_{3,5} \). For the same reason, \( c_1 = ar_{3,1} \).
Figure 14.3: Euler pencil and incircle
3. Circles $C_1$ and $C_5$ are tangent at $F = X_{11}$, the Feuerbach point. Thus cycles $(F), C_1, C_5$ belong to the same pencil, together with their common tangent $AR_{1,5}$. Representatives $U_f, V_1, V_5, V_{15}$ are aligned and so are $x_f, c_1, c_5, ar_{1,5}$.

4. Cycles $AR_{1,3}, AR_{1,5}, AR_{3,5}$ are on the same pencil (they concur in the radical center $X_k$) and their shadows $ar_{1,3}, ar_{1,5}, ar_{3,5}$ are aligned.

5. In fact line $c_1c_5$ is not representative of a specific pencil, but rather of the bundle generated by $C_1, C_3, C_5$. We have:

$$V_b \simeq \begin{pmatrix} (b + c - a)^2 \\ (c + a - b)^2 \\ (a + b - c)^2 \\ 4 \end{pmatrix}, V_5 \simeq \begin{pmatrix} b^2 + c^2 - a^2 \\ c^2 + a^2 - b^2 \\ a^2 + b^2 - c^2 \\ 4 \end{pmatrix}, V_3 \simeq \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix}$$

and therefore:

$$V_b \simeq \begin{pmatrix} (c - b) (b + c - a) (b^2 + c^2 - a^2) (b^2 + c^2 - ab - ac) \\ (a - c) (c + a - b) (c^2 + a^2 - b^2) (c^2 + a^2 - bc - ba) \\ (b - a) (a + b - c) (a^2 + b^2 - c^2) (a^2 + b^2 - ca - cb) \\ 4 (a - b) (b - c) (c - a) (a + b + c) \end{pmatrix}$$

From $V_b$, the well-known result $X_k = X_{676}$ and the obvious $r_k = |X_kX_f|$ can be re-obtained.

6. As it should be, $x_k, c_k, G$ are aligned (small insert, at the bottom of Figure 14.3b).

7. Consider $W_k$ at intersection between line ($V_k Siris$) and plane ($V_1, V_3, V_5$). This points represents a circle that is both concentric and orthogonal to $c_k$. This circle is therefore $(X_k, r_k)$ and its shadow $co$ belongs to both $Gx_k$ and $c_kc_1$. Moreover, division $G, x_k, co, c_k$ is harmonic.

### 14.9 The Brocard-Lemoine pencils

**Notation** 14.9.1. The cevians of a point $M$ are usually noted $M_a, etc$. When dealing with the incenter $I_0$, this would collide with $I_a, etc$, the usual notation of the excenters. Thus we will use $J_a$, etc to note the cevians of $I_0$ (i.e. the feet of the internal bisectors) and use $P_a$, etc to note the cocevians of $I_0$ (i.e. the feet of the external bisectors).

**Proposition 14.9.2.** Let $J_a$ and $P_a$ be the points on sideline $BC$ met by the interior and exterior bisectors of angle $A$. The circle $(E_a)$ having diameter $[J_a, P_a]$ goes through $A$ and is called the A-Apollonian circle. The B- and C- Apollonian circles are similarly constructed. Points $P_j$ are the cocevians of $I_0 = X(1)$, while points $E_a$ are the cocevians of $K = X(6)$.

The Apollonian circles belong to a same pencil whose base points are the isodynamic points $X(15), X(16)$, while the radical axis is the Brocard line. The orthogonal pencil contains the circumcircle and the Brocard 3-6 circle, while the radical axis is the Lemoine line.

**Proof.** These assertions depend of the following Lemmas.

**Lemma 14.9.3.** Lemoine Pencil. Point $J_a$ is the A cevian of $X(1)$. Thus $J_a \simeq 0 : b : c$ and $P_a \simeq 0 : b : -c$. Now, we take the wedge of the Veronese of $A, J_a, P_a$. We obtain:

$$V_b \simeq \left\{ \begin{pmatrix} 1, 0, 0, 0 \end{pmatrix}, \begin{pmatrix} 0, b(b + c), (b + c)c, -a^2bc \end{pmatrix}, \begin{pmatrix} 0, b(c - b), c(b - c), -a^2bc \end{pmatrix} \right\} \simeq \begin{pmatrix} 0 \\ -a^2c^2 \\ a^2b^2 \\ b^2 - c^2 \end{pmatrix}$$

Let us remember that $V_b$ is a column that describe a point, while each Veronese is a plane incident to this point. Here, it is obvious that the three $V_b$ are not independent from each other. Using the
The Brocard-Lemoine pencils

each circle cuts the others at 60° (see (14.17))

Figure 14.4: Lemoine and Brocard pencils

usual formalism to describe the corresponding pencil, one obtains:

\[
\text{Lemoine point} \equiv \left( V_a \wedge V_b \right) \simeq \begin{pmatrix}
0 & a^2 b^2 (b^2 - a^2) & a^2 c^2 (c^2 - a^2) & a^4 b c^2 \\
0 & a^2 b^2 (a^2 - b^2) & b^2 c^2 (c^2 - b^2) & a^2 b^2 c^4 \\
a^2 c^2 (a^2 - c^2) & b^2 c^2 (b^2 - c^2) & 0 & -a^4 b c^2 \\
-a^2 b c^2 & -a^2 c^2 & -a^4 b^2 c & 0
\end{pmatrix}
\]

where the index 'point' is used to remember that this object has to be multiplied by a point (i.e. a column) to determine if the point belongs to the pencil.

Lemma 14.9.4. In order to determine the point-circles that belong to the pencil, we solve

\[
\text{Ver}(x : y : z) \cdot Q_b^{-1} \cdot \text{Lemoine point} = 0.
\]

We obtain both umbilics, together with

\[
E_{\pm} \simeq \begin{pmatrix}
a^2 \pm ia^2 \sqrt{3} \\
-2b^2 \\
c^2 \pm ic^2 \sqrt{3}
\end{pmatrix}
\]

\[
\begin{pmatrix}
a^2 (2a^2 - b^2 - c^2) \pm i \sqrt{3}a^2 (b^2 - c^2) \\
b^2 (2b^2 - a^2 - c^2) \pm i \sqrt{3}b^2 (c^2 - a^2) \\
c^2 (2c^2 - b^2 - a^2) \pm i \sqrt{3}c^2 (a^2 - b^2)
\end{pmatrix}
\]

The fact that \( E_{\pm} \simeq X (187) \pm i\sqrt{3} X (512) \) is not real indicates that the pencil is an isoptic one. From these values, one sees that the line of centers is the Lemoine axis (eponymous property).

Lemma 14.9.5. The polar planes \( \Pi_j \) of the \( V_j \) are obtained by \( \Pi_j \simeq \left( V_j \wedge Q_b \right) \). Their pencil is described by

\[
\text{Brocard plane} \equiv \left( \Pi_a \wedge \Pi_b \right) \simeq \begin{pmatrix}
0 & 0 & 0 & b^2 c^2 \\
0 & 0 & 0 & a^2 c^2 \\
0 & 0 & 0 & a^2 b^2 \\
-b^2 c^2 & -a^2 c^2 & -a^2 b^2 & 0
\end{pmatrix}
\]

where the index "plane" is to remember that this object has to be multiplied by a plane (i.e. a row) to determine if the plane belongs to the pencil of planes. To obtain the matrix acting over points (here: the representatives of cycles), one has to use the dual. This leads to:

\[
\text{Brocard point} \simeq \begin{pmatrix}
0 & a^2 b^2 & -a^2 c^2 & 0 \\
a^2 b^2 & 0 & b^2 c^2 & 0 \\
a^2 c^2 & -b^2 c^2 & 0 & 0 \\
0 & 0 & 0 & 0
\end{pmatrix}
\]
Lemma 14.9.6. Using the same algorithm as above, we obtain the point-circles that belong to the pencil. We obtain both umbilics, together with points $X(15), X(16)$ the isodynamic points.

Proof. The expression obtained isn’t symmetric... and rather complicated. The best thing to do is using the Kimberling’s search keys. Thereafter, one can check that:

$$ F_{\pm} = \begin{pmatrix} a^2 (2a^2 - b^2 - c^2) + \frac{1}{16} a^2 \left( (b^2 + c^2) a^2 - b^4 - c^4 \right) \\ b^2 (2b^2 - c^2 - a^2) + \frac{1}{16} b^2 \left( (c^2 + a^2) b^2 - c^4 - a^4 \right) \\ c^2 (2c^2 - a^2 - b^2) + \frac{1}{16} c^2 \left( (a^2 + b^2) c^2 - a^4 - b^4 \right) \end{pmatrix} $$

i.e. $F_{\pm} \simeq X(187) \pm \sqrt{3} \text{OrtO} \cdot X(512)$ (see Proposition 14.5.6). From these values, one sees that the line of centers is the Brocard axis (eponymous property).

Lemma 14.9.7. One sees that the circumcircle $\Gamma$ and the 3-6 Brocard circle belong to the second pencil.

Remark 14.9.8. This pencil is reexamined at Section 18.3 (using Morley coordinates).

14.10 The Apollonius configuration

In the general case, it exists eight cycles $\Omega$ tangent to three given cycles $\Omega_1$, $\Omega_2$, $\Omega_3$ (not from the same pencil). A survey of this question is Gisch and Ribando (2004), while the usual decomposition into ten cases is Wikipedia: WillowW et al. (2006); Bogomolny (2009). The best space where this Apollonius problem can be discussed is $\mathbb{P}_2(\mathbb{R}^5)$ (cf Section 14.11). Nevertheless, most of the results can be formulated in $\mathbb{P}_2(\mathbb{R}^4)$... and it will appear that only one situation is really special (cycles through the same point), all the other belonging to the same general case.

14.10.1 Tangent cycles in the representative space

Proposition 14.10.1. Two cycles are tangent when their pencil line is tangent to the fundamental quadric. Therefore, the locus of the representatives of all cycles tangent to a given (real) cycle $\Omega$ represented by $V$ (not inside $Q$) is the cone whose vertex is $V$ and that goes through $Q \cap \text{polar}(V)$.

Proof. Two tangent cycles are defining a tangent pencil! 

Remark 14.10.2. When $\Omega$ is a point circle, its representative $U$ belongs to the fundamental quadric, and the cone of the tangent cycles degenerates into a doubly coated plane.

Definition 14.10.3. The Gram matrix $G_{p,q,...,r}$ of $X_p, X_q, \cdots, X_r \in \mathbb{R}^4$ is the matrix of all the products $\left<h,X_p\right>\left<X_q,X_r\right>$. In this context, notation $W_{pq} = \left<h,X_p\right>\left<X_q\right>$ and $w_p^2 = \left<h,X_p\right>\left<X_p\right>$ will be used, leading to

$$ G_{pq} = \begin{pmatrix} w_p^2 & W_{pq} \\ W_{pq} & w_q^2 \end{pmatrix} \quad (14.23) $$

Proposition 14.10.4. Two cycles $\Omega_1, \Omega_2$ are secant, tangent or external when signum $\det G_{12}$ is (respectively) $+1$, $0$ or $-1$.

Theorem 14.10.5. Special cases of the Apollonius problem are (1) cycles from the same pencil and (2) cycles through the same point (tangent bundle). Otherwise, representatives $V_j$ of the three given cycles and their common orthogonal cycle $\Omega_4$ form a basis that splits the problem into four pairs of solutions. One of the solutions is given by $V_0 = \sum k_j V_j$ where :

\[
\begin{align*}
  k_1 &= (w_2 w_3 - W_{23}) (-w_1 w_2 w_3 - w_1 W_{23} + w_2 W_{13} + w_3 W_{12}) \\
  k_2 &= (w_1 w_3 - W_{13}) (-w_1 w_2 w_3 + w_1 W_{23} - w_2 W_{13} + w_3 W_{12}) \\
  k_3 &= (w_1 w_2 - W_{12}) (-w_1 w_2 w_3 + w_1 W_{23} + w_2 W_{13} - w_3 W_{12}) \\
  k_4 &= \sqrt{-2} (w_2 w_3 - W_{23}) (w_1 w_3 - W_{13}) (w_1 w_2 - W_{12}) G_{123}/w_4
\end{align*}
\]
and the others are obtained by changing $k_4$ into $-k_4$ (inversion through $\Omega_4$) or changing the signs of $w_1, w_2, w_3$. A solution is real/imaginary or "unimaginable" (object that would have a non real center) according to the sign of $k_4^2$. Globally, the number of "imaginable" solutions changes when the tangency condition $\prod G_{jk}$ vanishes.

Proof. When $\Omega_j, j = 1, 2, 3$ is a basis of a non tangent bundle, then $\Omega_j, j = 1, 2, 3, 4$ is a basis of the whole representative space. The fundamental quadratic form is described, in this basis, by matrix $G_{1234}$ where $W_{1j} = 0$ for $j = 1, 2, 3$. Computing, in this basis, the tangency condition of $\Omega_0$ and any of the $\Omega_j$ leads to 0. Since the $w_j$ are defined as $\sqrt{W_{jj}}$ we have 4 choices of signs leading, due to the possibility of a global proportionality factor, to eight different values.

\subsection*{14.10.2 An example: the Soddy circles}

\textbf{Proposition 14.10.6. Soddy circles} are three mutually, externally, tangent circles. Let $A, B, C$ be their centers. Then the common orthogonal circle of the Soddy’s is the incircle of $ABC$ (see Oldknow, 1996).

Proof. Let $x$ be the radius of circle $(A)$, etc. We have $a = y + z, b = z + x, c = x + y$. Therefore $x = b + c - a$, etc. The contact point of the $(B)$, $(C)$ circles is $G_a \simeq 0 : y : z$, etc. As a result, the $G_j$ are the cevians of the Gergonne point $G_a = X(7)$, and the conclusion follows.

\textbf{Proposition 14.10.7.} The Apollonius configuration of the Soddy circles are twice each of them, a small circle (inside the intouch triangle) and an outer circle. The center of the smaller circle is called $X(176)$, the other is called $X(175)$. Let $H_a$ be the branch of hyperbola that goes through $A, G_a$ and has $B, C$ as focuses. Then the three branches through a vertex of the hyperbolas concur at $X(176)$, while the other three branches concur at $X(175)$.

Proof. This is clear from $a = y + z$, etc.

\textbf{Proposition 14.10.8.} Centers and radiuses of the Soddy circles are given by:

\[
\begin{align*}
nX(175) &= \frac{2s}{2s - (4R + r_0)} nX(1) - \frac{4R + r_0}{2s - (4R + r_0)} nX(7) \simeq \begin{bmatrix} a - \frac{2S}{b + c - a} \\ b \\ c \\ 0 \\ 1 \\ 1 \\ 1 \\ 2s - (4R + r_0) \end{bmatrix} \\
nX(176) &= \frac{2s}{2s + (4R + r_0)} nX(1) + \frac{4R + r_0}{2s + (4R + r_0)} nX(7) \simeq \begin{bmatrix} a + \frac{2S}{b + c - a} \\ b \\ c \\ 0 \\ 1 \\ 1 \\ 2s + (4R + r_0) \end{bmatrix}
\end{align*}
\]

(14.25)

Proof. Using (14.10), the representatives of circles $\Omega_1 \cdots \Omega_4$ are:

\[
\begin{pmatrix}
(b + c - a)^2 & (b + c - a)(b - 3c - a) & (b + c - a)(c - a - 3b) & (b + c - a) \\
(c + a - b)(a - b - 3c) & (c + a - b)^2 & (c + a - b)(c - 3a - b) & (c + a - b) \\
(a + b - c)(a - 3b - c) & (a + b - c)(b - 3a) & (a + b - c)^2 & (a + b - c) \\
-4 & -4 & -4 & 4
\end{pmatrix}
\]

Their Gram matrix is:

\[
\begin{pmatrix}
(b + c - a)^2 & - (b + c - a)(c + a - b) & -(a + b - c)(b + c - a) & 0 \\
-(b + c - a)(c + a - b) & (c + a - b)^2 & -(c + a - b)(a + b - c) & 0 \\
-(a + b - c)(b + c - a) & -(c + a - b)(a + b - c) & (a + b - c)^2 & 0 \\
0 & 0 & 0 & 16s^2
\end{pmatrix}
\]

Then (14.24) gives the decomposition of the Soddy’s circles on the $\Omega$ basis. We have:

\[
\begin{pmatrix}
1 \\
1 \\
1 \\
\frac{1}{2S}
\end{pmatrix}
\]

and the conclusion follows (since $a + b + c = 2s$).
Remark 14.10.9. We have cross ratio \((X_1, X_7, X_{175}, X_{176}) = -1\), while condition \(4R + r_0 = 2s\) is not excluded. In this case, the \(X(1)\) and \(X(7)\) are on \(\Omega_6\), the inner Soddy circle, while \(X(175)\) is at infinity and \(\Omega_5\) is a straight line. When \(ABC\) \((a = b = c = 1)\), then \(s = 3, R = \sqrt{1/3}, r_0 = \sqrt{1/12}\) and (14.25) provides a positive \(\rho_6\).

**Proposition 14.10.10.** The Soddy radiuses satisfy the following "curvature formula":

\[
\left( \frac{1}{\rho_1} + \frac{1}{\rho_2} + \frac{1}{\rho_3} + \frac{1}{\rho_6} \right)^2 = 2 \left( \frac{1}{\rho_1^2} + \frac{1}{\rho_2^2} + \frac{1}{\rho_3^2} + \frac{1}{\rho_6^2} \right)
\]

**Proof.** For an elementary proof: substitute and simplify. For a stratospheric (and interesting !) proof, see Pedoe (1967).

**Proposition 14.10.11.** Let \(\varepsilon_a, \varepsilon_b, \varepsilon_c\) etc be the cocevians and \(G_a, G_b, G_c\) etc be the cevians of \(G = X(7)\). Draw circles \(E_j\) centered at \(\varepsilon_j\) and going through \(G_j\). Then \(E_j\) cuts \(C_j\) at the contact points with the Soddy circles. Moreover the three \(E_j\) circles concur at \(X(3638)\) and \(X(3639)\).

**Proof.** It is easy to check that each of the \(E_j \cap C_j\) points belongs to one Soddy circle. Moreover, one has

\[
X(1323) \simeq \frac{2a^2 - a(b + c) - (b - c)^2}{b + c - a} ::
\]

\[
X(516) \simeq \frac{2a^2 - b^2 - c^2 + (bc - a^2)(b + c)}{2} ::
\]

\[
X(3638) = 3X(516) + 4\sqrt{3}X(1323)
\]

And it is easy to check that \(X(3638)\) belongs to the three \(E_j\) circles. Caveat: the Soddy points are not the common points of the \(E_j\) circles, but they are the Poncelets of the Soddy circles.

14.10.3 An other example: the not so Soddy circles

**Proposition 14.10.12.** The not so Soddy circles of a triangle are the circles centered at \(A\) with radius \(a = BC\), etc. Their common orthogonal circle is the Longchamps circle \(\lambda\) (i.e. the polar circle of the antimedial triangle).

**Proof.** The circle \(\gamma_a\), centered at \(A\) with radius \(a\) is described by:

\[
yza^2 + b^2zx + c^2yx + (x + y + z)(a^2x + y(a^2 - c^2) + z(a^2 - b^2)) = 0
\]

Its intersections with the circumcircle are:

\[
Q_b \simeq a^2 - c^2 : -b^2 : b^2 - a^2 \quad \text{and} \quad Q_c \simeq a^2 - b^2 : b^2 - a^2 : -c^2
\]

while its intersections with \(\gamma_b\) (resp. \(\gamma_c\)) are \(Q_c\) and \(U_c \simeq 1 : 1 : -1\) (resp. \(Q_b\) and \(U_b \simeq 1 : -1 : 1\)). Points \(U_a, U_b, U_c\) are on circle \((H, 2R)\) and form the antimedial triangle. Lines \(Q_aU_a\) are the altitudes of \(U_aU_bU_c\) and concur at \(L = H' = X(20)\). But they are also the radical axes of our circles. The radius \(r_L\) of the orthogonal circle is obtained by:

\[
r_L^2 = |AL|^2 - a^2, \quad \text{etc} = \frac{-S_aS_bS_c}{S^2} = 4(2R + \rho + s)(2R + \rho - s)
\]

**Proposition 14.10.13.** The Apollonius circles of the not so Soddy circles are obtained by extraversions (i.e. \(a \mapsto -a\)) from their central versions. These central circles are centered at the Soddy points \(X(175)\) and \(X(176)\).

**Proof.** The representatives and the Gram matrix of \(\gamma_a, \gamma_b, \gamma_c, \lambda\) are:

\[
\begin{pmatrix}
-a^2 & c^2 - b^2 & b^2 - c^2 & a^2 \\
 c^2 - a^2 & -b^2 & a^2 - c^2 & b^2 \\
b^2 - a^2 & a^2 - b^2 & -c^2 & c^2 \\
 1 & 1 & 1 & 1
\end{pmatrix}
\begin{pmatrix}
a^2 & S_c & S_b & 0 \\
 S_c & b^2 & S_a & 0 \\
 S_b & S_a & c^2 & 0 \\
 0 & 0 & 0 & -S_aS_bS_c \div S^2
\end{pmatrix}
\]

— pldx : Translation of the Kimberling’s Glossary into barycentrics —
Figure 14.5: Apollonius circles of the not so Soddy configuration.
we have:

\[
K \simeq \begin{pmatrix} 
(bc - S_a)(-bca - aS_a + bS_b + cS_c) \\
(ca - S_b)(-bca - bS_b + cS_c + aS_a) \\
(ab - S_c)(-bca - cS_c + aS_a + bS_b)
\end{pmatrix} \\
16S^3 \div (a + b + c)
\]

Organizing the obtained equations, we have:

\[
\gamma_6 = \lambda + 2(x + y + z)(ax + by + cz)\rho_6 \quad \text{where} \quad \rho_6 = \frac{+2s(2R + \rho + s)}{4R + \rho + 2s}
\]

\[
\gamma_5 = \lambda + 2(x + y + z)(ax + by + cz)\rho_5 \quad \text{where} \quad \rho_5 = \frac{-2s(2R + \rho - s)}{4R + \rho - 2s}
\]

This leads to \( \gamma_6 = (X_{176}, \rho_6) \), etc. Additionally, this proves that \( ax + by + cz = 0 \) is the radical axis of the three circles. Moreover, the Soddy conic (through \( A, B, C \), with focuses \( X(175), X(176) \) and perspector \( X(7) \)) is tangent to the Longchamps circle at the common points of \( \lambda, \gamma_6, \gamma_5 \) since we have:

\[
\text{conic} = \lambda + (ax + by + cz)^2
\]

---

### 14.10.4 The three excircles

Taking the three excircles as \( \Omega_1, \Omega_2, \Omega_3 \) leads to a well-known situation (Stevanovic, 2003).

1. The representative of the point-circle \( (X_4; 0) \) is:

\[
U_0 = [bc (b + c - a) ; ca (c + a - b) ; ab (a + b - c) ; a + b + c]
\]

while radius of the incircle is:

\[
r = \sqrt{\frac{(a + b - c)(c + a - b)(b + c - a)}{4(a + b + c)}}
\]

2. The representative of the incircle, given by (14.10), is:

\[
V_0 = [(a - b - c)^2 ; (a - b + c)^2 ; (a + b - c)^2 ; 4]
\]

3. Centers, radiuses and representatives \( V_a, V_b, V_c \) of the excircles are obtained by changing one of the sidelengths into its opposite in the respective formulas for the incircle.

4. The alt_Spieker circle is defined as the common orthogonal circle to the three excircles. From (14.18), this circle can be computed as:

\[
V_4 = [(c + a - b)(a + b - c) ; (a + b - c)(b + c - a) ; (b + c - a)(c + a - b) ; -4]
\]

5. The radius of this circle, as computed from (14.15), is:

\[
\omega_4 = \sqrt{\frac{b^2c + ab^2 + bc^2 + a^2b + ac^2 + a^2c + acb}{4(a + b + c)}}
\]

\[
= \frac{1}{2}\sqrt{r_0^2 + s^2}
\]

while the representative of the center is:

\[
U_4 = \begin{pmatrix} 
2a(b^2 + c^2) - abc + b^3 + c^3 - a^3 \\
2b(c^2 + a^2) - abc + c^3 + a^3 - b^3 \\
2c(a^2 + b^2) - abc + a^3 + b^3 - c^3
\end{pmatrix} \\
4 (a + b + c)
\]

and the center itself is:

\[
b + c : a + c : a + b = X_{10}
\]
6. The pairs of solutions of the Apollonius problem, as given by (14.24), are:

\[
\frac{S_1}{S_5} = \begin{pmatrix}
    b^2 + c^2 - a^2 & c^2 + a^2 - b^2 & a^2 + b^2 - c^2 & 4 \\
    a + b + c + \frac{2bc}{a} & a + b + c + \frac{2ca}{b} & a + b + c + \frac{2ab}{c} & -4
\end{pmatrix}
\]

\[
S_2 = \begin{pmatrix}
    1 \\
    0 \\
    0 \\
    0
\end{pmatrix}, \quad S_6 = \begin{pmatrix}
    (a + b + c)(b^2 + ab + ac + c^2) \\
    (b + c)(a - b - c)(a + b - c) \\
    (b + c)(a - b - c)(a - b + c) \\
    4(b + c)
\end{pmatrix}
\]

where point \( S_1 \) is the representative of the nine-points circle, centered at \( X_5 \) while \( S_5 \) is related to the Apollonius circle, centered at \( X_{50} \). Points \( S_2, S_3, S_4 \) are the representatives of \( \omega \), \( \omega_3 \), \( \omega_5 \) (obtained cyclically) are the representatives of the last three solutions.

### 14.10.5 The special case

**Proposition 14.10.14.** Let \( \Omega_1, \Omega_2, \Omega_3 \) be three cycles generating a bundle whose common orthogonal cycle is a point-cycle \( (\omega_5) \) and \( \omega_3 \) be any other point. The representative of one of the cycles tangent to \( \Omega_1, \Omega_2, \Omega_3 \) is given by \( V_0 = \sum_1^3 k_j V_j + 4U_4 \) where:

\[
k_1 = \left( \frac{w_2w_3 - W_{2,3}}{(w_1w_3 - W_{1,3})(w_1w_2 - W_{1,2})} \right) \frac{G_{1,2,3,4} - 2G_{2,3,4}}{\Delta_{2,3,4}^{1,2,3}}
\]

\( \Delta_{2,3,4}^{1,2,3} \) is the minor obtained by deleting row 1 and column 4 in \( G_{1,2,3,4} \) while \( k_2, k_3 \) are obtained cyclically. Three other cycles are obtained by changing one of the \( w_1, w_2, w_3 \) into its opposite. The other solutions are four times the point cycle \( \omega_5 \).

**Proof.** In this special case, \( G_{1,2,3} = 0 \) and \( \Omega_4 \) is chosen so that \( w_4 = 0 \). When assuming that \( \Omega_1, \Omega_2, \Omega_3 \) aren’t pairwise tangent, a direct substitution shows that \( \Omega_0 \) is tangent to any of the given cycles.

**Example 14.10.15.** Using \( \Omega_1 = 1 : 0 : 0 : 0 \) (representative of line \( BC \)) etc, leads to \( \omega_5 = \textit{Sirius} \). An efficient choice for \( \omega_4 \) is any vertex. Using, for example, \( U_4 = 0 : c^2 : b^2 : 1 \), one re-obtains easily the in/excircles.

**Example 14.10.16.** The Apollonius circles relative to the three circles \( (ABH), (BCH), (CAH) \) are \( (H, 0) \) four times, \( (H, 2R) \) once and three other circles, \( Ta, Tb, Tc \).

Circles \( Ta, Tb, Tc \) are ever external to each other, and their common orthogonal circle \( To \) is real. Condition of (external) tangency is:

\[
(a^2 - b^2)^2 - (a^2 + b^2)c^2 = 0
\]
(etc) or $ABC$ rectangular. The Apollonius circles of $Ta, Tb, Tc$ are $(ABH)$ etc, their inverses in $To$ and two others. Center $X = x : y : z$ and radius $\omega$ of the first one are:

$$x = (b^2 + c^2 - a^2) \times$$

$$\left( a^8 - 2 (b^2 + c^2) a^6 + 2 (b^4 - b^2 c^2 + c^4) a^4 - 2 (b^2 + c^2) (b^2 - c^2)^2 a^2 + (b^2 - c^2)^4 \right)$$

$$\omega = 2 b^6 + b^6 + c^6 - a^2 b^6 - a^2 c^6 - b^2 c^4 - c^4 a^2 + 4 a^2 b^2 c^2$$

while the second is less simple.

### 14.11 Elementary properties of the Triangle Lie Sphere

**Definition 14.11.1.** The Triangle Lie Sphere is the locus of points $X = x : y : z : t : \tau \in \mathbb{P}_2^2$ such that $\hat{t}X \cdot \mathcal{Q}_5 \cdot X = 0$ where fundamental matrix is defined by:

$$\mathcal{Q}_5 = \begin{bmatrix} \mathcal{Q} & 0 \\ 0 & -4S^2 \end{bmatrix} = \left[ \begin{array}{cccc} a^2 & -Sc & -Sb & -a^2Sa \\ -Sc & b^2 & -Sa & -b^2Sb \\ -Sb & -Sa & c^2 & -c^2Sc \\ -a^2Sa & -b^2Sb & -c^2Sc & a^2b^2c^2 \\ 0 & 0 & 0 & -4S^2 \end{array} \right]$$

**Proposition 14.11.2.** Using notations of Theorem 14.3.5, an element of the Lie sphere represents, when $t \neq 0$ and $\tau \neq 0$, an oriented circle:

$$Y_1 = \begin{bmatrix} \frac{b^2r_1^2 + c^2q_1^2 + 2 Sa q_1 r_1 - \omega_1^2 (p_1 + q_1 + r_1)^2}{c^2p_1^2 + a^2r_1^2 + 2 Sb r_1 p_1 - \omega_1^2 (p_1 + q_1 + r_1)^2} \\ \frac{a^2q_1^2 + b^2p_1^2 + 2 Sd q_1 - \omega_1^2 (p_1 + q_1 + r_1)^2}{(p_1 + q_1 + r_1)^2} \\ 2 \omega_1 (p_1 + q_1 + r_1) \end{bmatrix}$$

(where signum of $\omega_1$ defines the orientation) or, when $t = 0$ and $\tau \neq 0$, an oriented line:

$$Y_3 = \begin{bmatrix} u_3; v_3; w_3; 0; \pm \frac{1}{2S} \sqrt{\Delta \cdot \mathcal{M} \cdot \hat{t}\Delta} \end{bmatrix}$$

cf (10.3) for definition of $\mathcal{M}$, or, when $\tau = 0$, a non-oriented point.

**Proof.** All these results follow directly from Theorem 14.3.5.

**Proposition 14.11.3.** Two oriented cycles are tangent if and only if $\hat{t}Y_j \cdot \mathcal{Q}_5 \cdot Y_k$ vanishes.

**Proof.** This result is the rationale behind the former definitions. Using notations of Theorem 14.3.5, we obtain for two circles:

$$\hat{t}Y_1 \cdot \mathcal{Q}_5 \cdot Y_2 = -\left( |P_1 P_2|^2 - (\omega_1 - \omega_2)^2 \right) \times 8S^2 \left( p_1 + q_1 + r_1 \right) (p_2 + q_2 + r_2)^2 \quad (14.27)$$

By continuity, the Proposition holds also when lines are involved. For the sake of completeness, one can nevertheless compute:

$$\hat{t}Y_1 \cdot \mathcal{Q}_5 \cdot Y_3 = -\left( \frac{p_1 v_3 + q_1 v_3 + r_1 w_3}{p_1 + q_1 + r_1} + \omega_1 \frac{1}{2S} \sqrt{\Delta_3 \cdot \mathcal{M} \cdot \hat{t}\Delta_3} \right) \times 8S^2 \left( p_1 + q_1 + r_1 \right)^2 \quad (14.28)$$

for a circle and a line, and (after multiplication by conjugate quantity):

$$-\left( (v_4 - w_4) u_3 + (v_4 - u_4) v_3 + (-v_4 + u_4) w_3 \right)^2 \times 8S^2$$

for two lines $Y_3, Y_4$. In the three cases, this is the condition of tangency times a non vanishing factor.

— pldx : Translation of the Kimberling’s Glossary into barycentrics ——
Remark 14.11.4. When using the Lie representation, objects that don’t belong to $Q_5$ are meaningless, while $Q_5$ itself is obtained by double coating the outside of $Q$ in $\mathbb{P}_R(\mathbb{R}^4)$. Therefore, imaginary circles are lost: when the radius decreases to 0 in an isotomic pencil, the differentiable continuation is going back to positive radiuses (with the other orientation) and not escaping to imaginary values.

Proposition 14.11.5. Let $\Omega_0$ be a cycle, but not a point-circle, $\sigma$ the inversion in cycle $\Omega_0$ as described in Theorem 14.7.4, $V_1 \in \mathbb{P}_R(\mathbb{R}^4)$ a representative of cycle $\Omega_1$ and $Y_1 = (V_1, \tau_1) \in \mathbb{P}_R(\mathbb{R}^5)$ a representative of one of the corresponding oriented cycles. Then applications:

$$
\sigma^+ : \sigma^+ (V_1, \tau_1) = (\sigma (V_1), +\tau_1) \\
\sigma^- : \sigma^- (V_1, \tau_1) = (\sigma (V_1), -\tau_1)
$$

are describing inversions wrt each of the oriented cycles corresponding to $\Omega_0$. The conservation law (14.22) can be rewritten in order to describe a projective invariant by inversion, namely:

$$
\frac{t Y_1 \cdot Q_5 \cdot Y_2}{4S^2 \tau_1 \tau_2} = \frac{|P_1 P_2|^2 - (\omega_1 - \omega_2)^2}{2\omega_1 \omega_2}
$$

or, for a circle and a line,

$$
\frac{t Y_1 \cdot Q_5 \cdot Y_3}{4S^2 \tau_1 \tau_3} = 1 + \frac{p_1 u_3 + q_1 v_3 + r_1 w_3}{\omega_1 (p_1 + q_1 + r_1) \tau_3}
$$

and for two lines:

$$
\frac{t Y_3 \cdot Q_5 \cdot Y_4}{4S^2 \tau_3 \tau_4} = 1 \pm \cos (\Delta_3, \Delta_4)
$$

Remark 14.11.6. Searby (2009) illustrates how this invariant (named $\epsilon_{jk}$) can be used to summarize tangencies in various situations.
Chapter 15

Morley spaces, or how to use complex numbers

Our aim in this chapter is to describe how to translate into complex numbers all of the methods we have described in the previous chapters. This is equivalent to give the complex version of all the operators we have already described.

In fact, some of these operators have a very simple form when using complex numbers and can be introduced directly. Just after that, we will prove the equivalence between "the Morley version" and "the barycentric version" of these operators.

15.1 How to deal with complex conjugacy?

When writing points $M$ as $(\xi, \eta)$ and describing curves $C$ by polynomials $P$ so that $M \in C$ when $P(\xi, \eta) = 0$, it is efficient to use homogeneous coordinates $(X, Y, T)$ and describe curves by homogeneous polynomials

$$P_n(X, Y, T) = T^n P\left(\frac{X}{T}, \frac{Y}{T}\right)$$

where $n = dg(P)$ is the degree of the curve.

**Theorem 15.1.1** (Bezout). Two algebraic curves $C(P_n)$ and $C(P_m)$ have exactly $m \times n$ common points when polynomials $P_n$ and $P_m$ have no non-constant common factor. To obtain this result, all the points have to be taken into account, including points with non real coordinates as well as points at infinity, and also considering the multiplicities of the solutions (Bezout, 1764).

In order to reformulate this result using complex affixes, the following proposition is needed.

**Proposition 15.1.2.** Define the complex polynomial $Q_n$ of an algebraic curve by

$$Q_n(Z, T, \overline{Z}) = P_n\left(\frac{Z + i \overline{Z}}{2}, \frac{Z - i \overline{Z}}{2i}, T\right)$$

where $Z$ (read it as "big $z$"), $\overline{Z}$ (read it as "big $\zeta$") and $T$ are algebraic variables. Define $\text{conj}(Q_n)$ as the complex polynomial associated to polynomial $\overline{P_n}$ (obtained by complex conjugation of the coefficients). Then

$$\text{conj}\left(\sum_{p+q+r} c_{p,q,r} Z^p \overline{Z}^q T^r\right) = \sum_{p+q+r} c_{p,q,r} Z^q \overline{Z}^p T^r$$

in other words, conjugate the coefficients and exchange $Z$ with $\overline{Z}$.

**Remark 15.1.3.** An algebraic variable is a placeholder used to write polynomials. What could be the $\mathbb{C}$-conjugate of a placeholder? Therefore, a notation was to be found in order to satisfy the following constraints:

1. Have capital letters, since polynomials are usually written $P(X_1, X_2, \cdots, X_k)$
2. Avoid indices, and deal with the fact that letter "big $\zeta$" has the same shape as letter "big $z$" (nevertheless, read $\overline{Z}$ as "big zeta").

3. Have a robust cursive version, to facilitate hand computations: a $Z$ is clearly a $Z$, while a $\overline{Z}$ is clearly a $\overline{Z}$!

4. Enforce nevertheless the fact that $Z$ is not the $C$-conjugate of $Z$.

This leads to the following definition.

**Definition 15.1.4.** The Morley space is $\mathbb{P}_C(C^3)$: a point is a complex triple defined up to a non-zero complex factor. A point of the Morley space that can be written as:

$$
\zeta_P \simeq \begin{pmatrix} z_P \\ 1 \\ \overline{z_P} \end{pmatrix}
$$

for some $z_P \in \mathbb{R}^3$ is referred as the Morley-affix of the ordinary point $P$ (shortened as : $\zeta_P$ is a finite point). A point of the Morley space that can be written as

$$
\zeta_\vartheta \simeq \begin{pmatrix} \exp (+i \vartheta) \\ 0 \\ \exp (-i \vartheta) \end{pmatrix}
$$

for some $\vartheta \in \mathbb{R}$ is referred as the Morley-affix of the angle whose measure is $\vartheta + k \pi$ (shortened as : $\zeta_\vartheta$ is a direction). Taken together, finite points and directions are referred as the visible points. All other points of the Morley space are described as being invisible.

**Remark 15.1.5.** The invisible points of the Morley space correspond to the points that cannot be written as $(x : y : t)$ with $x, y, t \in \mathbb{R}$ in the $\xi, \eta, t$ representation. In this context they are referred as "non real". In the context of the Morley-space, this designation is no more suitable, and another word must be used.

**Remark 15.1.6.** From an abstract point of view, it could be tempting to define the Morley space as $\mathbb{P}_R(C \times C \times \mathbb{R})$, i.e. to describe points by triples $(z, \zeta, t)$ such that $z, \zeta \in \mathbb{C}$ and $t \in \mathbb{R}$, each triple being defined up to a real proportionality factor. But in real life, enforcing $t \in \mathbb{R}$ can only be done by carrying annoying factors. We better have the following definition.

**Proposition 15.1.7.** An algebraic curve is said to be "reduced to a point" when it contains only one visible point, and "visible" when it contains an infinite number of visible points. The polynomial of an irreducible visible curve must be proportional to its conjugate.

**Remark 15.1.8.** From an abstract point of view, using a constant factor to enforce $\text{conj}(P) = P$ is always possible. But in real life, this can only be done by carrying annoying factors and has to be avoided. Before any simplification, a polynomial obtained from a determinant, as the equation of a line or a circle, verifies $\text{conj}(P) = -P$.

**Proposition 15.1.9.** In the Morley space, the equation of a visible line is $\pi Z + a \overline{Z} + b T = 0$ (with $b \in \mathbb{R}$). Its point at infinity is:

$$
a : 0 : -\pi = \cos \vartheta + i \sin \vartheta : 0 : \cos \vartheta - i \sin \vartheta = \omega^2 : 0 : 1
$$

where $\vartheta$ is the oriented angle from the real axis to the line. This is an angle between straight lines and not an angle between vectors. We have the formula:

$$
\omega^2 = -\text{coeff} (\Delta, \overline{Z}) \div \text{coeff} (\Delta, Z)
$$

**Proof.** Straight line $y = px + q$ can be rewritten as $(p + i)Z + (p - i)\overline{Z} + 2qT = 0$. Point at infinity is $[p + i, 2q, p - i] \wedge [0, 1, 0]$. \qed

**Proposition 15.1.10.** As a consequence, collinearity of three points is (collin-cocycl)

$$
\sum_{j=1}^{3} \det ([z_j, \overline{z_j}, 1]) = 0
$$
15.2 The usual operators of the Morley space

Definition 15.2.1. Complex affix of a point. Let $\xi_P, \eta_P$ be the Cartesian coordinates of a point $P$ in the euclidian plane. The $\mathbb{C}$-affix of point $P$ is defined as

$$z_P = \xi_P + i\eta_P$$

Remark 15.2.2. In this definition, quantity $i$ is a quarter turn. Since two quarter turns performed one after another is nothing but one half turn, we have $i^2 = -1$. This equation has another solution, namely $-i$, the quarter turn in the opposite orientation. Obviously, the very choice of a frame to obtain Cartesian coordinates like $\xi_P, \eta_P$ ensures a choice of orientation of the plane: when you look at a plane from above, you measure angles by placing your protractor onto the plane, seeing $z_P = \xi_P + i\eta_P$. But the guy that looks at the plane from below will put his protractor on to the other face of the plane, and will therefore see $z_P = \xi_P - i\eta_P$ instead of $z_P$.

Remark 15.2.3. The former affine point of view, describing points of $\mathbb{R}^2$ by vectors $(\xi_P, \eta_P, 1)$ can be restated as:

$$P : \begin{pmatrix} z_P \\ \bar{z}_P \end{pmatrix} = \begin{pmatrix} \xi_P + i\eta_P \\ 1 \\ \xi_P - i\eta_P \\ 1 \end{pmatrix} = \mathbf{Y} \begin{pmatrix} \xi_P \\ \eta_P \end{pmatrix}$$

where \(
\mathbf{Y} = \begin{pmatrix} 1 & +i & 0 \\ 0 & 0 & 1 \\ 1 & -i & 0 \end{pmatrix}\)

collecting together the from above and the from below points of view.

Theorem 15.2.4. In the Morley space, the area of triangle $ABC$ is given by:

$$\text{area}(ABC) = \begin{vmatrix} -\frac{1}{4}i & z_A & z_B & z_C \\ 1 & 1 & 1 & \bar{z}_A & \bar{z}_B & \bar{z}_C \end{vmatrix}$$

(15.1)

Proof. Despite it’s great consequences, this result has a very short proof: formula (7.6) using Cartesian coordinates has to be modified by factor $-i/2$ since:

$$\det \begin{pmatrix} 1 & +i & 0 \\ 0 & 0 & 1 \\ 1 & -i & 0 \end{pmatrix} = 2i$$

\[\square\]

Proposition 15.2.5. In the Morley space, the line at infinity is described by

$$\mathcal{L}_z \simeq \begin{bmatrix} 0 & 1 & 0 \end{bmatrix}$$

(15.2)

Proof. The value is obvious. And can also seen as $[0, 0, 1] \cdot \mathbf{Y}^{-1}$.

\[\square\]

Proposition 15.2.6. In the Morley space, the matrix $W_z$ of the operator that changes the affix $D$ of a line, into the affix $P$ of the point at infinity of that line can be written as:

$$P = W_z \cdot D$$

where

$$W_z = \begin{pmatrix} 0 & 0 & -1 \\ 0 & 0 & 0 \\ +1 & 0 & 0 \end{pmatrix}$$

(15.3)

Proof. This is only $P = D \wedge \mathcal{L}_z$.

\[\square\]
Proposition 15.2.7. In the Morley space, the matrix \( \text{Pyth}_z \) of the quadratic form that gives the squared length of a vector in the \( V \) space can be written as:

\[
\text{Pyth}_z = \frac{1}{2} r^2 \begin{pmatrix}
0 & 0 & 1 \\
0 & -2 & 0 \\
1 & 0 & 0
\end{pmatrix}
\]

(15.4)

Proof. Here, equalities are required since a length is not defined up to a proportionality factor. Remember that an element of \( V \) is obtained as the difference of the normalized columns of two finite points, and therefore looks like \( z_1/t_1 - z_2/t_2 \), \( 0, \overline{z_1}/t_1 - \overline{z_2}/t_2 \). The obtained formula is nothing but the usual \( |\cdot|^2 = \zeta \overline{\zeta} \). The strange looking \(-2\) that acts on the \( t = 0 \) component is the counterpart for the better looking form of the \( \text{Pyth}_z \) matrix. And the \( R \) factor that appears in the formula will only make sense in the Lubin’s context. Otherwise, the radius of the "unit circle" is supposed to be 1.

\[
\text{Pyth}_z = \frac{1}{2} r^2 \begin{pmatrix}
0 & 0 & 1 \\
0 & -2 & 0 \\
1 & 0 & 0
\end{pmatrix}
\]

(15.4)

Proposition 15.2.8. In the Morley space, the matrix \( \text{OrtO}_z \) of the operator that transforms a direction into its orthogonal direction while transforming the circumcenter into \( 0 : 0 : 0 \), is given by:

\[
\text{OrtO}_z = i \begin{pmatrix}
1 & 0 & 0 \\
0 & 0 & 0 \\
0 & 0 & -1
\end{pmatrix} \approx Y \begin{pmatrix}
0 & +1 & 0 \\
-1 & 0 & 0 \\
0 & 0 & 0
\end{pmatrix} \cdot Y^{-1}
\]

(15.5)

Proof. The Cartesian formula \( x : y : 0 \mapsto +y : -x : 0 \) is well-known. Then the change of basis formulas have to be applied. Eigenvectors of \( \text{OrtO}_z \) are both umbilics \( \Omega_y \approx 1 : 0 : 0, \Omega_x \approx 0 : 0 : 1 \) and \( 0 : 1 : 0 \) (the circumcenter); umbilic \( \Omega_y \) is rotated by a quarter of turn, and umbilic \( \Omega_x \) is rotated by the opposite amount.

Proposition 15.2.9. Orthodir. In the Morley space, the matrix \( \text{M}_z \) of the orthodir operator that changes the affix \( D \) of a line, into the affix \( P \) of the point at infinity is the orthogonal direction, can be written as:

\[
P = \text{M}_z \cdot .i \quad \text{where} \quad \text{M}_z \doteq i \begin{pmatrix}
0 & 0 & 1 \\
0 & 0 & 0 \\
1 & 0 & 0
\end{pmatrix} = \text{OrtO}_z \cdot W_z
\]

(15.6)

Proof. Obvious from definitions. Coefficient \( i \) is kept here in order to obtain a better looking tangent formula.

Theorem 15.2.10. Tangent of two lines. In the Morley space, the oriented angle from a visible line \( \Delta_1 \) to a visible line \( \Delta_2 \) is characterized by:

\[
\tan \left( \Delta_1, \Delta_2 \right) = \frac{\Delta_1 \cdot W_z \cdot .i \Delta_2}{\Delta_1 \cdot \text{M}_z \cdot .i \Delta_2}
\]

(15.7)

where \( W_z \), \( \text{M}_z \) are exactly as given in (15.3) and (15.6) (not up to a proportionality factor).

Proof. Consider two Cartesian lines \( y = p_1x + m_1 \) and \( y = p_2x + m_2 \). Then \( \tan (D_1, D_2) \) is obviously \((p_2 - p_1) / (1 + p_1 p_2)\) since the slope of a line is the tangent of the angle from the \( x \) axis to the line. It remains to see that \( D_1 \) is written \([p_j, -1, m_j]\) in the Cartesian frame, and then \([p_j + i, 2 m_j, p_j - i]\) in the Morley frame... and the formula gives the right value. Numerator tell us when lines are parallel, and denominator when they are orthogonal.

Proposition 15.2.11 (Laguerre Formula). Consider two (distinct) lines \( \Delta_1, \Delta_2 \) and their common point \( M \). Then

\[
\text{cross_ratio} (M \Omega_y, M \Omega_x, \Delta_1, \Delta_2) = \exp \left( 2i \left( \Delta_1, \Delta_2 \right) \right)
\]

Proof. Tedious proof: compute and check using (15.7).
Proof. Better proof: the four lines are cutting the line at infinity, and we have:

\[
\text{cross}_\text{ratio} \left( \begin{bmatrix} 1 & 0 & \tau \\ 0 & 0 & 1/\tau \\ 0 & 1 & 1/\sigma \end{bmatrix} \right) = \frac{\sigma^2}{\tau^2}
\]

\[\Box\]

Proposition 15.2.12. In the Morley space, distance from point \( P \) to line \( \Delta \) is given by:

\[
\text{dist} (P, \Delta) = \frac{\Delta \cdot P}{(L_z : P) \sqrt{2i \cdot M_z \cdot t \Delta}}
\]

where \( L_z = \begin{bmatrix} 0 & 1 & 0 \end{bmatrix} \) and \( M_z \) is as given in (15.6) (not up to a proportionality factor).

Remark 15.2.13. Formula (15.8) is invariant when barycentrics of \( P \) or \( \Delta \) are modified by a proportionality factor. Denominators are enforcing the fact that \( P \) is supposed to be at finite distance, and \( \Delta \) is not supposed to be the infinity line. The square root is the operator norm of the application

\[
\phi : Z : T : \mathbb{Z} \rightarrow (zZ + tT + \zeta \overline{Z}) / (T)
\]

Replace \( Z : T : \overline{Z} \) by \( Z + r \tau : T : \overline{Z} + r/\tau \). Derive \( \tau \) and obtain \( r z - r \zeta/\tau^2 \). Solve in \( \tau \) and obtain \( \tau = \sqrt{\zeta/z} \). So that \( \Delta \phi = 2\sqrt{z\zeta} \).

15.3 Lubin representation of first degree

Proposition 15.3.1. The Morley-affix of a point \( P \) whose barycentrics are \( p : q : r \in \mathbb{P} \mathbb{R} (\mathbb{R}^3) \) with respect to triangle \( ABC \) is given by:

\[
\zeta_P \simeq \frac{p \pm \frac{q + r}{2} - \frac{q - r}{2}}{p \pm \frac{q + r}{2} + \frac{q - r}{2}}
\]

Proof. When \( p + q + r \neq 0 \), this is nothing but the usual definition of a barycenter, and \( \zeta_P \) is a Morley finite point. When \( p + q + r = 0 \), \( P \) is at infinity and \( \zeta_P \) is a Morley direction. The condition \( p, q, r \in \mathbb{R} \) ensures that no invisible points in the Morley-space can be generated from a real point in the Kimberling space.

Remark 15.3.2. Formula (15.9) requires a lot of conjugacies... and it is well-known that conjugacy is a terrific process that introduces branching points. Therefore a method is needed to transform conjugacy into an holomorphic process, at least for a sufficiently large subset of points.

Definition 15.3.3. The Lubin parametrization are obtained by assuming that the circumcircle of triangle \( ABC \) is nothing but the unit circle of the complex plane, together with the relations:

\[
z_A = \alpha^n, z_B = \beta^n, z_C = \gamma^n
\]

Since our interest is directed toward central objects, we will largely use the so-called symmetric functions:

\[
s_1 = \alpha + \beta + \gamma =, s_2 = \alpha \beta + \beta \gamma + \gamma \alpha, s_3 = \alpha \beta \gamma, s_4 = i (\alpha - \beta) (\beta - \gamma) (\gamma - \alpha)
\]

\[
\sigma_1 = z_A + z_B + z_C, \sigma_2 = z_A z_B + z_B z_C + z_A z_C, \sigma_3 = z_A z_B z_C, \sigma_4 = i (z_A - z_B) (z_B - z_C) (z_C - z_A)
\]

Remark 15.3.4. Quantity \( s_4 \) (i times the Vandermonde of the three numbers) is skew-symmetric and verifies:

\[
s_4^2 = -s_1^2 s_2^2 + 4 s_1 s_3^3 + 4 s_2^3 - 18 s_3 s_1 s_2 + 27 s_3^3
\]

so that \( s_4 \) will not appear by a power greater than one. The "small" symmetric functions \( s_1, s_2, s_3, s_4 \) depend on the degree of the representation, while the "big" ones \( \sigma_1, \sigma_2, \sigma_3, \sigma_4 \) depend only on the triangle.
Proposition 15.3.5. Forward and backward matrices. Using the Lubin-1 parametrization, we have:

\[
L_u = \begin{pmatrix}
\alpha & \beta & \gamma \\
1 & 1 & 1 \\
1/\alpha & 1/\beta & 1/\gamma
\end{pmatrix}; \quad \det \begin{bmatrix} \sigma_4 \\ \sigma_3 \end{bmatrix} = \frac{4i}{R^2} S
\tag{15.11}
\]

\[
L_u^{-1} = \frac{1}{i\sigma_4} \begin{pmatrix}
\alpha (\beta - \gamma) & \alpha (\gamma^2 - \beta^2) & \alpha \beta \gamma (\beta - \gamma) \\
\beta (\gamma - \alpha) & \beta (\alpha^2 - \gamma^2) & \alpha \beta \gamma (\gamma - \alpha) \\
\gamma (\alpha - \beta) & \gamma (\beta^2 - \alpha^2) & \alpha \beta \gamma (\alpha - \beta)
\end{pmatrix}
\]

Theorem 15.3.6. Forward substitutions. Suppose that barycentrics \( p : q : r \) of point \( P \) depends rationally on \( a^2, b^2, c^2, S \). Then Morley-affix of \( P \) is obtained by substituting the identities:

\[
S = \frac{i R^2 (\alpha - \beta) (\beta - \gamma) (\gamma - \alpha)}{4 \alpha \beta \gamma}, \quad a = \frac{R (\beta - \gamma)}{\sqrt{-\beta \gamma}}, \quad b = \frac{R (\gamma - \alpha)}{\sqrt{-\gamma \alpha}}, \quad c = \frac{R (\alpha - \beta)}{\sqrt{-\alpha \beta}}
\tag{15.12}
\]

into \( p : q : r \) and applying (15.9). The result obtained is a rational fraction in \( \alpha, \beta, \gamma \) whose degree is +1. When \( P \) is a triangle center, \( \zeta_P \) depends only on \( \sigma_1, \sigma_2, \sigma_3 \). When \( P \) is invariant by circular permutation, but not by transposition, a \( \sigma_4 \) term appears.

Proof. Cancellation of radicals is assured by condition \( p, q, r \in \mathbb{Q} (a^2, b^2, c^2, S) \). Elimination of \( R \) comes from homogeneity. Symmetry properties are evident. \( \square \)

Proposition 15.3.7. Backward substitutions. Let \( z_P \in \mathbb{C} (\alpha, \beta, \gamma) \) be an homogeneous rational fraction, supposed to be the complex affix of a finite point. Then \( \deg (z_P) = 1 \) is required. Alternatively, let \( \omega^2 \in \mathbb{C} (\alpha, \beta, \gamma) \) be an homogeneous rational fraction, supposed to describe the Morley affix of a direction. Then \( \deg (\omega^2) = 2 \) is required. When these conditions are fulfilled, the ABC-barycentrics \( p : q : r \) of these objects can be obtained as follows. Compute the corresponding vector:

\[
\begin{pmatrix} p \\ q \\ r \end{pmatrix} = \begin{bmatrix} \omega^2 \\ \frac{1}{z_P} \end{bmatrix}
\]

then apply substitutions

\[
\beta = \frac{+2i(a^2 + b^2 - c^2)}{a^2b^2} S + \frac{a^4 + b^4 + c^4 - 2(a^2 + b^2)c^2}{2a^2b^2} \alpha
\]

\[
\gamma = \frac{-2i(c^2 + a^2 - b^2)}{a^2c^2} S + \frac{a^4 + b^4 + c^4 - 2(a^2 + c^2)b^2}{2a^2c^2} \alpha
\]

to this vector and simplify the obtained expression using the Heron formula:

\[
S^2 = \frac{1}{16} (a + b + c)(b + c - a)(c + a - b)(a + b - c)
\]

Proof. Transform \( \alpha, \beta, \gamma \) into \( \alpha \delta, \beta \delta, \gamma \delta \). Since this transform is a similarity, barycentrics must remain unchanged and the \( z_P \) affix is turned by \( \delta \). On the other hand, polynomial \( z_P \) is multiplied by \( \delta^k \) where \( k = \deg (z_P) \). Concerning the directions, \( \arg (\omega^2) \) is twice the angle with the real axis, and degree 2 is required.

Alternatively, the degrees of rows of \( \zeta_P \) are \( +k, -k, 0 \) while the degrees of columns of \( L_u^{-1} \) are \( -1, 0, +1 \). Quantities \( p, q, r \) will therefore be a sum of terms whose degrees are respectively \( k - 1, 0, 1 - k \). But homogeneity is required in order that a transformation \( \beta = B \alpha, \gamma = C \alpha \) can eliminate \( \alpha \), leading to \( k = 1 \).

To obtain the substitution formulas, compute \( \beta \) from \( c^2 \alpha \beta = -R^2 (\alpha - \beta)^2 \). A choice of branch (a sign for \( i \)) has to be chosen. Exchange \( b \) and \( c \) (and therefore change \( S \) into \( -S \)) and obtain the corresponding \( \gamma \). \( \square \)

Remark 15.3.8. The substitution formulas can be written as:

\[
\beta = \left( \frac{2S^2}{a^2b^2} + i \frac{4S}{a^2b^2} - 1 \right) \alpha ; \quad \gamma = \left( \frac{2S^2}{a^2c^2} - i \frac{4S}{a^2c^2} - 1 \right) \alpha
\]

January 3, 2024 21:08 published under the GNU Free Documentation License
i.e. $\beta = \alpha \exp(2iC)$, $\gamma = \alpha \exp(-2iB)$. This result is indeed symmetric, since $\tilde{B} = (BC, BA)$ while $\tilde{C} = (CA, CB)$.

Remark 15.3.9. When starting with a symmetric Morley-affix, the obtained $p : q : r$ remains symmetric in $\alpha$, $\beta$, $\gamma$. The given substitutions are breaching the symmetry of individual coefficients $p, q, r$, that can only be reestablished by cancellation of asymmetric common factors between the $p, q, r$. Most of the time, its more efficient to proceed by numerical substitution and use the obtained search key to identify the point (and proceed back to obtain a proof of the result).

Proposition 15.3.10. The Kimberling search key associated to a visible finite point defined by its Morley affix (short= Morley's search key) is obtained by substituting:

$$z_A = 1 \quad z_B = -\frac{391}{729} - \frac{104}{729} \sqrt{35} \quad z_C = \frac{401}{1521} - \frac{248}{1521} \sqrt{35}$$

into $\mathbb{Z}/\mathbb{T}$ and then computing:

$$\text{searchkey} \left( \frac{\mathbb{Z}}{\mathbb{T}} \right) = \Re \left( \left( \frac{157}{840} \sqrt{35} - i \frac{22}{3} \right) \frac{\mathbb{Z}}{\mathbb{T}} + \frac{321}{280} \sqrt{35} \right)$$

Proof. Kimberling’s search keys are associated with triangle $a = 6$, $b = 9$, $c = 13$. The radius of the circumcircle is $R = (351/280) \sqrt{35}$. One can see that sidelengths of triangle $\alpha\beta\gamma$ are $6/R$, $9/R$, $13/R$. We apply these substitutions to obtain the numerical value of the "wayback" matrix, and then use (6.1).

Remark 15.3.11. When using Lubin-n with $n > 1$, adequate substitutions have to be used to calculate $\mathbb{Z}/\mathbb{T}$.

Proposition 15.3.12. The Morley's searchkey of a visible point at infinity ($T = 0$) is obtained from $\Omega = \mathbb{Z}/\overline{\mathbb{Z}}$ by:

$$1108809 \left( 241 + 16i\sqrt{35} \right) \Omega^2 - 907686 \left( 157 + 176i\sqrt{35} \right) \Omega + (-224394311 + 30270800i\sqrt{35})$$

$$38919159 \Omega^2 - (106136082 + 118980576i\sqrt{35}) \Omega + (19397664i\sqrt{35} - 371888361)$$

Another method is identifying the isogonal conjugate of the given point, which is simply: $\sigma_3/\Omega : -1 : \Omega/\sigma_3$.

Proof. The searchkey of an point at infinity is: $\frac{z}{a} \times \left( \frac{a}{z} + \frac{b}{y} + \frac{c}{z} \right)$, leading to this tremendous expression. But after all, this formula is not designed for hand computation but rather to a floating evaluation by a computer... .

15.4 Comparing barycentric and Morley spaces

Fact 15.4.1. Concerning the line at infinity (15.2), we have

$$L_z \doteq \begin{bmatrix} 0 & 1 & 0 \end{bmatrix} = L_\infty \cdot L_u^{-1}$$

Fact 15.4.2. Concerning the operator that takes the direction of a line (15.3), we have

$$W_z = \begin{bmatrix} 0 & 0 & -1 \\ 0 & 0 & 0 \\ +1 & 0 & 0 \end{bmatrix} = L_u \cdot W \cdot L_u \div \det L_u$$

Fact 15.4.3. Concerning the quadratic form acting on the line at infinity (15.4), we have

$$Pyth_z \doteq \frac{1}{2} R^2 \begin{bmatrix} 0 & 0 & 1 \\ 0 & -2 & 0 \\ 1 & 0 & 0 \end{bmatrix} = \frac{1}{2} L_u^{-1} \cdot Pyth \cdot L_u^{-1}$$

Remark 15.4.4. Here, strong equalities are required, not equalities up to a proportionality factor. Values of $a, b, c$ that appear in $[Pyth]$ have to be substituted (see (15.12)).
Fact 15.4.5. Concerning the orthopoint transform (15.5), we have

\[
\text{OrtO}_z = i \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -1 \end{pmatrix} \simeq Lu \cdot \text{OrtO}_z \cdot Lu^{-1}
\]

Remark 15.4.6. Values of \(a, b, c\) that appear in \(\text{OrtO}\) have to be substituted (see (15.12)). The exact application of the change of basis formula leads to a global factor \(i\) that can be canceled.

Fact 15.4.7. Concerning the orthodir transform (15.6) that changes the affix \(D\) of a line, into the affix \(P\) of the point at infinity in the orthogonal direction, we have

\[
\mathcal{M}_z = -i \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix} = \text{OrtO}_z \cdot \mathcal{W}_z = \text{Lu} \cdot \mathcal{M}_z \cdot \text{Lu}^{-1} \div \det \text{Lu}
\]

Proof. With the given definition, the proportionality factor (acting over the right member of the \(\simeq\)) is exactly the same factor that was acting over \(\mathcal{W}_z\).

Fact 15.4.8. Concerning the tangent of two lines, formulas (7.20) and (15.7) have exactly the same shape, i.e.

\[
\tan(\Delta_1, \Delta_2) = \frac{\Delta_1 \cdot \mathcal{W}_z \cdot \text{t} \Delta_2}{\Delta_1 \cdot \mathcal{M}_z \cdot \text{t} \Delta_2}
\]

because the same proportionality factors were used for both \(\mathcal{W}_z\) and \(\mathcal{M}_z\).

15.5 Some examples of first degree

Example 15.5.1. The circumcenter \(O\). By definition, \(\zeta_O = 0 : 0 : 1\). The preceding transformations are giving:

\[
\begin{pmatrix} p \\ q \\ r \end{pmatrix} \simeq \begin{pmatrix} \alpha (\beta - \gamma)(\beta + \gamma) \\ \beta (\gamma - \alpha)(\alpha + \gamma) \\ \gamma (\alpha - \beta)(\alpha + \beta) \end{pmatrix} \simeq \begin{pmatrix} a^2 (b^2 + c^2 - a^2) \\ b^2 (c^2 + a^2 - b^2) \\ c^2 (a^2 + b^2 - c^2) \end{pmatrix}
\]

Example 15.5.2. Symmedian point \(X(6)\), aka Lemoine point.

1. Consider the middle \(A'\) of segment \([B, C]\) and define the \(A\) symmedian as the line \(\Delta_A\) that goes through \(A\) and verifies \(\angle(AB, \Delta_A) = \angle(AA', AC)\). We will use \(P\) instead \(\zeta_P\) since this is more readable ... and hand-writable. A remark: symmetry wrt bisectors would be irrelevant, since bisectors are unreachable in the Lubin-1 representation!

2. We have \(A' = B + C\), \(AA' = A \wedge A'\) etc. and our equations are

\[
\Delta_A \cdot A = 0 \\
\tan(AB, \Delta_A) + \tan(AC, AA') = 0
\]

solving this system, then permuting, gives:

\[
\Delta_A \simeq \begin{pmatrix} 2 \alpha - \beta - \gamma \\ (\alpha \gamma + \alpha \beta - 2 \beta \gamma) \alpha \\ -2 \alpha^2 + 2 \beta \gamma \end{pmatrix}
\]

\[
\Delta_B \simeq \begin{pmatrix} 2 \beta - \gamma - \alpha \\ (\alpha \beta + \beta \gamma - 2 \alpha \gamma) \beta \\ -2 \beta^2 + 2 \alpha \gamma \end{pmatrix}
\]

3. Intersecting two symmedians gives a symmetric result. Therefore, the three symmedians are concurrent at some point. This point is well-known as Lemoine point, and we have:

\[
K = \zeta(6) = \Delta_A \wedge \Delta_B = \begin{pmatrix} 2 \sigma_2^2 - 6 \sigma_3 \sigma_1 \\ \sigma_2 \sigma_1 - 9 \sigma_3 \\ 2 \sigma_1^2 - 6 \sigma_2 \end{pmatrix}
\]
4. Going back to barycentrics, we obtain the well-known result:

\[
X(6) \simeq \begin{pmatrix}
\alpha (\gamma - \beta)^2 \\
\beta (\alpha - \gamma)^2 \\
\gamma (\alpha - \beta)^2
\end{pmatrix} \simeq \begin{pmatrix}
\alpha^2 \\
b^2 \\
c^2
\end{pmatrix}
\]

**Example 15.5.3.** The Kiepert parabola.

1. Morley equation of the circumcircle is \( Z \overline{Z} - T^2 = 0 \). The Morley affix \( \Delta_P \) of the polar line of point \( K = z : 1 : \zeta \) wrt the circumcircle is therefore given by:

\[
\Delta_P \equiv \left[ z \ 1 \ \zeta \right] \cdot \begin{pmatrix}
0 & 0 & 1 \\
0 & -2 & 0 \\
1 & 0 & 0
\end{pmatrix}
\]

2. The coefficients of the tangential conic determined by five given lines \( [u_j, v_j, w_j] \) are obtained as:

\[
\bigwedge_{j=1.5} [u_j^2, v_j^2, w_j^2, u_jv_j, v_jw_j, w_ju_j]
\]

by universal factorization of the corresponding \( 6 \times 6 \) determinant. Let us consider the inconic tangent to the infinity line (parabola) and to the circumpolar of point \( K \). Using \( (BC) \simeq B \wedge C \), etc together with the previous equation, we obtain the symmetric matrix:

\[
C^* \simeq \begin{pmatrix}
2\sigma_3 z^2 - 2\sigma_2 \sigma_3 z \zeta + 4\sigma_3^2 \zeta \\
-\sigma_1 \sigma_2 \sigma_3 \zeta + 2\sigma_2 z - 2\sigma_3 \sigma_1 \zeta \\
-\sigma_1 \sigma_2 \sigma_3 \zeta + 2\sigma_2 z - 2\sigma_3 \sigma_1 \zeta
\end{pmatrix}
\]

Degrees of all these expressions are:

\[
dg (C^*) = \begin{pmatrix}
5 & 4 & 3 \\
4 & 2 & . \\
3 & 2 & 1
\end{pmatrix}
\]

3. A focus is a point such that both isotropic lines through that point are tangent to the conic. Writing that \( Q = (Z : T : \overline{Z}) \wedge (1 : 0 : 0) \) satisfies \( Q \cdot C^* \cdot Q = 0 \) and the similar with the other umbilic gives two equations whose solution is

\[
F(z) \simeq \begin{pmatrix}
z^2 - \sigma_2 z \zeta + 2\sigma_3 \zeta \\
2z - \sigma_1 z \zeta + \sigma_3 \zeta^2 \\
z^2 - \sigma_1 z \zeta + \sigma_3 \zeta^2
\end{pmatrix}
\]

and this point is on the circumcircle.

4. Take \( K \) at the Lemoine point \( X(6) \). Its circumpolar line is called the Lemoine axis. One obtains the Kiepert parabola, where:

\[
C^* \simeq \begin{pmatrix}
2\sigma_2 \sigma_3 & \sigma_3 \sigma_1 & 0 \\
\sigma_3 \sigma_1 & 0 & -\sigma_2 \\
0 & -\sigma_2 & -2\sigma_1
\end{pmatrix}, \quad C \simeq \begin{pmatrix}
-\sigma_2^2 & 2\sigma_3 \sigma_1^2 & -\sigma_2 \sigma_3 \sigma_1 \\
2\sigma_3 \sigma_1^2 & -4\sigma_2 \sigma_3 \sigma_1 & 2\sigma_3^2 \sigma_3 \\
-\sigma_2 \sigma_3 \sigma_1 & 2\sigma_3^2 \sigma_3 & -\sigma_1^2 \sigma_3^2
\end{pmatrix}
\]

5. As stated in Proposition 12.3.15, the triangle of the circle-polars of the sidelines of triangle \( T \) is described by matrix \( C^* \cdot T^* \). Both triangle are in perspective (lines \( AA' \), etc are concurrent). The perspector is obtained as \( AA' \wedge BB' \) and concurrence is verified by the symmetry of the result. It is well-known that \( P = X(99) \), the Steiner point.

\[
F = \frac{\sigma_2}{\sigma_1}, \quad P = \frac{\sigma_3 \sigma_1^2 - 3\sigma_2 \sigma_3}{\sigma_2^2 - 3\sigma_3 \sigma_1}
\]
6. Applying the preceding transformations, the \( p : q : r \) associated with \( z = \sigma_2/\sigma_1 \) is obtained as:

\[
\begin{pmatrix}
p \\
q \\
r
\end{pmatrix}
\simeq
\begin{pmatrix}
\alpha (\beta - \gamma) (\alpha \beta - \gamma^2) (\gamma \alpha - \beta^2) \\
\beta (\gamma - \alpha) (\beta \gamma - \alpha^2) (\alpha \beta - \gamma^2) \\
\gamma (\alpha - \beta) (\gamma \alpha - \beta^2) (\beta \gamma - \alpha^2)
\end{pmatrix}
\simeq
\begin{pmatrix}
a^2 \\
(b + c)(b - c) \\
(c + a)(c - a) \\
(a + b)(a - b)
\end{pmatrix}
\]

and we can identify \( X(110) \), the focus of the Kiepert parabola.

**Example 15.5.4.** Isogonic, isodynamic and Napoleon.

1. Define \( j = \exp(2i\pi/3) \). Start from triangle \( ABC \). Construct \( P_A \) such triangle \( P_ABC \) is equilateral. More precisely, the \( z \) affix of \( P_A \) is such that \( z + j \beta + j^2 \gamma = 0 \), deciding of the orientation. The three lines \( AP_A, BP_A, CP_C \) are concurrent leading to the first isogonic center \( X(13) \). Changing \( j \) into \( j^2 \) leads to the second isogonic point \( X(14) \). A simple computation leads to:

\[
\frac{9 \sigma_2 \sigma_3 - 12 \sigma_3 \sigma_1^2 + 3 \sigma_1 \sigma_2^2}{6 \sigma_2^2 - 18 \sigma_3 \sigma_1} \pm \sqrt{3} \frac{\sigma_4 \sigma_2}{6 \sigma_2^2 - 18 \sigma_3 \sigma_1}
\]

2. When trying to transform the former expression into barycentrics, the formal computer is poisoned by the following fact. Quantity \( \sigma_4 \) describes the orientation of the triangle, while the choice of \( \pm \sqrt{3} \) depends on the orientation of the whole plane. We better generalize the problem using \( \tan(AB, AA') = K \). This leads to:

\[
\zeta_K \simeq \begin{pmatrix}
4K(\sigma_2^2 - 3\sigma_3 \sigma_1) + \sigma_4 (K^2 + 1) \sigma_1 \\
2K (\sigma_2 \sigma_1 - 9\sigma_3) + \sigma_4 (3 + K^2) \sigma_2 \\
4K (\sigma_1^2 - 3\sigma_2) + \sigma_4 (K^2 + 1) \frac{\sigma_2}{\sigma_3}
\end{pmatrix}
\]

\[
\begin{pmatrix}
p \\
q \\
r
\end{pmatrix}
\simeq
\begin{pmatrix}
\gamma - \beta \\
(\beta + \gamma) K + i(\gamma - \beta) \\
(\alpha + \gamma) K + i(\alpha - \gamma)
\end{pmatrix}
\simeq
\begin{pmatrix}
1 \\
2S - S_a K \\
2S - S_b K
\end{pmatrix}
\]

3. And we obtain a lot of results when changing \( K \), and even more by isogonal conjugacy. In the following table, line \( K \) lists usual values for the tangent of an angle, while the other two lines give the Kimberling number of the corresponding points. The \( P_K \) points are on the Kiepert RH (more details in Proposition 13.21.1).

<table>
<thead>
<tr>
<th>( K )</th>
<th>-\sqrt{3}</th>
<th>-1</th>
<th>\frac{1}{2}</th>
<th>\frac{1}{2}\sqrt{3}</th>
<th>0</th>
<th>\frac{1}{2}\sqrt{3}</th>
<th>\frac{1}{2}</th>
<th>\sqrt{3}</th>
<th>\infty</th>
</tr>
</thead>
<tbody>
<tr>
<td>( P_K )</td>
<td>13</td>
<td>485</td>
<td>3316</td>
<td>17</td>
<td>2</td>
<td>18</td>
<td>3317</td>
<td>486</td>
<td>14</td>
</tr>
<tr>
<td>isog ( P_K )</td>
<td>15</td>
<td>371</td>
<td>3311</td>
<td>61</td>
<td>6</td>
<td>62</td>
<td>3312</td>
<td>372</td>
<td>16</td>
</tr>
</tbody>
</table>

**Fact 15.5.5.** As of \( j_{\text{max}} < 3587 \), there are 367 points whose barycentrics contain \( R \). Among them:

1. 49 contain other literal radicandes
2. 85 are are in \( \mathbb{C}(a, b, c, S) \) but aren’t in \( \mathbb{C}(a^2, b^2, c^2, S) \)
3. 233 are in fact inside \( \mathbb{C}(a^2, b^2, c^2, S) \). When changing \( R \) into \(-R\), three aren’t paired 2981, 3381, 3382: there are 115 pairs. For 46 of them, the isogonal of a known pair is again a known pair.

January 3, 2024 21:08 published under the GNU Free Documentation License
15.6 Lubin representation of second degree

When dealing with half angles, we have to introduce the mid-arcs on the circumcircle of $ABC$, i.e. the circumcevians of the in-excenters.

**Proposition 15.6.1. Lubin-2 parametrization.** When using parametrization $z_A = \alpha^2$, etc., the mid-arcs are $\pm \alpha \beta, \pm \beta \gamma, \pm \gamma \alpha$. But there are only four choices of sign since

$$\text{product of midarcs } = (-1) \times \text{product of vertices}$$

must be enforced. When using the symmetric choice, i.e. $-\alpha \beta, -\beta \gamma, -\gamma \alpha$ then lines $AI_a, BI_b, CI_c$ concur at $-\alpha \beta - \beta \gamma - \gamma \alpha = -s_2$.

**Remark 15.6.2. Lemoine transform.** As already stated at Theorem 2.1.9, the Lemoine transforms are obtained by $\alpha \mapsto -\alpha$ or $\beta \mapsto -\beta$ or $\gamma \mapsto -\gamma$ when using the Lubin-2 parametrization. And then, the Lubin-1 points (aka the strong points) remain unchanged under these actions.

**Proof.** Here again, the fact that $L_o \circ L_h = L_o$ comes from the homogeneity required for the formulas of interest. Remember: a theorem is a proposition with the biggest consequences, not something difficult to prove.

**Proposition 15.6.3. Forward-2 and backward-2 matrices.** Using the Lubin-2 parametrization, we have:

$$Lu_2 = \begin{pmatrix} \alpha^2 & \beta^2 & \gamma^2 \\ 1 & 1 & 1 \\ 1/\alpha^2 & 1/\beta^2 & 1/\gamma^2 \end{pmatrix}; \quad \det Lu_2 = \frac{i \sigma_4}{\sigma_3} = \frac{i}{s_3} \frac{s_1 s_2 - s_3}{s_3} = \frac{4i}{R^2} S \quad (15.13)$$

$$Lu_2^{-1} = \frac{1}{i \sigma_4} \begin{bmatrix} \alpha^2 (\beta^2 - \gamma^2) & \alpha^2 (\gamma^4 - \beta^4) & \sigma_3 (\beta^2 - \gamma^2) \\ \beta^2 (\gamma^2 - \alpha^2) & \beta^2 (\gamma^4 - \alpha^4) & \sigma_3 (\gamma^2 - \alpha^2) \\ \gamma^2 (\alpha^2 - \beta^2) & \gamma^2 (\beta^4 - \alpha^4) & \sigma_3 (\alpha^2 - \beta^2) \end{bmatrix}$$

**Proof.** These formulas come from (15.11). Remember that $s_1 = \alpha + \beta + \gamma$ while $\sigma_1 = z_A + z_B + z_C$, etc.

**Theorem 15.6.4. Forward-2 substitutions.** Suppose that barycenters $p : q : r$ of point $P$ depends rationally on $a, b, c, S$. Then Morley-affix of $P$ is obtained by substituting the identities:

$$S = \frac{i}{4} R^2 (\alpha^2 - \beta^2) (\gamma^2 - \alpha^2) (\beta^2 - \gamma^2); \quad a = iR \left( \frac{\gamma}{\beta} - \frac{\beta}{\gamma} \right); \quad b = iR \left( \frac{\alpha}{\beta} - \frac{\beta}{\gamma} \right); \quad c = iR \left( \frac{\beta}{\alpha} - \frac{\alpha}{\beta} \right) \quad (15.14)$$

into $p : q : r$ and applying (15.9). The result obtained is a rational fraction in $\alpha, \beta, \gamma$ whose degree is $+2$. When $P$ is a triangle center, $z_P$ depends only on $s_1, s_2, s_3$. When $P$ is invariant by circular permutation, but not by transposition, a $s_4$ term appears.

**Proof.** Elimination of $R$ comes from homogeneity. Symmetry properties are evident. Sign chosen for $a$ is irrelevant, but signs of $b, c$ must be chosen accordingly.

**Proposition 15.6.5. Backward substitutions.** Let $z_P \in \mathbb{C}(\alpha, \beta, \gamma)$ be an homogeneous rational fraction, supposed to be the Lubin-2 affix of a finite point. Then $\deg(z_P) = 2$ is required. Alternatively, let $\omega^2 \in \mathbb{C}(\alpha, \beta, \gamma)$ is an homogeneous rational fraction, supposed to describe the Lubin-2 affix of a direction. Then $\deg(\omega^2) = 4$ is required. When these conditions are fulfilled, the $ABC$-barycenters $p : q : r$ of these objects can be obtained as follows. Compute the corresponding vector:

$$\begin{pmatrix} p \\ q \\ r \end{pmatrix} = Lu_2^{-1} \begin{pmatrix} z_P \\ 1 \\ \overline{z_P} \end{pmatrix} \quad \text{or} \quad \begin{pmatrix} p \\ q \\ r \end{pmatrix} = Lu_2^{-1} \begin{pmatrix} \omega^2 \\ 1 \end{pmatrix}$$

then apply substitutions

$$\alpha = -1; \quad \beta = \frac{S_c + 2iS}{ab}; \quad \gamma = \frac{S_b - 2iS}{ac} \quad (15.15)$$

to this vector and simplify the obtained expression using the Heron formula:

$$S^2 = \frac{1}{16} (a + b + c) (b + c - a) (c + a - b) (a + b - c)$$
Proof. Result about degrees follows Proposition 15.3.7. Substitutions are \( \alpha = -1 \), \( \beta = \exp(\pi i) \), \( \gamma = \exp(-\pi i) \). This result is indeed symmetric, since \( B = (BC, BA) \) while \( C = (CA, CB) \).

### 15.7 Poncelet representation

**Notation 15.7.1.** In this section \( z_M \) denotes the Lubin(2) affix of a point \( M \), while \( \zeta_M \) denotes the Poncelet affix of the same point \( M \). The respective "coordinates in the view from below" will be noted as \( \overline{z_M} \) and \( \overline{\zeta_M} \), leading to

\[
M \overset{\text{Lubin}}{\simeq} \left( \frac{z_M}{1 + \overline{z_M}} \right) \overset{\text{Poncelet}}{\simeq} \left( \frac{\zeta_M}{1 + \overline{\zeta_M}} \right)
\]

**Definition 15.7.2.** The parameters of this representations are the contact points of the incircle, described as \( \rho, \sigma, \tau \) in a frame using this circle as unit circle. Thus \( p = 1/\rho \), etc.

**Proposition 15.7.3.** This representation describes the triangle \( ABC \) by:

\[
\begin{bmatrix}
\rho_m \\
\end{bmatrix} = \begin{bmatrix}
\frac{2\tau \sigma}{\sigma + \tau} & \frac{2\rho \tau}{\tau + \rho} & \frac{2\rho \sigma}{\rho + \sigma} \\
1 & 1 & 1 \\
2(\sigma + \tau)^{-1} & 2(\tau + \rho)^{-1} & 2(\rho + \sigma)^{-1}
\end{bmatrix}
\]

The algebraic direct substitutions are:

\[
a = \frac{2i\rho (\sigma - \tau)}{(\tau + \rho)(\rho + \sigma)} r_0; \quad b = \frac{2i\sigma (\tau - \rho)}{(\rho + \sigma)(\sigma + \tau)} r_0; \quad c = \frac{2i\tau (\rho - \sigma)}{(\sigma + \tau)(\tau + \rho)} r_0
\]

\[(15.16)
\]

while the backward substitutions are:

\[
\sigma = \frac{-2i\rho S}{ab} - \rho \left( \frac{a^2 + b^2 - c^2}{2ab} \right); \quad \tau = \frac{2i\rho S}{ac} - \rho \left( \frac{a^2 - b^2 + c^2}{2ac} \right)
\]

\[(15.17)
\]

**Proposition 15.7.4.** Going back from the Poncelet affix \( \zeta_M \) of a point \( M \) to the Lubin-2 affix \( z_M \) of this point only requires the similarity:

\[
z_M = -s_2 + \frac{1}{2} (s_2 s_1 - s_3) \zeta_M = -s_2 - s_3 \frac{r_0}{R} \overline{\zeta_M}
\]

**Proof.** Due to homogeneity, we can multiply all the \( \rho, \sigma, \tau \) by a same non vanishing factor, and enforce \( \rho = \alpha \). Then substituting (15.14) into (15.17) leads to:

\[
\begin{cases}
\rho = \alpha; \quad \sigma = \beta; \quad \tau = \gamma; \quad r_0 = -R \frac{\beta + \gamma}{2 \alpha \beta \gamma} (\alpha + \gamma) (\alpha + \beta)
\end{cases}
\]

And then, it only remains a change of the projective basis, that is given by:

\[
\text{subs}(\rho = \alpha, Lu \cdot [Pon^{-1}]) = \begin{bmatrix}
0 & -s_2 & 1/2 (s_2 s_1 - s_3) \\
0 & 1 & 0 \\
-2 s_3 / s_2 & s_1 / s_3 & 0
\end{bmatrix}
\]

\[
\square
\]
15.8 Poulbot’s points (using the Lubin-4 parametrization)

Proposition 15.8.1. When using half angles, i.e. $A/2$, etc., we can introduce $a$, etc such that $z_A = a^4$, etc. The intermediate points on the circumcircle can be described as:

$$
\beta^4; i\beta^3\gamma,-\beta^2\gamma^2,-i\beta\gamma^3; \gamma^4; \gamma_3^3\alpha;\gamma^2\alpha^2;\gamma\alpha^3;\alpha^4;\alpha^3\beta;\alpha^2\beta^2;\alpha\beta^3;\beta^4
$$

And then we have:

$$
\frac{\cot A}{2} = \frac{bc + S_a}{2S}; \cos \frac{A}{2} = \frac{\sqrt{b + c + a\sqrt{b + c - a}}}{2\sqrt{b+c}}; \sin \frac{A}{2} = \frac{\sqrt{a + b - c\sqrt{a - b - c}}}{2\sqrt{b+c}}
$$

$$
\cos \frac{A}{2} = i\frac{\beta^2 - \gamma^2}{2\beta\gamma}; \sin \frac{A}{2} = -\frac{\beta^2 + \gamma^2}{2\beta\gamma}
$$

Proof. Everything goes like in Lubin-2: product of points 2,5,8 must be $i$ times the product of points 1,4,7, etc.

Remark 15.8.2. The following quantity belongs to Lubin-2:

$$
\sin \left(\frac{A}{2}\right) \sin \left(\frac{B}{2}\right) \sin \left(\frac{C}{2}\right) = \frac{r_1}{4R} = \frac{S_3 - S_1 S_2}{8S_3}
$$

Remember that $s_1 = \alpha + \beta + \gamma$, $S_1 = \alpha^2 + \beta^2 + \gamma^2$, $\sigma_1 = \alpha^4 + \beta^4 + \gamma^4$.

Example 15.8.3. Consider the circles going through the incenter $I$ and tangent to $AB$ and $AC$.

1. Clearly the centers of these circles, say $A_j$, must be on the $AI$ line. Thus:

$$
A_j = \mu A + (1 - \mu) I = \left[ \mu \frac{\alpha^4 + (1 - \mu) \left( -\alpha^2 \beta^2 - \alpha^2 \gamma^2 - \beta^2 \gamma^2 \right)}{\alpha^4 - (1 - \mu) \frac{1}{\alpha^2 \beta^2 \gamma^2}} \right]
$$
2. Equating the distance to $I$ and the distance to $AB$ (15.8) we obtain:

$$\pm \mu \frac{(\alpha^2 + \gamma^2) (\alpha^2 + \beta^2) \gamma}{\alpha^2 \beta \gamma} = (1-\mu) \frac{(\beta^2 + \gamma^2) (\alpha^2 + \gamma^2) (\alpha^2 + \beta^2)}{2 \alpha^2 \beta^2 \gamma^2}$$

leading to $\mu = (\beta^2 + \gamma^2) / (\beta^2 + 2 \beta \gamma + \gamma^2)$, and to:

$$A_1 = \begin{bmatrix}
S_2 \alpha^3 - \alpha s_2^3 + 2 S_2 s_3 \\
-\alpha^4 + S_1 \alpha - 2 s_3 \\
1 \\
S_1 s_3 - \alpha^2 s_3 + 2 S_1 \alpha^3 \\
\alpha^3 s_3 (-\alpha^3 + S_1 \alpha - 2 s_3)
\end{bmatrix} ; A_0 = \begin{bmatrix}
S_2 \alpha^3 - \alpha s_2^3 - 2 S_2 s_3 \\
-\alpha^4 + S_1 \alpha + 2 s_3 \\
1 \\
S_1 s_3 - \alpha^2 s_3 - 2 S_1 \alpha^3 \\
\alpha^3 s_3 (-\alpha^3 + S_1 \alpha + 2 s_3)
\end{bmatrix}$$

Here $s_3 = \alpha \beta \gamma$ while $S_1 = \alpha^2 + \beta^2 + \gamma^2$, etc. Point $A_1$ is unchanged by $\alpha \mapsto -\alpha$ (since this leads also to $s_3 \mapsto -s_3$). But $\beta \mapsto -\beta$ changes only $s_3$ and exchanges $A_1$ and $A_0$.

3. Obviously, circles $(A_1)$ and $(A_0)$ are tangent at $I$.

4. Let us note $X_{jk}$ the second intersection of circle $Y_j$ and circle $Z_k$. For example, $B_{31}$ is $(C_3) \cap (A_1)$. We obtain:

$$z(A_{11}) = \frac{2 \alpha^3 \beta \gamma (\beta^2 + \gamma^2) + 2 \alpha \beta^3 \gamma^3 + \beta^2 \gamma^2 (\beta + \gamma) (\beta^2 + \gamma^2) + \alpha^2 (\beta^2 + \beta \gamma + \gamma^2) (\beta^2 - \beta \gamma + \gamma^2) (\beta + \gamma)}{\alpha^2 (\beta + \gamma) - 2 \alpha \beta \gamma}$$

And then, we use $\beta + \gamma = s_1 - \alpha$ and $\beta \gamma = s_2 - \alpha s_1 + \alpha^2$. This leads to "huge" polynomials in $\alpha$, that can be reduced using the relation $\alpha^3 - s_1 \alpha^2 + s_2 \alpha - s_3 = 0$. As a result:

$$z(A_{11}) = \frac{2 s_1^3 s_3 - s_1^2 s_2 + s_3^3}{-s_1 s_2 s_3 + s_3^2} \alpha + \frac{-2 s_1^3 s_3 + s_1^2 s_2 + 4 s_1^2 s_2 s_3}{s_2 s_3^2}$$

5. And now consider circles $(A, A_{jk}, I)$, $(B, B_{kn}, I)$, $(C, C_{nj}, I)$. On the Figure, we can see that these circles concur in a second point. Under the action of the Klein group of the $\alpha \mapsto -\alpha$ transforms, the parity of $j + k + n$ is kept, as described in Table 15.1: two situations are encountered.

6. Construction. The center $B_0$ of circle $(B, B_{00}, I)$ can be obtained as $A_0 C_0 \cap M_0 M_c$ where $A_0, C_0$ are the centers described above, and $M_0, M_c$ are the mid-arcs relative to the incenter $I_0$.

7. Spoiler (Veronese map). Use $z(A) = \alpha^4$, $z(I) = 2 s_1 s_3 - s_2^2$ and $z(A_{11})$ as above. Take the Veronese of these points, and then the wedge of these three rows. Reduce this column, and take the remainder of each element wrt $\alpha^3 - s_1 \alpha^2 + s_2 \alpha - s_3$. Now, the representative of circle $(I, A, A_{11})$ is written as

$$V_a = V_1 + \alpha V_2 + \alpha^2 V_3$$

were $V_1, V_2, V_3$ are symmetric in $\alpha, \beta, \gamma$. One can see that the family $V_1, V_2, V_3$ is not independent, so that the $V_a, V_b, V_c$ belong to a same pencil.
Figure 15.1: Poulbot’s points

orange: 11 circles, blue 00 circles, magenta: 01 or 10 circles
8. Spoiler (pencil of circles). Determine the point-circles in this pencil, i.e. determine \( K \) so that 
\[
(V_1 + K V_2) \cdot \left( \frac{Q}{z} \right) \cdot (V_1 + K V_2) = 0.
\]
This equation factors gently, giving \( K \) and therefore the point-circles. Going back to the equations in \( Z, T, Z \), we obtain factored equations:
\[
\left( s_1^2 - 2 s_2 \right) T + \left( 2 s_1^3 s_2 - s_1^3 s_2 - 2 s_1^3 s_2 - 3 s_1^3 s_2 s_3 \right) Z + \left( s_1^3 s_2 - 2 s_1 s_2^2 \right) Z = 0
\]
and conjugate. This characterizes the base points of an isoptic pencil. Thus the \( jkn = 111 \) case leads to Poulbot’s points of first kind (Ayme et al., 2014):
\[
z_U = -2 s_1^2 s_2 s_3 + \frac{s_1^3 s_2^3 + 2 s_1^3 s_2^3 + 3 s_1^3 s_2^3 s_3 - 2 s_1 s_2^2 - 10 s_1 s_2 s_3 + 4 s_2 s_3 + 4 s_2^3}{s_1^3 s_2 - 2 s_1 s_2^2 - 3 s_1^3 s_3 + 4 s_2 s_3}
\]
and it can be seen that this point is on the Bevan circle \( J_a J_b J_c \). So are the other three.

9. In the same vein, the \( jkn = 000 \) case leads to Poulbot points of second kind:
\[
z_W = \frac{s_1^2 s_2^3 - 2 s_1^2 s_2 s_3 - 2 s_2^2 + 3 s_1 s_2^2 s_3 + 2 s_1 s_2^2 - 2 s_2 s_3}{s_1^2 s_2 - 2 s_2^2 + s_1 s_3}
\]
and it can be seen that this point is on the Bevan circle \( J_a J_b J_c \). So are the other three.

15.9 More about the foci of a conic
Chapter 16

Collineations

16.1 Definition

Definition 16.1.1. A collineation is a reversible linear transformation of the barycentrics, i.e. $U = \phi(P)$ determined by:

$$
\begin{pmatrix}
  u \\
  v \\
  w
\end{pmatrix}
\simeq
\begin{pmatrix}
  m_{11} & m_{12} & m_{13} \\
  m_{21} & m_{22} & m_{23} \\
  m_{31} & m_{32} & m_{33}
\end{pmatrix}
\begin{pmatrix}
  p \\
  q \\
  r
\end{pmatrix}
$$

where $\simeq$ is a reminder of the fact that barycentrics are determined up a proportionality factor.

Proposition 16.1.2. A collineation is determined by two ordered lists four points: $P_i, i = 1, 2, 3, 4$, $U_i, i = 1, 2, 3, 4$ such that no triples of $P$ points are on the same line, and the same for the $Q$ points.

Proof. If $\phi$ is reversible, then $\det M \neq 0$ is required and the $\phi(P_i)$ haven’t alignments when the $P_i$ haven’t. Conversely, we have the following algorithm. $\square$

Algorithm 16.1.3. Collineationalgorithm. Let be given the two lists of points $P_i, i = 1, 2, 3, 4$, $U_i, i = 1, 2, 3, 4$. With obvious notations, the question is to find the $m_{ij}$ (not all being 0) and the $k_i$ (none being 0) in order to ensure:

$$
\begin{pmatrix}
  u_1 & u_2 & u_3 & u_4 \\
  v_1 & v_2 & v_3 & v_4 \\
  w_1 & w_2 & w_3 & w_4
\end{pmatrix}
\begin{pmatrix}
  k_1 & 0 & 0 & 0 \\
  0 & k_2 & 0 & 0 \\
  0 & 0 & k_3 & 0 \\
  0 & 0 & 0 & k_4
\end{pmatrix}
= 
\begin{pmatrix}
  m_{11} & m_{12} & m_{13} \\
  m_{21} & m_{22} & m_{23} \\
  m_{31} & m_{32} & m_{33}
\end{pmatrix}
\begin{pmatrix}
  p_1 & p_2 & p_3 & p_4 \\
  q_1 & q_2 & q_3 & q_4 \\
  r_1 & r_2 & r_3 & r_4
\end{pmatrix}
$$

This system has 13 unknowns and 12 equations, since a global proportionality factor remains undetermined. The $k_i$ are determined (up to a global proportionality factor) by

$$
\frac{1}{k_i} \det Q = \det P \det M_{\neq i}
$$

where a $3 \times 4$ matrix subscribed by an $\neq i$ refer to the square matrix obtained by deleting the $i$-th column. Thereafter, $M$ is easy to obtain. To summarize:

$$
\begin{pmatrix}
  k_i = \frac{\det Q \cdot \det P_{\neq i}}{\det P_{\neq i}} & M = Q \cdot K \cdot \left(\frac{P_{\neq i}}{P_{\neq i}}\right)^{-1}
\end{pmatrix}
$$

With the given hypotheses, transformation $\phi$ is clearly reversible.

Remark 16.1.4. An efficient choice of the $P_i, Q_i$ is eight centers, or a central triangle and a center for the $P$ and the corresponding $Q$. In such a case, any center is transformed into a center, and homogeneous curves into homogeneous curves of the same degree.
16.2 Involutory collineations

**Proposition 16.2.1.** Let \( M_1, M_2, N_1, N_2 \) be four (different points). The collineation \( \psi \) that swaps the \((M_1, M_2)\) pair and also the \((N_1, N_2)\) pair is involutory. The line through the crossed intersections \( M_1N_1 \cap M_2N_2 \) and \( M_1N_2 \cap M_2N_1 \) is a line of fixed points (the axis of \( \psi \)). The paired intersection, i.e. point \( P = M_1M_2 \cap N_1N_2 \) is an isolated fixed point (the pole of \( \psi \)). Reciprocally, given an axis \( \Delta \) and a pole \( P \) (outside of the axis), we obtain an involutory transform \( U \mapsto X = \psi(U) \) by requiring \( P, U, X \) aligned together with \( (P, U, PU \cap \Delta, X) = -1 \) (harmonic conjugacy).

**Proof.** Consider \( \psi \) defined by \( A \leftrightarrow P \) and \( B \leftrightarrow C \). Its matrix \( \psi \) can be obtained by the general Alg. 16.1.3. Then the cevian triangle of \( P \) provides a diagonalization basis and we have:

\[
\psi \simeq \begin{pmatrix} 1 & 0 & 0 \\ q & 0 & -q \\ p & r & 0 \end{pmatrix} ; \quad TP = \begin{pmatrix} 0 & p & p \\ q & 0 & q \\ r & r & 0 \end{pmatrix}
\]

\[
T_T^{-1} \cdot \psi \cdot T_T = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & r & 0 \end{pmatrix}
\]

In the general case, we can chose matrix \( \psi \) to enforce \( \det \psi = 1 \). Then minimal polynomial is \( \mu^2 - 1 \) while characteristic polynomial is \( \chi(\mu) = (\mu - 1)^2(\mu + 1) \). \( \square \)

16.3 Usual affine transforms as collineations

**Remark 16.3.1.** Umbilics have been defined in Subsection 14.1.2. A possible choice can be described as \( \Omega^\pm \simeq abc X_{512} \pm iR X_{511} \). The exact value is given in (14.2).

**Proposition 16.3.2. Translation.** The matrix of the translation \( U \mapsto U + \vec{V} \) where \( \vec{V} = (p, q, r) \) is given by:

\[
\begin{pmatrix} 1 + p & p & p \\ q & 1 + q & q \\ r & r & 1 + r \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} + \vec{V} \cdot L_\infty \quad (16.1)
\]

**Proof.** Use \( P = A, B, \Omega^+, \Omega^- \) and \( Q_1 = C, D, \Omega^+, \Omega^- \). Characteristic polynomial is \( \chi(\mu) = (\mu - 1)^2(\mu - 1 - p - q - r) \)

For a translation, \( p + q + r = 0 \), and the matrix is not diagonalizable. \( \square \)

**Remark 16.3.3.** The translation operator is linear, meaning that \( M (\vec{V}_1) + M (\vec{V}_2) = M (\vec{V}_1 + \vec{V}_2) \).

**Proposition 16.3.4. Homothety.** When \( p + q + r \) is different from \( 0 \) and \( -1 \) then (16.1) characterizes the homothety centered at point \( P = p : q : r \) with ratio \( \mu = 1 / (1 + p + q + r) \).

**Proof.** The factor is the reciprocal of the eigenvalue \( \lambda_P \) since a fixed point should be described by \( \lambda = 1 \) : all the eigenvalues have to be divided by \( \lambda_P \). This result can also be obtained by direct examination of \( f(P) f(U) \).

**Remark 16.3.5.** When computing \( P \), the column \( ^t(p, q, r) \) can be viewed as "defined up to a proportionality". This does not apply to the computation of \( \mu \). In any case, we are re-obtaining (7.27).

**Remark 16.3.6.** The matrix \( \pi_\Delta \) of the orthogonal projector onto line \( \Delta \simeq [p, q, r] \) is:

\[
\pi_\Delta = \Delta \cdot M \cdot ^t \Delta - M \cdot ^t \Delta \cdot \Delta
\]

while the matrix \( \sigma_\Delta \) of the orthogonal reflection wrt line \( \Delta \simeq [p, q, r] \) is:

\[
\sigma_\Delta = \Delta \cdot M \cdot ^t \Delta - 2 M \cdot ^t \Delta \cdot \Delta
\]

These formulas are recalled from Section 7.8, where more details are given.
Proposition 16.3.7. The matrix of the rotation centered at finite point $P = p : q : r$ with angle $\phi$ is:

$$(p + q + r) \Phi = \begin{pmatrix} p & p & p \\ q & q & q \\ r & r & r \end{pmatrix} + \begin{pmatrix} r + q & -p & -p \\ -q & r + p & -q \\ -r & -r & q + p \end{pmatrix} \cos \phi + \frac{\text{OrtO}}{L_{\infty} \cdot P - P \cdot L_{\infty}} \sin \phi$$

Proof. It suffices to check what happens to $P, \Omega^+, \Omega^-$: they are fixed points, with respective eigenvalues: 1, exp$(+i\phi)$, exp$(-i\phi)$, while the global factor $p + q + r$ is a remainder of the constraint $P \notin L_{\infty}$.

Stratospherical proof. A rotation with angle $\phi$ is multiplication by $\Phi = \cos \phi + i \sin \phi$ in the complex plane. Therefore, rotation with center $P$ and angle $\phi$ can be written as:

$$\Phi(X) = P + [1 \cos \phi + i \sin \phi] PX$$

Since matrix $\text{OrtO}$ describes a "project and turn" action, we have $[\text{OrtO}]^3 = -\text{OrtO}$ so that $[i = \text{OrtO}]^3 \sin \phi$. Multiplying, we get $[\text{OrtO}]^4 = -[\text{OrtO}]^2$ and $[\text{OrtO}]^2$ is a projective space $\mathcal{V}$. This gives $[1 = -\text{OrtO}]^2$. Canceling the denominators, we obtain:

$$\Phi \begin{pmatrix} x \\ y \\ z \end{pmatrix} \simeq \begin{pmatrix} p \\ q \\ r \end{pmatrix} + \begin{pmatrix} p \\ q \\ r \end{pmatrix} \begin{pmatrix} \sin \phi \text{OrtO} - \cos \phi [\text{OrtO}]^2 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} p \\ q \\ r \end{pmatrix}$$

leading to the required matrix:

$$\Phi = \text{proj} + \begin{pmatrix} \sin \phi \text{OrtO} - \cos \phi [\text{OrtO}]^2 \end{pmatrix} \cdot (1 - \text{proj}) \text{ proj} = \frac{1}{L_{\infty} \cdot F} (P \cdot L_{\infty})$$

Proposition 16.3.8. Similarity. When $A, B, C, D$ are points at finite distance, with $A \neq B$, $C \neq D$ it exists two similarities $\phi, \psi$, respectively called direct and skew, that sends $A \rightarrow C$ and $B \rightarrow D$. As collineations, they are characterized by:

$$\phi = \text{collinear} (\, A, B, \Omega^+, \Omega^-; C, D, \Omega^+, \Omega^-)$$

$$\psi = \text{collinear} (\, A, B, \Omega^+, \Omega^-; C, D, \Omega^+, \Omega^-)$$

Proof. The group of all the similarities is the stabilizer subgroup of the pair $\{\Omega^+, \Omega^-\}$ under the action of the group of all the collineations. This comes from the fact that any similarity transforms circles into circles, and therefore must preserve the umbilical pair.

Proposition 16.3.9. Similarity (Morley plane). Spoiler: in the Morley plane, the matrix of the similarity $\sigma$ defined by center $M \simeq z : t : \zeta$, ratio $k$ and turn $\tau$ is:

$$\sigma \simeq \begin{bmatrix} k\tau & \frac{z}{t} (1 - k\tau) & 0 \\ 0 & 1 & 0 \\ 0 & \frac{\zeta}{t} (1 - k) & k \frac{\tau}{7} \end{bmatrix}$$

Stratospherical proof. Take $\Omega_y, M, \Omega_x$ as basis and say that eigenvalues are $k\tau, 1, k/\tau$.

Computational proof. Say that $\sigma = \text{collinear} (\Omega_y, \Omega_x, M, N, \Omega_y, \Omega_x, M, N')$ while $|MN|^2 = k^2 |MN|^2$ and $\tan(MN, MN') = i (1 - r^2) / (1 + r^2)$. Solve for $N'$ and obtain the former result. In fact there is another solution, obtained by $k\tau \mapsto -k\tau$. But we must obtain the unit matrix when $k\tau = 1$. 

— pldx : Translation of the Kimberling's Glossary into barycentric ——
16.4 Barycentric multiplication as a collineation

**Proposition 16.4.1.** Barycentric multiplication by \( P = p : q : r \) is what happens to the plane when using collineation \( \phi : (A, B, C, X_2) \mapsto (A, B, C, P) \). In other words:

\[
X \ast _b P = \begin{bmatrix} P \end{bmatrix} \cdot X \quad \text{where} \quad \begin{bmatrix} P \end{bmatrix} = \begin{pmatrix} p & 0 & 0 \\ 0 & q & 0 \\ 0 & 0 & r \end{pmatrix}
\]

**Remark 16.4.2.** Obviously, trilinear multiplication can be described using collineations involving \( X_1 \).

**Proposition 16.4.3.** The collineation whose matrix is diagonal, with elements \( U \div P \) transforms \((A, B, C, P)\) into \((A, B, C, U)\) and circumconic \( CC(P) \) into \( CC(U) \).

**Proof.** Direct examination. One obtains:

\[
\begin{pmatrix} u & 0 & 0 \\ 0 & v & 0 \\ 0 & 0 & w \end{pmatrix} \cdot \begin{pmatrix} 0 & w & v \\ w & 0 & u \\ v & u & 0 \end{pmatrix} \cdot \begin{pmatrix} u & 0 & 0 \\ 0 & v & 0 \\ 0 & 0 & w \end{pmatrix} = \begin{pmatrix} 0 & r & q \\ r & 0 & p \\ q & p & 0 \end{pmatrix}
\]

**Construction 16.4.4.** The following recipe constructs \( F = D \ast _b E \).

1. Points \( A, B, C, D, E \) given
2. Line \( ab \) through \( A, B \)
3. Line \( bc \) through \( B, C \)
4. Line \( ca \) through \( C, A \)
5. Point \( X_1 \) Intersection of \( bc, Line[A, D] \)
6. Point \( X_2 \) Intersection of \( bc, Line[A, E] \)
7. Point \( K_x \) Intersection of \( ab, Line[X_2, ca] \)
8. Point \( H_x \) Intersection of \( ca, Line[X_1, ab] \)
9. Point \( Y_1 \) Intersection of \( ca, Line[B, D] \)
10. Point \( Y_2 \) Intersection of \( ca, Line[B, E] \)
11. Point \( H_y \) Intersection of \( ab, Line[Y_1, bc] \)
12. Point \( K_y \) Intersection of \( bc, Line[Y_2, ab] \)
13. Point \( Q_x \) Intersection of \( Line[B, H_x], Line[C, K_x] \)
14. Point \( Q_y \) Intersection of \( Line[C, H_y], Line[A, K_y] \)
15. Point \( F \) Intersection of \( Line[A, Q_x], Line[B, Q_y] \)

**Proof.** The idea is to construct parallelograms \( AK_1X_1H_1, AK_2X_2H_2 \) and use them as pantographs. Using \( D \approx p : q : r \) and \( E \approx u : v : w \), we have:

\[
\begin{array}{cccccccccc}
X_1 & X_2 & K_x & H_x & Y_1 & Y_2 & H_y & K_y & Q_x & Q_y & F \\
0 & 0 & w & q & p & u & p & 0 & wq & up & up \\
qu & v & v & 0 & 0 & 0 & r & u & qu & wr & qw \\
wr & 0 & r & r & w & 0 & wr & wr & wr & wr & wr \\
\end{array}
\]
16.5 Complement and anticomplement as collineations

Proposition 16.5.1. Complement is what happens to the plane when using collineation \((A, B, C, X_2) \mapsto (A_2B_2C_2, X_2)\) where \(A_2B_2C_2\) is the medial triangle. In other words:

\[
\text{complem} (X) = \begin{bmatrix} C \end{bmatrix} \cdot X \quad \text{where} \quad C = \begin{bmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{bmatrix}
\]

\[
\text{anticomplem} (X) = \begin{bmatrix} C^{-1} \end{bmatrix} \cdot X \quad \text{where} \quad C^{-1} = \begin{bmatrix} -1 & 1 & 1 \\ 1 & -1 & 1 \\ 1 & 1 & -1 \end{bmatrix}
\]

Proof. Direct computation. 

Proposition 16.5.2. The cevian collineation wrt point \(P\) is defined as collineation \((A, B, C, P) \mapsto (A_PB_PC_P, P)\). Its matrix is:

\[
\phi_P = \begin{bmatrix} P^{-1} \end{bmatrix} \cdot \begin{bmatrix} C \end{bmatrix} \cdot \begin{bmatrix} P^{-1} \end{bmatrix} = \begin{bmatrix} 0 & p/q & p/r \\ q/p & 0 & q/r \\ r/p & r/q & 0 \end{bmatrix}
\]

where \(P^{-1}\) is to be understood as the reciprocal of matrix \(P\).

Proof. Composition of the two former collineations. 

Proposition 16.5.3. We have the following relation between conics:

\[
\text{complem} (X) \in \text{conicev} (\text{isotom} (U) , X_2) \iff X \in \text{conicir} (\text{complem} (U))
\]

16.6 Collineations and cevamul, cevadiv, crossmul, crossdiv

In this Section 16.6, the start point will ever be Table 3.2 (II) i.e. \(T_1 = C_P\) (the cevian of \(P\)), \(T_2 = ABC\), \(T_3 = A_U\) (the anticevian of \(U\)).

Proposition 16.6.1. Start as described, and use \(\phi_U = \begin{bmatrix} U \end{bmatrix} \cdot \begin{bmatrix} C \end{bmatrix} \cdot \begin{bmatrix} U^{-1} \end{bmatrix}\). This collineation is tailored so that \(\phi(T_3) = ABC\), \(\phi(T_2) = C_U\) and \(\phi(T_1)\) is the cevian triangle of \(\phi(P)\) wrt \(C_U\). Then:

\[
\phi \cdot \text{cevadiv} (P, U) = \text{crossdiv} (\phi \cdot P, \phi \cdot U) = \begin{bmatrix} w^2 \end{bmatrix} : \begin{bmatrix} v^2 \end{bmatrix} : \begin{bmatrix} u^2 \end{bmatrix} = P_U^#
\]

\[
\text{cevamul} (\phi^{-1}X, \phi^{-1}U) = \phi^{-1} \cdot \text{crossmul} (X, U) = \begin{bmatrix} w^2 \\ v^2 \\ u^2 \\ r^2 \end{bmatrix} = X_U^#
\]

— pkdx : Translation of the Kimberling’s Glossary into barycentrics —
Proof. Direct computation. The symmetry between \( U, X \) is broken by using \( \phi_U \).

\[
\text{Exercise 16.6.2. Explain why one obtains } \text{sqrtdiv}(U, P) \text{ and } \text{sqrtdiv}(U, X), \text{reverting the symmetry.}
\]

Proposition 16.6.3. Start as described and use \( \psi_P = \left[ \begin{array}{c} P \\ C \end{array} \right] \cdot \left[ \begin{array}{c} C^{-1} \\ P^{-1} \end{array} \right] \). This collineation is tailored so that \( \psi(T_1) = ABC, \psi(T_2) = A_P \) and \( \psi(T_3) \) is the anticevian of \( \psi(U) \) wrt \( A_P \). Triangles \( \psi(T_3) \) and \( \psi(T_2) \) are perspective wrt \( \psi(U) \) while \( \psi(T_3) \) and \( \psi(T_1) \) are perspective wrt \( \psi(X) \) and :

\[
\psi \cdot \text{cevadiv} \left( \psi^{-1} \cdot P, \psi^{-1} \cdot U \right) = \text{sqrtdiv} (P, U) = \frac{P^2}{u} \cdot \frac{u^2}{v} \cdot \frac{r^2}{w} = U_P^\#
\]

Proposition 16.6.4. Start as before, but use instead collineation \( (ABC, X_2) \mapsto (\text{ceva}(P), P) \) i.e. \( \psi_P = \left[ \begin{array}{c} C^{-1} \\ P^{-1} \end{array} \right] \). Then \( \psi(T_1) \) is ABC while \( \psi(T_2) \) is the anticomplement of \( \psi(U) \). Perspector between \( \psi(T_1) \) and \( \psi(T_2) \) is \( \psi(U) = X_2 \), perspector between \( \psi(T_2) \) and \( \psi(T_3) \) is \( \psi(P) = \text{anticomplem}(U/P) \) while perspector between \( \psi(T_1) \) and \( \psi(T_3) \) is isotomic conjugate of the former. In other words :

\[
X = \text{cevadiv}(P, U) = (\psi^{-1} \circ \text{isotom} \circ \psi)(U)
\]

16.7 Cevian conjugacies

Definition 16.7.1. The psi-Kimberling collineation of pole \( P \) is the collineation \( \psi_P \) such that \( ABC \mapsto \text{cevian}(P) \) and \( X_1 \mapsto P \). Therefore :

\[
\psi_P(U) = \begin{array}{c} P \\ b \end{array} \text{ complem} (U \mapsto_b X_1) \\
= \begin{array}{c} P \\ b \end{array} \frac{v}{w} : q \left( \frac{u}{a} + \frac{w}{c} \right) : r \left( \frac{u}{a} + \frac{v}{b} \right) \\
\psi_P^{-1}(U) = \begin{array}{c} X_1 \\ b \end{array} \text{ anticomplem} (U \mapsto_b P) \\
= a \left( \frac{-u}{p} + \frac{v}{q} + \frac{w}{r} \right) : b \left( \frac{u}{p} - \frac{v}{q} + \frac{w}{r} \right) : c \left( \frac{u}{p} + \frac{v}{q} - \frac{w}{r} \right)
\]

Remark 16.7.2. It is clear that \( \psi_P, \psi_P^{-1} \) are type-keeping when \( X_1 (a : b : c), P(p : q : r) \) and \( U(u : v : w) \) are transformed. Moreover, \( \psi_P(A) = A_P = 0 : q : r, \psi_P(X_1) = P \) (from the very definition) while \( \psi_P(-a : b : c) = A \) is obvious.

This \( \psi_P \) collineation has been used by Kimberling (2002a) to construct some new functions, following the patterns :

\[
\phi \mapsto \psi_P \circ \phi \circ \psi_P^{-1} \quad \text{or} \quad \phi \mapsto \psi_P^{-1} \circ \phi \circ \psi_P
\]

1. **cevadivision** of \( P \) by \( U \) can be re-obtained as \( \psi_P \circ \text{isogon} \circ \psi_P^{-1} \). The result \( X \) is the perspector of \( \text{cevian}(P) \) and \( \text{anticevian}(U) \). More about this operation in Section 3.11. One has the formulas :

\[
X = \left( -uqr + vrp + wqp \right) u : \left( uqr - vrp + wqp \right) v : \left( uqr + vrp - wqp \right) w \\
P = (vz + wy)^{-1} : (uz + wx)^{-1} : (yu + xv)^{-1}
\]

(a) cevadivision and ceva-multiplication are both type-keeping with respect to \( P \) and \( U \). Using isogonal conjugation result in the disappearing of \( a, b, c \) from the equations.

(b) fixed points are \( P = \psi_P(X_1) \) and the three vertices \( A = \psi_P(-a : b : c) \), since \( \pm a : \pm b : \pm c \) are the fixed points of \( \text{isogon} \). A brute force resolution leads also to the cevians of \( P \). A Taylor expansion around \( 1 : 0 : 0 \) shows that vertices are really fixed points of cevadivision, while a Taylor expansion around \( 0 : p : q \) shows that undetermined \( \psi_P(0 : q : r) = 0 : 0 : 0 \) must be determined as \( \psi_P(0 : q : r) = 0 : -q : r \).
2. **alephdivision** of $P$ by $U : \psi_P^{-1} \circ isogon \circ \psi_P$ (Hyacinthos #4111, Oct. 11, 2001). Formulas (cyclically):

$$x \simeq a \left( p^2 r^2 v^2 + q^2 p^2 w^2 - q^2 r^2 a^2 \right) + \frac{p^2 r^2 b^2 + q^2 r^2 c^2 - q^2 r^2 a^2}{bc} (vaw + ubw + cuv)$$

$$p^2 : q^2 : r^2 = \frac{1}{(bw + cu)(bz + cy)} : \frac{1}{(cu + aw)(cx + az)} : \frac{1}{(av + bu)(ay + bx)}$$

Therefore, the alephmultiplication gives four result, one inside the triangle $ABC$ and three outside.

3. **bethdivision** of $P$ by $U : \psi_P \circ sym3 \circ \psi_P^{-1}$ (Hyacinthos #4146, Oct. 26, 2001) where involution $sym3$ is the reflection in the circumcenter $X_3$. This involution $sym3$ is related to the Darboux cubic. Barycentrics are:

$$x \simeq a u - \frac{p (c + b)(a + c - b)}{q (-a + b + c)} v - \frac{p (c + b)(b + a - c)}{r (-a + b + c)} w$$

$$= -a u + \frac{\bar{p}}{\bar{q}} (c + b) v + \frac{\bar{p}}{\bar{r}} (c + b)$$

where $\bar{p} : \bar{q} : \bar{r} = P * X_7$

(a) This operation is type-keeping with respect to $P, U$.

(b) The fixed points of $U \mapsto \beta(P, U)$ are obtained by $\psi_P$ from the fixed points of $sym3$.

They are $\psi_P(X_3)$ together with $\psi_P(L_\infty)$, namely the line: $u/\bar{p} + v/\bar{q} + w/\bar{r} = 0$.

(c) Bethdivision of $X_{21} = a(b + c - a)/(b + c)$ by the circumcircle gives the circumcircle.

(d) Bethdivision of $P$ by $U$ is $P$ if and only if $U = P * X_{57}$.

(e) Bethmultiplication is not simple (equation of third degree).

**Exercise 16.7.3.** What are the situations where the discriminant vanishes?

4. **gimeldivision** of $P$ by $U : \psi_P^{-1} \circ sym3 \circ \psi_P$. Using barycentrics, one obtains:

$$F(U) = 16\sigma^2 U - \alpha \beta X_1 + 2\alpha \left( X_{48}_b * isot(P) \right)$$

where

$$16\sigma^2 = (b + c - a)(a + c - b)(b + a - c)(b + a + c)$$

$$\alpha = \frac{q + r}{a} + \frac{r + p}{b} + \frac{p + q}{c}$$

$$\beta = \frac{(b^2 + c^2 - a^2)a^2}{p} + \frac{(c^2 + a^2 - b^2)b^2}{q} + \frac{(a^2 + b^2 - c^2)c^2}{r}$$

(a) The fixed points of $U \mapsto \gamma(P, U)$ are obtained by $\psi_P^{-1}$ from the fixed points of $sym3$.

They are $\psi_P^{-1}(X_3)$ together with $\psi_P(L_\infty)$, namely the line: $u bc (q + r) + v ca (r + p) + w ab (p + q) = 0$.

(b) Gimel multiplication leads to three points on the triangle sides, and three other points.

5. **mimosa** aka "much ado about nothing" $X(1707)-X(1788)$. As with other names in ETC, the name Mimosa is that of a star. Define mimosa $(P)$ as $\psi_P^{-1}(X_3)$. Using barycentrics, one obtains:

$$mimosa(p : q : r) = u : v : w$$

where

$$u = a \left( \frac{(-a^2 + b^2 + c^2)a^2}{p} + \frac{b^2(a^2 - b^2 + c^2)}{q} + \frac{c^2(a^2 + b^2 - c^2)}{r} \right)$$

and also:

$$mimosa(P) = ccevadiv \left( X_{92}_b * P, X_1 \right)$$

$$mimosa^{-1}(U) = ccevamul \left( U, X_1 \right) + X_{63}$$

---

pldx : Translation of the Kimberling's Glossary into barycentrics ---
Then, marvelously, the Mimosa transform $M(X)$ arises in connection with the equation $\text{gimeldiv}(P,X) = X$. And there are too many such cases of gimel conjugates for all to be itemized in ETC... Here is a list of pairs $(I,J)$ for which $X(J) = M(X(I))$.

1 46 20 1712 48 43 71 846 85 1729
2 19 21 4 54 47 72 191 86 1730
3 1 27 1713 55 1721 73 1046 88 1731
4 920 28 1714 56 1722 74 1725 89 1732
6 1707 29 1715 57 1723 75 1726 90 90
7 1708 31 1716 58 1724 77 57 95 92
8 1158 35 1717 60 580 80 1727 97 48
10 1710 37 1719 63 9 81 579 98 1733
19 1711 40 1720 69 63 84 1728 99 1577

6. **zosma**, yet another star. $X(1824)-X(1907)$. The Zosma transform of a point $X$ is the isogonal conjugate of the inverse mimosa transform of $X$.

7. **dalethdivision** of $P$ by $U$:

\[
\phi_P \circ \text{hirst}_1 \circ \psi_P^{-1}
\]

where $\text{hirst}_1(X) = \text{hirstpoint}(X_1, X)$ and thus $U \neq P$. Using barycentrics, one obtains:

\[
x \simeq \left( \frac{w}{r} - \frac{v}{q} \right)^2 p - \left( \frac{u}{p} + \frac{v}{q} + \frac{w}{r} \right) u - 3 \frac{u^2}{p}
\]

(a) This operation is type-keeping with respect to $P, U$.

(b) The locus of fixed points of $\text{hirst}_1$ is the circumconic $cc(X_1)$.

Therefore, the locus of fixed points of $\text{daleth}_P$ is the conic $cvc(P,P)$ tangent to the sidelines of $ABC$ at the cevian points of $P$.

8. **hedivision** of $P$ by $U$:

\[
\psi_P^{-1} \circ \phi \circ \psi_P
\]

where $\phi(X) = \text{hirstpoint}(X_1, X)$ and thus $U \neq \psi_P^{-1}(P)$. Using barycentrics, one obtains:

\[
x \simeq -p \left( \frac{v}{b} + \frac{w}{c} \right)^2 + qa \left( \frac{u}{a} + \frac{w}{c} \right)^2 + ra \left( \frac{u}{a} + \frac{v}{b} \right)^2
+ \frac{rq a^2}{c b p} \left( \frac{u}{a} + \frac{v}{b} \right) \left( \frac{u}{a} + \frac{w}{c} \right) - \frac{qc p}{b r} \left( \frac{u}{a} + \frac{w}{c} \right) \left( \frac{v}{b} + \frac{w}{c} \right) - \frac{br p}{qc} \left( \frac{u}{a} + \frac{v}{b} \right) \left( \frac{v}{b} + \frac{w}{c} \right)
\]

(a) The locus of fixed points is a conic, but not a conic of cevians.

### 16.8 Miscellany

#### 16.8.1 Poles-of-lines and polar-of-points triangles

In what follows, indexes are to be taken modulo $3$.

**Definition 16.8.1. Polars-of-points triangle.** Consider the general triangle $T$ with vertices $T_i$, for $i = 1, 2, 3$. Taking the tripolars, we obtain a trigone. Taking the dual, we obtain the (may be degenerate) polars-of-points triangle $T_U = \text{pntpoltri}(T)$. Its vertices are:

\[
U_i = \text{tripolar}(T_{i+1}) \wedge \text{tripolar}(T_{i+2})
\]

**Definition 16.8.2. Poles-of-lines triangle.** Consider the general triangle $T$ with vertices $T_i$, for $i = 1, 2, 3$. Taking the dual trigone and then the tripoles, we obtain the (may be degenerate) poles-of-lines triangle $T_P = \text{linpoltri}(T)$. Its vertices are:

\[
P_i = \text{tripole}(T_{i+1} \wedge T_{i+2})
\]

**Remark 16.8.3.** An example of flat line-polar triangle is given by triangles sharing the circumcircle of $ABC$.
Lemma 16.8.4. The determinants of these triangles are:

\[ \det T_U = (\det T_{\text{isot}})^2 \quad \det T_P = (\det T\det T_{\text{isot}})^2 \prod_0 \text{Adjoint}(T) \]

where \(\det T_{\text{isot}}\) is either the determinant of the triangle of the isotomics or the determinant of the trigone of the tripolars, and \(\prod_0\) is the condition expressing that two vertices of \(T\) are aligned with a vertex of \(ABC\).

Proposition 16.8.5. When isotomic conjugates are collinear, \(T_U\) is totally degenerate. Otherwise \(\text{ptnptoltri}(T)\) is a triangle. Point-polarity is type-keeping (and both tribes share the same formula).

Proof. For example, the polar of \(P_1\) is the line \(x/p_1 + y/q_1 + z/r_1 = 0\), and the polars of \(P_2\) and \(P_3\) are defined cyclically. Then \(U_1\) is obtained as the common point of the last two lines. \(\square\)

Proposition 16.8.6. When \(T\) is flat (aligned points), then \(T_P\) is totally degenerate. When isotomic conjugates are collinear, \(T_P\) is flat, i.e. simply degenerate. Otherwise, \(\text{linptoltri}(T)\) is a triangle. The line-polarity transform is type-keeping (and both tribes share the same formula).

Proof. For example \(U_2 \land U_3\) gives the barycentrics of line \(U_2 U_3\), while tripole is transpose and invert : being the product of two type-crossing transforms, \(\text{linptoltri}\) is type-keeping. \(\square\)

Proposition 16.8.7. Line-polar and point-polar transforms are converse of each other... in the generic case. More precisely, \(\text{ptnptoltri}(\text{linptoltri}(T))\) gives \(T\) times \(\det T\), going back to any non degenerate triangle. On the contrary, \(\text{linptoltri}(\text{ptnptoltri}(T))\) gives \(T\) times \(1/\det T_{\text{isot}}\): the converse relation holds certainly when isotomic conjugates of points \(P_i\) aren’t collinear and points \(P_i\) aren’t collinear either and points \(P_i\) aren’t on the sidelines.

16.8.2 Unary cofactor triangle, eigencenter

Definition 16.8.8. The unary cofactor triangle of triangle \(U_i\) \((i = 1, 2, 3)\) is the triangle whose vertices are the isoconjugates of the vertices of the line-polar triangle of the points \(U_i\). This operator is type-crossing over the \(U_i\), but is nevertheless type-keeping over all involved points when using any fixed point \(F\) instead of \(P = F^2\). Using barycentrics:

\[ X_i = ^i(U_{i+1} \land U_{i+2}) * P \]

Proposition 16.8.9. When triangle \(U_1U_2U_3\) is degenerate (collinear vertices), then triangle \(X_1X_2X_3\) is totally degenerate (reduced to a point). Apart this situation, the unary cofactor transform is involutory.

Definition 16.8.10. Eigencenter. Any triangle \(U_1U_2U_3\) and its unary cofactor \(X_1X_2X_3\) are perspective. Their perspector is called the eigencenter of these triangles (formula don’t simplify, and has Maple-length 945).

When the original triangle is the cevian or the anticevian of a point \(U\), formula shorten into :

\[ \text{eigencenter}(C_U) = \text{anticomplem} \left( \left( U * U \right)_P \right)_P * U_P = \text{cevadiv}(U, U^*) \]

\[ \text{eigencenter}(A_U) = \text{anticomplem} \left( U * U \right)_P * P \]

These points are called, respectively, the eigentransform and the antieigentransform of point \(U\) (see Section 20.4.8).
Chapter 17

Cremona group and isoconjugacies

Notation 17.0.1. From now on, $a, b, c, d$ are four complex numbers involved in a Cremona homographic transforms, not to be confused with the $a, b, c$ which are used to note the sidelengths of a given triangle. In this context, $a', b', c', d'$ are four other (independent) complex variables, the relations $a' = \pi$, etc being assumed only for visible objects.

The context should be sufficient to avoid any confusion between $a$ and $a$, etc. Moreover, a careful reader will not confuse upright characters with italic ones.

17.1 Homographic Cremona transforms of the projective plane

Definition 17.1.1. The upper spherical map of the Morley space $\mathbb{P}_C(C^3)$ is the projection $Z : T : \mathbb{Z} \mapsto Z : T$, while the lower spherical map is the projection $Z : T : \mathbb{Z} \mapsto \mathbb{Z} : T$. Each of them sends $\mathbb{P}_C(C^3)$ onto (yet another copy) of the Riemann sphere $\mathbb{P}_C(C^2)$.

Definition 17.1.2. An homography $\psi$ is an element of $\text{PGL}_C(C^2)$. Such object can be seen as $z \mapsto \psi(z) = \frac{az + b}{cz + d}$ acting on the Riemann sphere $\overline{C}$.

Construction 17.1.3. Construct the middle of a subtangent $[J, K]$, even if $E$ is inside the conic (so that only $\text{pol}E$ and $BC$ are available, see Figure 17.1). Obtain $T$, the contact point and join to the center $O$. Then $OT$ cuts $FG = \text{pol}E$ at some point $L$. And finally, $EL$ cuts $BC$ at $M$ which is the required middle of $[J, K]$. See pappus (2017) for some context, and Construction 25.10.12 for an application.

![Figure 17.1: Construct the middle of a subtangent](image-url)
Consider the inscribed conic whose auxiliary line is spherical maps. Let $z_{F} = \alpha$ and let $z_{G} = 1/\alpha$. The tangents at $F$ and $G$ cut at a point $E$ on the real axis, and $x_{E} = 2\alpha/ (1 + \alpha^{2})$. We have $FG \simeq [\alpha : -1 - \alpha^{2} : \alpha]$.

Then define $J = E + kE$, $K = E + k'E$. If we require that $JK$ remains tangent to $C$, this induces an homography between parameters:

$$k' = \frac{-4k + 4}{(\alpha^{2} + 2 - \alpha^{2})k + 4} = \frac{x_{E}^{2}k - x_{E}^{2}}{x_{E}^{2}k - x_{E}^{2}}$$

Straightforward computations are leading to:

$$z_{T} = \frac{k(1 - \alpha^{2}) - 2\alpha}{k(\alpha^{2} - 1) - 2\alpha^{2}}; \quad z_{L} = \frac{(\alpha^{2} - 1)^{2}k^{2} + 4(\alpha^{2} - 1)k + 4}{(\alpha^{2} - 1)^{2}k^{2} + 4\alpha^{2}}$$

$$z_{M} = \frac{(\alpha^{2} - 1)k + 2}{2(\alpha^{2} + 1)}$$

And we can check that $z_{L} + z_{K} = 2z_{M}$.

The key point is that line $FG$ is the polar of $E$ and therefore remains visible even if $E$ goes inside the circle, while $\alpha$ ceases to be a "true" turn and $J, K$ become conjugate invisible points on the visible line $BC \simeq \left[\frac{k(\alpha^{2} - 1) - 2\alpha^{2}}{\alpha(k(\alpha^{2} - 1) + 2)}, \frac{\alpha(k(\alpha^{2} - 1) + 2)}{k(\alpha^{2} - 1) - 2\alpha^{2}}\right]$. \hfill \qedhere

\begin{remark}
Some French writers are using 'homography' as a synonym to collineations acting on $\mathbb{P}_{\mathbb{C}}(\mathbb{C}^{3})$. This is as stupid as possible. At High Schools, students have acquired (or should have acquired) some behavioral skills about these $\psi$ functions acting on the complex plane $\mathbb{C}$ and it's Alexandrov compactification $\mathbb{C}$. When introducing new concepts, here another set of points at infinity, one has to preserve what is already acquired.
\end{remark}

\begin{definition}
A Cremona homography combines two ordinary homographies, each of them acting on its own Riemann sphere. This can be seen as:

$$\begin{pmatrix}
\psi \\
\psi'
\end{pmatrix}
\begin{pmatrix}
Z \\
T \\
\overline{Z}
\end{pmatrix}
\simeq
\begin{pmatrix}
aZ + bT \\
cZ + dT \\
a'\overline{Z} + b'T \\
1 \\
c'\overline{Z} + d'T
\end{pmatrix}
\simeq
\begin{pmatrix}
(aZ + bT)(c'\overline{Z} + d'T) \\
(cZ + dT)(c'\overline{Z} + d'T) \\
(a'\overline{Z} + b'T)(cZ + dT)
\end{pmatrix}
$$

A visible homography is such that visible points are transformed into visible points. This implies the obvious relations of complex conjugacy between the coefficients of $\psi$ and $\psi'$.

\begin{theorem}
Consider a proper conic $C$ in the projective plane and two fixed tangents $\Delta_{1}, \Delta_{2}$ to that conic. A moving tangent $\Delta$ cuts $\Delta_{1}$ at $M$ and $\Delta_{2}$ at $N$. If we adopt two linear parametrization, $k$ on $\Delta_{1}$ and $K$ on $\Delta_{2}$, the relation $M \mapsto N$ induces an homography between parameters $k$ and $K$. Moreover correspondence $\Delta_{1} \mapsto \Delta_{2} : M \mapsto N$ can be extended into a transform $\Psi$ that acts into $\mathbb{P}_{\mathbb{C}}(\mathbb{C}^{3})$ and looks like a pair of homographies $\bar{\psi}, \bar{\psi}$, each of them acting onto one of the spherical maps.
\end{theorem}

\begin{proof}
Consider the inscribed conic whose auxiliary line is $Q = [u, v, w]$. Consider lines $AB, AC$ and parametrize by $M = kA + (1-k)B$, $N = KB + (1-K)C$. Assume that $MN$ is tangent to $C$ and obtain:

$$K = \frac{kw}{(u + v + w)k - v}$$

\end{proof}
This proves the first part. Using the Lubin transmutation, we have:

\[
M_z \doteq \begin{pmatrix} Z \\ T \\ \mathbb{Z} \end{pmatrix} \approx \begin{pmatrix} k \\ 1-k \\ 0 \end{pmatrix} \approx \begin{pmatrix} \frac{1}{k} + \frac{1}{\beta} \\ k(1-k) \end{pmatrix}
\]

\[
N_z \doteq \begin{pmatrix} Z' \\ T' \\ \mathbb{Z} \end{pmatrix} \approx \begin{pmatrix} K \\ 0 \\ 0 \end{pmatrix} \approx \begin{pmatrix} \frac{k(u+\nu+v)}{\alpha} + \frac{ku+vk-v}{\gamma} \end{pmatrix}
\]

Identifying with respect to parameter \( k \), we are conducted to define

\[
\psi = \begin{pmatrix} (\gamma (u+\nu) + \alpha \mu) \\ u+\nu + w \\ - (\beta \nu u + \gamma \alpha \nu + \alpha \beta w) \end{pmatrix}
\]

in order to have:

\[
\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} Z' \\ T' \\ \mathbb{Z} \end{pmatrix} = \begin{pmatrix} \psi_{11} & \psi_{12} & 0 \\ \psi_{21} & \psi_{22} & 0 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} Z \\ T \\ \mathbb{Z} \end{pmatrix}
\]

\[
\begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} Z' \\ T' \\ \mathbb{Z} \end{pmatrix} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & \psi_{22} & \psi_{21} \\ 0 & \psi_{12} & \psi_{11} \end{pmatrix} \begin{pmatrix} Z \\ T \\ \mathbb{Z} \end{pmatrix}
\]

**Theorem 17.1.7.** (Continued). Finally, the four focuses of \( C \) are the fixed points of the transform \( \Psi \) that acts into \( \mathbb{P}_2(C^2) \) while the projections of the focuses onto the upper spherical map are the two ordinary fixed points of \( \psi \in \text{PGL}_2(C^2) \).

**Proof.** Fixed points of \( \psi \) are the roots of:

\[
(u+\nu + w)Z^2 - (\alpha (w+u) + \beta (w+u) + \gamma (u+\nu)) ZT + (\beta \gamma u + \gamma \alpha v + \alpha \beta w) T^2 = 0
\]

This equation can be rewritten into:

\[
\frac{u}{Z - \alpha T} + \frac{\nu}{Z - \beta T} + \frac{w}{Z - \gamma T} = 0
\]

and this equation characterizes the focuses of the conic in the upper map. \( \Box \)

**Lemma 17.1.8.** Let \( \psi \) be an element of \( \text{PGL}_2(C^2) \), i.e. an homography \( z \mapsto (az + b)/(cz + d) \) of the Riemann sphere. If we assume that the fixed points \( f_1, f_2 \) of \( \psi \) are not equal, then \( \psi \) is characterized by a number \( \mu \in \mathbb{C} \) that is neither 0 nor \( \infty \). We call it the **multiplier** of \( \psi \) and we have:

\[
\mu \doteq \text{cross ratio} (f_1, f_2, z, \psi(z)) = \frac{c f_2 + d}{c f_1 + d} = \psi' (f_1) = \frac{1}{\psi'' (f_2)}
\]

**Proof.** To see that \( \text{cross ratio} (f_1, f_2, z_1, \psi(z_1)) = \text{cross ratio} (f_1, f_2, z_2, \psi(z_2)) \), use the fact that \( \text{cross ratio} (f_1, f_2, z_1, z_2) \) is invariant by an homography. \( \Box \)

**Lemma 17.1.9.** A more symmetric quantity is:

\[
\sigma \doteq \mu + \frac{1}{\mu} - 2 = \frac{(\mu - 1)^2}{\mu} = \frac{(a - d)^2 + 4bc}{ad - bc}
\]

**Proof.** Direct computation. \( \Box \)

**Remark 17.1.10.** An homography \( z \mapsto (az + b)/(cz + d) \) of the Riemann sphere is involutory when \( a + d = 0 \).

—— pldx : Translation of the Kimberling’s Glossary into barycentrics ——
Proposition 17.1.11. The focal transform:
\[ \begin{align*}
Z : T : \mathbb{Z} \mapsto & \left( f_1 - \mu f_2 \right) Z + \left( \mu - 1 \right) f_1 f_2 T : 1 : \left( g_1 - \mu g_2 \right) \mathbb{Z} + \left( \mu - 1 \right) g_1 g_2 T \\
& (1 - \mu) Z + (\mu f_1 - f_2) T 
\end{align*} \]

is transmuted into the following standardized transform:
\[ \psi_0 : \mathbb{Z} \mapsto (1 + \mu) Z + \left( 1 - \mu \right) T : 1 : (1 + \nu) \mathbb{Z} + (1 + \mu) T \]

by similarity:
\[ \begin{pmatrix} f_1 - f_2 & f_1 + f_2 & 0 \\ 0 & 2 & 0 \\ 0 & g_1 + g_2 & g_1 - g_2 \end{pmatrix} \]

while \( \psi_0 \) can be factored into:
\[ \psi_0 = \text{trans} \circ \sigma \circ \text{multi} \circ \text{trans} \]

where the Cremona transform \( \sigma \), the multiplication and the translation (the same translation is applied once and again, not once and the reverse afterward) are defined by:
\[ \sigma \left( \begin{pmatrix} Z \\ T \\ \mathbb{Z} \end{pmatrix} \right) = \left( \begin{pmatrix} 1/Z \\ 1/T \\ 1/\mathbb{Z} \end{pmatrix} \right), \quad \text{multi} = \left( \begin{pmatrix} \sigma_\mu & 0 & 0 \\ 0 & -4 & 0 \\ 0 & 0 & \sigma_\nu \end{pmatrix} \right), \quad \text{trans} = \left( \begin{pmatrix} 1 & \mu + 1 & 0 \\ \mu & \mu - 1 & 0 \\ 0 & 1 & 0 \\ 0 & \nu + 1 & \nu - 1 \end{pmatrix} \right) \]

Proof. Direct computation. \( \square \)

17.2 Defining the Cremona transforms

Definition 17.2.1. The Cremona group is defined as the set of the bi-rational transforms of \( \mathbb{P}(\mathbb{C})^2 \). Therefore a transformation \( \Psi \in \text{Cremona} \) can be written as:
\[ \psi \left( \begin{pmatrix} Z \\ T \\ \mathbb{Z} \end{pmatrix} \right) = \psi_1 \left( \begin{pmatrix} Z \\ T \\ \mathbb{Z} \end{pmatrix} \right) : \psi_2 \left( \begin{pmatrix} Z \\ T \\ \mathbb{Z} \end{pmatrix} \right) : \psi_3 \left( \begin{pmatrix} Z \\ T \\ \mathbb{Z} \end{pmatrix} \right) \]

where the \( \psi_i \) are three homogeneous polynomials of the same degree. And the existence of another transform \( \Phi \in \text{Cremona} \) is assumed so that, at least formally, \( (\Phi \circ \Psi) \left( \begin{pmatrix} Z \\ T \\ \mathbb{Z} \end{pmatrix} \right) \simeq \left( \begin{pmatrix} Z \\ T \\ \mathbb{Z} \end{pmatrix} \right) \).

Exercise 17.2.2. What can be said about the degrees when two transforms are inverse of each other? See Diller (2011)

Definition 17.2.3. Given a Cremona transform, we define the indeterminacy points and the exceptional curves by:
\[ \text{Ind} \left( \psi \right) = \left\{ M \mid \psi_1 (M) = \psi_2 (M) = \psi_3 (M) = 0 \right\} \]
\[ \text{Exc} \left( \psi \right) = \left\{ M \mid \det \left( \frac{\partial (\psi_1, \psi_2, \psi_3)}{\partial (Z, T, \mathbb{Z})} \right) = 0 \right\} \]

Exercise 17.2.4. What do you think about: A quadratic transformation \( f \in \text{Cremona} \) acts by blowing up three (indeterminacy) points \( \text{Ind} \left( f \right) = \{ p_1^+, p_2^+, p_3^+ \} \) and blowing down the (exceptional) lines joining them. Typically, the points and the lines are distinct, but they can occur with multiplicity. Then \( f^{-1} \) is also a quadratic transformation and \( \text{Ind} \left( f^{-1} \right) = \{ p_1, p_2, p_3 \} \) consists of the images of the three exceptional lines?

Remark 17.2.5. This simple relation between indeterminacy points and exceptional curves does not hold for higher degree transforms.

Theorem 17.2.6. The Cremona group is generated by collineations and the “inverse everything” transform \( \sigma : \left( \begin{pmatrix} Z \\ T \\ \mathbb{Z} \end{pmatrix} \right) \mapsto (T \mathbb{Z} : \mathbb{Z} \mathbb{Z} : ZT) \).

Proof. A detailed proof can be found in Alberich-Carramiñana (2002), and an historical sketch is given in Déserti (2009a). The idea is to separate infinitely neighbor points in \( \text{Ind} \left( \psi \right) \) if required and then proceed to a descending recursion over the cardinal of \( \text{Ind} \left( \psi \right) \). \( \square \)
17.2 Cremona group and isoconjugacies

17.3 Working out some examples

*** references Alexander (1916) Clobe (1922) Trkovska (2008) should be introduced here ***

Example 17.3.1. Transform $\rho$ is $\rho \left( Z : T : \overline{Z} \right) = \left( Z\overline{Z} : T \overline{Z} : T^2 \right)$. This is an involution. The set $\text{Ind} (\rho)$ contains $1 : 0 : 0$ (twice) and $0 : 0 : 1$. The exceptional locus is the reunion of the contraction lines $T = 0$ and $\overline{Z} = 0$. The decomposition of this transform can be conducted as described in Table 17.1.

<table>
<thead>
<tr>
<th>1</th>
<th>$\left( \begin{array}{c} x z \ y z \ y^2 \end{array} \right)$</th>
<th>$\left( \begin{array}{c} 1 \ 0 \ 0 \end{array} \right) = \left( \begin{array}{c} 1 \ 0 \ 0 \end{array} \right)$, $\left( \begin{array}{c} 0 \ 0 \ 1 \end{array} \right)$</th>
<th>$\left( \begin{array}{c} 1 \ 1 \ 0 \end{array} \right)$, $\left( \begin{array}{c} 0 \ 1 \ 0 \end{array} \right)$, $\left( \begin{array}{c} 0 \ -1 \ 1 \end{array} \right)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>2</td>
<td>$\left( \begin{array}{c} (x + y) z \ y z \ (y - z) y \end{array} \right)$</td>
<td>$\left( \begin{array}{c} 1 \ 0 \ 0 \end{array} \right)$, $\left( \begin{array}{c} 0 \ 0 \ 1 \end{array} \right)$</td>
<td>$\sigma$</td>
</tr>
<tr>
<td>3</td>
<td>$\left( \begin{array}{c} (y - z) y \ (x + y) (y - z) \ (x + y) z \end{array} \right)$</td>
<td>$\left( \begin{array}{c} 1 \ 0 \ 0 \end{array} \right)$, $\left( \begin{array}{c} 0 \ 0 \ -1 \end{array} \right)$, $\left( \begin{array}{c} 1 \ 1 \ 0 \end{array} \right)$, $\left( \begin{array}{c} 1 \ 1 \ 0 \end{array} \right)$</td>
<td></td>
</tr>
<tr>
<td>4</td>
<td>$\left( \begin{array}{c} (y - z) y \ (x + y) (y - z) \ (x + y) y \end{array} \right)$</td>
<td>$\left( \begin{array}{c} 1 \ 0 \ 0 \end{array} \right)$, $\left( \begin{array}{c} 0 \ 0 \ -1 \end{array} \right)$, $\left( \begin{array}{c} 1 \ 1 \ 0 \end{array} \right)$</td>
<td>$\sigma$</td>
</tr>
<tr>
<td>5</td>
<td>$\left( \begin{array}{c} x + y \ y \ y - z \end{array} \right)$</td>
<td>none</td>
<td>$\left( \begin{array}{c} 1 \ -1 \ 0 \end{array} \right)$, $\left( \begin{array}{c} 0 \ 1 \ 0 \end{array} \right)$, $\left( \begin{array}{c} 0 \ 1 \ -1 \end{array} \right)$</td>
</tr>
<tr>
<td>6</td>
<td>$\left( \begin{array}{c} x \ y \ z \end{array} \right)$</td>
<td>none</td>
<td></td>
</tr>
</tbody>
</table>

Table 17.1: Reduction of a Cremona transform

Example 17.3.2. Transform $\mu$ is $\mu \left( Z : T : \overline{Z} \right) = \left( Z\overline{Z} : Z^2 - T \overline{Z} : \overline{Z}^2 \right)$. This is an involution. The points of indeterminacy form a sequence of infinitely close neighbor $\mu_3 \ll \mu_2 \ll \mu_1 = 0 : 1 : 0$. Similarly, the exceptional locus is line $\overline{Z} = 0$, counted three times. Due to this specificity, its Cremona factorization is longer. A description of this process is given in Déserti (2008–2009), leading to a nine steps process $(\phi_5 \sigma \phi_4 \sigma \phi_3 \sigma \phi_2 \sigma \phi_1)$.

17.4 Isoconjugacy and sqrtdiv operator

Definition 17.4.1. Let us recall the heuristic definition of the sqrtdiv operator that was already given at Definition 1.4.10. We start from triangle $ABC$ and ‘fix’ a point $F$ not on the sidelines. If we call $f : g : h$ it’s $ABC$-barycentrics, then $U^\#_F$, the sqrtdiv image of $U \simeq u : v : w$ is defined by :

$$\text{sqrtdiv}_F (U) \doteq U^\#_F \doteq \frac{f^2}{u} : \frac{g^2}{v} : \frac{h^2}{w}$$

(17.1)
Remark 17.4.2. Maps $\text{sqrtdiv}$ and $\text{sqrtmul}$ are related with cevian nests (Table 3.2, case III).

Proposition 17.4.3. The operator $\text{sqrtdiv}$ is globally type-keeping and therefore is a pointwise transform. Seen as a $U \mapsto U^\#_P$ transform, we have clearly a Cremona-wise involution. The fixed points are the four $f \pm f \equiv g \pm h$, i.e. $F$ and its associates under the Lemoine transforms wrt triangle $ABC$. In other words, any of the $F_j$ and the vertices of its anticevian triangle.

Proof. Obvious from definition. \hfill \Box

Corollary 17.4.4. When $U$ is on line $F_j F_k$, so is $U^\#_P$ and we have :

$$\text{cross\_ratio} \left( F_j, F_k, U, U^\#_P \right) = -1$$

Proof. We have : $\det \left( F_1, F_2, U^\#_P \right) = (-gh/vw) \times \det \left( F_1, F_2, U \right)$. \hfill \Box

Construction 17.4.5. When a pair $P, P^\#$ is known, the image $M^\#$ of a given point $M$ can be constructed by ruler only. As a proof, the coordinates of the corresponding points are given, using $P \simeq p : q : r, P^\# \simeq u : v : w, M \simeq x : y : z$

1. $E_0 \equiv AM \cap CP^\# \simeq [uy, vy, vz]$
2. $E_1 \equiv PM \cap BC \simeq [0, qx - py, rx - pz]$
3. $E_2 \equiv AP \cap E_0 E_1 \simeq [puy, qvx, rvx]$
4. $E_3 \equiv AB \cap E_0 E_2 \simeq [uy (pz - rx), vx (qz - ry), 0]$
5. $M^\# \equiv CE_2 \cap E_3 P^\# \simeq \left[ \frac{pu}{x}, \frac{qv}{y}, \frac{rv}{z} \right]$

Theorem 17.4.6. Formal definition of the $\text{sqrtdiv}$ operator. Start from four independent points $F_1, F_2, F_3, F_4$ (three of them are never on the same line). Call $ABC$ their diagonal triangle, i.e. define

$$A = F_1 F_4 \cap F_2 F_3 ; B = F_2 F_4 \cap F_1 F_3 ; C = F_3 F_4 \cap F_1 F_2$$

For any point $U$ in the plane, draw both conics :

$$C_{12} \equiv \mathcal{C}(U, F_1, F_2, F_1 F_4 \cap F_2 F_3, F_2 F_4 \cap F_1 F_3) = \mathcal{C}(U, F_1, F_2, A, B)$$
$$C_{34} \equiv \mathcal{C}(U, F_3, F_4, F_3 F_4 \cap F_2 F_3, F_2 F_3 \cap F_1 F_4) = \mathcal{C}(U, F_3, F_4, A, B)$$

Their fourth intersection is independent of the order chosen for the set $\{F_j\}$ and is the $\text{sqrtdiv}_P$ image of $U$ that was formerly defined wrt $ABC$, the diagonal triangle of the $F_j$.

Proof. Choose an order over set $\{F_j\}$, use the above defined triangle $ABC$ as the reference triangle and let $f : g : h$ be the barycentrics of $F_4$ in this context. Then $F_1, F_2, F_3$ is the anticevian triangle of $F_4$ wrt $ABC$, enforcing $F_1 = -f : g : h$, etc. Compute the conics using the usual $6 \times 6$ determinant and obtain :

$$C_{12} \simeq \begin{bmatrix} 0 & h^2 w (fv + gu) & -g (fgw^2 + h^2 uv) \\ h^2 w (fv + gu) & 0 & -f (fgw^2 + h^2 uv) \\ -g (fgw^2 + h^2 uv) & -f (fgw^2 + h^2 uv) & 2fgw (fv + gu) \end{bmatrix}$$
$$C_{34} \simeq \begin{bmatrix} 0 & h^2 w (gu - fv) & g (fgw^2 - h^2 uv) \\ h^2 w (gu - fv) & 0 & f (h^2 uv - fgw^2) \\ g (fgw^2 - h^2 uv) & f (h^2 uv - fgw^2) & 2fgw (fv - gu) \end{bmatrix}$$

Computing their re-intersection is straightforward, and the symmetry of the result implies the independence from the way the set $\{F_j\}$ was ordered. \hfill \Box

Construction 17.4.7. The fixed points. Given generic $A, B, C, U, V$ the fixed points of the $ABC$-isoconjugacy that exchanges $U, V$ are obtained as intersection of conics $\gamma_U$ and $\gamma_V$ where $\gamma_U$ is the conic that goes through $U$, the anticevians of $U$ and cevadiv $(V, U)$, etc. Remark: line $UV$ is tangent at $U$ to $\gamma_U$ and at $V$ to $\gamma_V$. 

January 3, 2024 21:08 published under the GNU Free Documentation License
Proof. Equation of $\gamma_U$ is $(g^2 r^2 - h^2 q^2) x^2 + (h^2 p^2 - f^2 r^2) y^2 + (f^2 q^2 - g^2 p^2) z^2 = 0$. It is obvious that the four $F_j$ belong to this conic.

\[ \square \]

**Construction 17.4.8.** The fourth harmonic (harmonic conjugate of point $U \in (F_1, F_2)$ wrt points $F_1, F_2$) can be constructed by choosing two arbitrary points $F_3, F_4$ and then using the $\text{sqrtdiv}$ operator having the four $F_j$ as fixed points. If we chose $F_3, F_4$ in order to obtain the middle of $F_1 F_2$ and the two umbilics as triangle $ABC$, we obtain the usual construction using one circle from the pencil admitting $F_1 F_2$ as base points and one from the pencil admitting $F_1 F_2$ as limit points.

**Definition 17.4.9.** Isoconjugacy. Forget that $f^2 : g^2 : h^2$ are the square of the barycentrics of four points, and consider them as the barycentrics of a new point, called the pole $P = p : q : r$ of the transform. Then the isoconjugate of $U = u : v : w$ wrt pole $P$ is obtained by:

\[ U_P^* = (u : v : w)_P^* = p v w : q w u : r w \]  

(17.2)

**Remark 17.4.10.** This transform was introduced in order to unify isotonic conjugacy (Section 3.4) and isogonal conjugacy into a common frame... and also to deal with the reluctance towards imaginary focuses. When using barycentrics, isotonic conjugacy is obtained with $P = X_2$ and isogonal conjugacy with $P = X_6$. When using trilinear, you have to use (respectively) $P = X_{75}$ and $P = X_1$. When $X_2$ is special, its isogonal conjugate $X_6$ is special too. When $X_1$ is special, its isotonic conjugate $X_{75}$ is special too.

**Remark 17.4.11.** Isoconjugacy $U \mapsto U^*$, considered as a function of $U$ alone is type-crossing, so that it is not so clear to say that this mapping has four fixed points (real or not), namely the points $F_i = \pm \sqrt{b} : \pm \sqrt{q} : \pm \sqrt{r}$.

**Remark 17.4.12.** Most of the time, barycentric multiplication appears as the result of two successive isoconjugacies, according to:

\[ (X_6^*)_b^* = X_b \ast U_P^* \]

For example, $\text{isot (isog} (X)) = (X_6)_b^* \ast \text{isot} (X_6)$ while $\text{isog (isog} (X)) = (X_2)_6^* = X_b \ast \text{isog} (X_2)$

### 17.4.1 Some other constructions

**Remark 17.4.13.** A construction has already be given at Construction 3.12.1. It "suffices" to draw $T_2$ as the anticevian of $F$ and then $T_3$ as the anticevian of $U$ wrt triangle $T_2$. Then $\text{sqrtdiv}_V (U)$ is the perspector of $T_1 = ABC$ with $T_3$. But constructing anti-cevian triangles is not so easy.

**Fact 17.4.14.** Let be given $A, B, C, U, V$ in generic position (without alignments). Then the $ABC$-isoconjugacy $\psi$ that exchanges $U, V$ can be constructed as follows. Once for ever, point $P$ is chosen on line $UV$ and its cevian triangle $A_b B_b C_b$ is obtained, and triangle $T_1$ is constructed as:

\[ A_1 = VA \cap UA, B_1 = VB \cap UB, C_1 = VC \cap UC \]

Then consider a point $X$ not on the sidelines, define triangle $T_2$ by $A_2 = A_1 X \cap BC$ (etc) and triangle $T_3$ as cross $(T_1, T_2)$ i.e. $A_3 = B_1 C_2 \cap C_1 B_2$ (etc). It happens that triangle $ABC$ and $A_3 B_3 C_3$ are perspective, and this perspector is the required point $\psi (X)$.

**Proof.** Put $V = p : q : r$ and $U = u : v : w$. Express $R$ as $R = V - \rho U$. Existence of $A_1$ requires $w q - v r \neq 0$, i.e. $A, U, V$ not collinear. The result is:

\[ T_1 = \begin{pmatrix} \rho u & p & p \\ q & \rho v & q \\ r & r & \rho w \end{pmatrix}, \quad T_2 = \begin{pmatrix} 0 & \rho v w - p y & \rho v x - p z \\ \rho w y - q z & 0 & \rho w x - q z \\ \rho u z - r x & \rho u z - r y & 0 \end{pmatrix} \]

\[ T_3 = \begin{pmatrix} p (y w + z v) - \rho v w x & \rho u z & p y \\ q v z & q (z u + x w) - \rho w v y & q x v \\ r w y & r u x & r (x v + y u) - \rho w u z \end{pmatrix} \]

and the perspector is: $p u / x : q v / y : r w / z$ as required. In (Dean and van Lamoen, 2001), $\psi$ was called reciprocal conjugacy.

\[ \square \]
Fact 17.4.15. Another construction, with same hypotheses \((A,B,C,U,V)\) given, without alignments). Consider the conic \(\gamma\) that goes through \(A,B,C,U,V\), and the traces of \(UV\) on the sidelines, i.e. \(T_a \equiv UV \cap BC\), etc. For a moving point \(X\), call shadows of \(X\) the re-intersections \(X_a, X_b, X_c\) of lines \(AX, BX, CX\) and the conic \(\gamma\). Then traces and shadows are collinear, i.e. \(Y_a\) is the second intersection of \(T_aX_a\) with \(\gamma\), etc.

Proof. Direct computation. Here again \(wq - vr \neq 0\), etc is required. \(\square\)

17.4.2 Morley point of view

Proposition 17.4.16. Isogonal conjugacy. The Morley affix of the isogonal conjugate of point \(P = Z : T : \overline{Z}\) is given by :

\[
isog \left( \begin{array}{c} Z \\ T \\ \overline{Z} \end{array} \right) \simeq \begin{pmatrix} \sigma_3 \overline{Z}^2 - ZT - \sigma_1 \overline{Z}T + \sigma_1 T^2 \\ T^2 - \overline{Z} \overline{T} \\ \frac{1}{\sigma_3} Z^2 - \frac{\sigma_1}{\sigma_3} ZT - \overline{Z}T + \frac{\sigma_2}{\sigma_3} T^2 \end{pmatrix}
\] (17.3)

Proof. This formula can be stated using the representation of first degree. Start from point \(P\) and obtain \(\Delta_A\), the \(A\)-isogonal of line \(AP\), by solving :

\[
\Delta_A \cdot A = 0 \\
tan(AB, \Delta_A) + tan(AC, AP) = 0
\]

Compute \(\Delta_A \land \Delta_B\) and obtain a symmetric expression, proving that \(\Delta_C\) goes also through this point. One can check that \(isog(\Omega^+) = \Omega^-\) and vice versa. \(\square\)

Exercise 17.4.17. Use unimodular \(\alpha, \beta, \gamma\) to describe the reference triangle of the complex plane. Let \(P \approx Z : T : \overline{Z}\) be the generic point, \(P^*\) its isogonal conjugate wrt \(ABC\) and \(U = (P + P^*)/2\), let \(O\) be the circum-center, and \(N = X(5)\) the Euler center. Use formula (17.3), (15.7), and the usual law \(s \star t = (s + t)/(1 - st)\) to check that quantity :

\[
\psi (P) \doteq \text{tands}(Ox, OP) \star \text{tands}(Ox, OP^*) \star \text{tands}(Ox, NU)
\]

is independent of \(P\). One can also use barycentrics, and use \(BC\) instead of \(Ox\) as reference line. The question is: what does this prove?

Proposition 17.4.18. The four in-excenters of triangle \(ABC\) are enumerated by the polynomial :

\[
\zeta^4 - 2 \sigma_2 \zeta^2 + 8 \sigma_3 \zeta + (\sigma_2^2 - 4 \sigma_1 \sigma_3)
\] (17.4)

Proof. The fixed points of the isogonal transform are obtained by solving the equation

\[
T \isog(M) - (T^2 - \overline{Z} \overline{Z}) \ M = 0.
\]

This gives two polynomials of degree 3, that are not self-conjugate but conjugate of each other. The corresponding algebraic curves are not visible. Intersecting these curves, we obtain nine points. Among them, are both umbilics. The umbilical pair is fixed, but a given umbilic is not fixed. They are nevertheless appearing since both \(T\) and \(T^2 - \overline{Z} \overline{Z}\) are vanishing here. When computing the \(\overline{Z}\) resultant of these polynomials, we obtain an eight degree polynomial that factors into :

\[
T \times \prod_3 (Z - \alpha T) \times \text{poly}_4(Z, T)
\]

The umbilical pair is represented by \(T\), vertices are appearing and it remains the required polynomial of degree 4: this gives the \(2 + 3 + 7 = 9\) intersections of two degree 3 curves. To be sure of what happens, we can compute the \(T\) resultant of both polynomials and obtain :

\[
\overline{Z} \overline{Z} \times \prod_3 (Z - \alpha^2 \overline{Z}) \times \text{poly}_4(Z, \overline{Z})
\]

where each point is represented by a specific factor. \(\square\)

Remark 17.4.19. When substituting \(\alpha = \alpha^2\), etc (i.e. using the Lubin representation of second degree), polynomial (17.4) splits, with roots \(\pm \beta \gamma \pm \gamma \alpha \pm \alpha \beta\) as required.
17.4.3 The isogonal Morley formula

Proposition 17.4.20. Points $Z : T : \overline{Z}$ and $z : t : \zeta$ are isogonal conjugate of each other if and only if:

\[
\begin{cases}
\sigma_3 \zeta \overline{Z} + tZ + zT - \sigma_1 tT &= 0 \\
zZ + (\zeta T + t\overline{Z}) \sigma_3 - \sigma_2 tT &= 0
\end{cases}
\] (17.5)

Since these equations are complex conjugates of each other, only one relation is required for visible points.

Proof. This is Corollary 10.6.3. A method to find such an expression $\vartheta$ is as follows. We search coefficients such that:

\[zZc_{00} + tZc_{01} + \zeta Zc_{02} + zTc_{10} + tTc_{11} + \zeta Tc_{12} + z\overline{Z}c_{20} + t\overline{Z}c_{21} + \zeta \overline{Z}c_{22} = 0\]

when conjugacy occurs. This formula must be symmetric in $M$ and $M^*$, so that $c_{10} = c_{01}$, $c_{20} = c_{02}$, $c_{12} = c_{21}$. Writing that in-excenters are fixed points gives us 4 equations, leaving two indeterminate coefficients. One obtains, for example, $c_{01} \vartheta + c_{12} \vartheta'$. And we can go back to Lubin-1. But $\vartheta'$ happens to be the complex conjugate of $\vartheta$. Exactly as written, equation $\vartheta = 0$ defines a line. But this equation is not self-conjugate, and only a point of the line is visible. When cutting by $\text{conj} \vartheta$, we obtain a point (and a 2nd degree equation, as it should be).

Exercise 17.4.21. Apply the same method to the isotomic conjugacy, obtain the only possible formula... and conclude.

17.4.4 isoconjkim

In the old ancient times, Kimberling (1998) introduced another definition of the isoconjugacy, in an attempt to unify isotomic and isogonal conjugacies into a broader concept. Thereafter, this definition was changed into the one given above. To avoid confusion with (17.2), we will use the term isoconjkim to describe the older concept.

Definition 17.4.22. isoconjkim. For points outside the sidelines of $ABC$, the Kimberley $P$-isoconjkim of $U$ is the point $X$ such the product of the trilinears of $P,U,X$ gives $1 : 1 : 1$. Restated into barycentrics, this gives:

\[P \ast U \ast X = X_1 \ast X_1 \ast X_1\] (17.6)

In this transformation, $P$ and $U$ play the same role so that isoconjkim acts like a multiplication. On the other hand $U$ and $X$ play also the same role, so that isoconjkim$_P$ is involutory. Description $X = \text{isoconjkim}_P(U)$ reflects the fact that second point of view is the more useful.

Example 17.4.23. Here is a list of various isoconjkim transformations. The isogonal conjugation is $X_1$-isoconjkim while the isotonic conjugation is $X_{31}$-isoconjkim.

<table>
<thead>
<tr>
<th>$P$</th>
<th>barycentrics</th>
<th>trilinears</th>
<th>pole</th>
<th>bar(pole)</th>
<th>fixed</th>
</tr>
</thead>
<tbody>
<tr>
<td>$X$ (1)</td>
<td>$a^2 \frac{1}{a}$</td>
<td>$1 \frac{1}{a}$</td>
<td>$X$ (6)</td>
<td>$a^2$</td>
<td>$X$ (1)</td>
</tr>
<tr>
<td>$X$ (2)</td>
<td>$a^3 \frac{1}{a}$</td>
<td>$a \frac{1}{a}$</td>
<td>$X$ (31)</td>
<td>$a^3$</td>
<td>$X$ (365)</td>
</tr>
<tr>
<td>$X$ (3)</td>
<td>$(a^2 / \cos A) \frac{1}{a}$</td>
<td>$(1 / \cos A) \frac{1}{a}$</td>
<td>$X$ (19)</td>
<td>$a/S_a$</td>
<td>$X$ (???)</td>
</tr>
<tr>
<td>$X$ (4)</td>
<td>$a^2 \cos A \frac{1}{a}$</td>
<td>$\cos A \frac{1}{a}$</td>
<td>$X$ (48)</td>
<td>$a^2 S_a$</td>
<td>$X$ (???)</td>
</tr>
<tr>
<td>$X$ (6)</td>
<td>$a \frac{1}{a}$</td>
<td>$(1/a) \frac{1}{a}$</td>
<td>$X$ (1)</td>
<td>$a$</td>
<td>$X$ (366)</td>
</tr>
<tr>
<td>$X$ (19)</td>
<td>$a \cos A \frac{1}{a}$</td>
<td>$(\cos A/a) \frac{1}{a}$</td>
<td>$X$ (3)</td>
<td>$a^2 S_a$</td>
<td>$X$ (???)</td>
</tr>
<tr>
<td>$X$ (31)</td>
<td>$1 \frac{1}{a}$</td>
<td>$(1/a^2) \frac{1}{a}$</td>
<td>$X$ (2)</td>
<td>$1$</td>
<td>$X$ (2)</td>
</tr>
<tr>
<td>$X$ (48)</td>
<td>$(a / \cos A) \frac{1}{a}$</td>
<td>$1/(a \cos A) \frac{1}{a}$</td>
<td>$X$ (4)</td>
<td>$1/S_a$</td>
<td>$X$ (???)</td>
</tr>
</tbody>
</table>

— pldx : Translation of the Kimberling’s Glossary into barycentrics ——
17.5 Antigonal conjugacy

17.5.1 Angular coordinates

Remark 17.5.1. In this section, \( \alpha \), etc are not the later defined Lubin affixes. This remark can be understood as an anti-spoiler!

Definition 17.5.2. Let \( M \) be a point neither at infinity nor on the circumcircle. The three angles \( \alpha = (MB,MC) \), etc are called the angular coordinates of \( M \). According to the following proposition, the point \( M \) is characterized by \( \alpha, \beta, \gamma \).

Proposition 17.5.3. Let be given three finite numbers \( \cot \alpha, \cot \beta, \cot \gamma \). The locus of the points such that \( \cot (MB,MC) = \cot \alpha \) is a circle \( \mu_a \). The three circles \( \mu_j \) are concurrent in a point \( M \) if and only if

\[
(\cot \alpha - \cot A) (\cot \beta - \cot B) (\cot \gamma - \cot C) \neq 0
\]

and then we have : \( M \simeq \frac{1}{\cot (\alpha - \cot (A))} : \frac{1}{\cot (\beta - \cot (B))} : \frac{1}{\cot (\gamma - \cot (C))} \)

Conversely, if \( M = p : q : r \) then

\[
\cot (\alpha - \cot (A)) = -\frac{a^2qr + b^2pr + c^2pq}{2S(p + q + r)} \frac{1}{p}
\]

(17.7)

Proof. Obtain \( \mu_a \simeq S_a - 2S \cot \alpha : 0 : 1 \), etc from the very definition of \( \mu_a \). Then compute \( \nu = \bigwedge_3 (\mu_a, \mu_b, \mu_c) \) and require that \( \nu \cdot Q_b - 1 \cdot t \nu = 0 \). This gives a product of four factors. Condition \( \cot \alpha = \cot A \) leads to \( A \) or to the whole circumcircle, and is to be discarded. The last factor is the denominator of the well known addition formula. Thereafter, obtaining \( M \) is straightforward.

Example 17.5.4. Here are some points having simple angular coordinates

\[
\begin{array}{cccccccc}
X(1) & X(80) & X(36) & X(1) & X(265) & X(3) & X(4) & X(186) \\
-A& +\pi & -A & -\frac{\pi}{2} & -2A & 2A & A & 3A \\
a & aX(80) & aX(1) & 33599 & aX(3) & aX(265) & x & 5962 \\
i & 10260 & iX(1) & iX(36) & 5961 & x & iX(186) & iX(4) \\
g & gX(1) & 36 & 80 & gX(1) & gX(186) & gX(4) & gX(3) & gX(265) \\
A + \frac{\pi}{3} & A + \frac{\pi}{3} & A + \frac{\pi}{3} & A + \alpha & -\alpha & A + \alpha & A - \alpha \\
a & aX(13) & aX(14) & 11600 & 11601 & \ast & \ast & \ast \\
i & 6105 & 6104 & iX(16) & iX(15) & \ast & \ast & \ast \\
g & gX(16) & gX(15) & gX(13) & gX(14) & \ast & \ast & \ast \\
\end{array}
\]

Proposition 17.5.5. When \( M, M' \) are isogonal conjugates \( (g) \), then \( \alpha + \alpha' = A \), etc. When \( M, M' \) are inverse in the circumcircle \( (i) \), then \( \alpha + \alpha' = 2A \), etc.

Proof. Obvious from the definitions. One can also substitute the isogonal formula into \( 1/\cot(\alpha + \alpha' - A) \), or the invincircum formula into \( 1/\cot(\alpha + \alpha' - 2A) \)... and conclude using 17.7.

17.5.2 Antigonal conjugacy

Definition 17.5.6. Points \( M, N \) are antigonal conjugates when \( (MB, MC) + (NB, NC) = 0 \), etc. Using the former notations, this is \( \alpha + \alpha' = 0 \), etc.

Proposition 17.5.7. The antigonal transform is involutory and can be factored into :

\[
antigon = \text{isogon} \circ \text{invincircum} \circ \text{isogon}
\]
Proof. Obvious from the definition and Proposition 17.5.5.

Proposition 17.5.8. Let $A', B', C'$ be the reflections of a point $P$ in the sidelines $BC, CA, AB$ of the reference triangle. Then circles $ABC', A'B'C', A'BC, A'BC$ concur into a common point $Q$ Figure 17.2 (Schoute, 1882). This point $Q$ is antigon $(A, B, C, P)$, the antigonal conjugate of $P$ wrt $ABC$. When point $P$ is given either by $P_b \simeq p : q : r$ (barycentrics) or by $P_z \simeq Z : T : \overline{Z}$ (Lubin affixes), point $Q$ is given by:

\[
Q_b \simeq \frac{(a^2 - b^2 - c^2)p^2 + (a^2 - b^2)pr + a^2qr}{(b^2 - c^2 - a^2)q^2 + (b^2 - c^2)qr + (b^2 - a^2)qp + b^2rp} = Q_z \simeq \frac{-\sigma_3 Z^2 + \sigma_2 ZT + \sigma_1 Z^2 - \sigma_1 ZT + \sigma_3 T^2 + (\sigma_3 - \sigma_1 \sigma_2) ZT^2 + (\sigma_1^2 - \sigma_2) T^3}{\sigma_3 Z^2 - ZT - \sigma_2 ZT + \sigma_1 T^2 - \sigma_3 Z^2 + \sigma_2 T^2 + T \sigma_1 \sigma_2 Z - (\sigma_3 - \sigma_2 - \sigma_3) T^2 Z - \sigma_2 \sigma_3 ZT^2 + (\sigma_3^2 - \sigma_3) T^3}
\]

Proof. Direct computation for $Q_b$, then usual transform for $Q_z$. Angular coordinates are obvious from the symmetry... and can be checked using the $\tan$ formula.

![Figure 17.2: Antigonal conjugacy](image)

Proposition 17.5.9. Moreover, the conic that goes through $A, B, C, P, Q$ is a rectangular hyperbola. Therefore, it goes through $Q$ is antigon $(A, B, C, P)$ and through $H \simeq X(A, B, C, 4)$ but goes also through antigon $(P, B, C, A)$, etc and $X(P, B, C, 4)$, etc. Moreover, $(P, Q)$ are antipodes on this conic (and so are $(A, aA)$, etc. Circle $A'B'C'$ is centered at isogon $(P)$ and goes also through $Q$.

Proof. Direct computation. This result confirms the involutory nature of this transform.

Proposition 17.5.10. Antigonal conjugacy is a 5th degree Cremona transform. The indeterminacy set contains six points: $A, B, C, H$ and both umbilics. The exceptional locus is the reunion of six conics. Each of them goes through five points of $\text{Ind}(\text{antigon})$: circumcircle is blown-down into...
H, circle BCH (centered at B + C − O) is blown-down into A, idem for the other two vertices. Finally, curve:
\[ \gamma_y = Z^2 - \sigma_1 Z T - \sigma_3 \bar{Z} T + \sigma_2 T^2 \]
that goes through A, B, C, H, \( \Omega_y \) is blown-down into \( \Omega_y \) (the same umbilic), idem for the other umbilic.

**Proof.** Direct computation.

**Proposition 17.5.11.** Define the ig transform as \( \text{ig} = \text{invincircum} \circ \text{isogon} \) then point \( \text{ig} (A, B, C, P) \) is independent of the ordering of points A, B, C, P (Hyacinthos 20929).

**Proof.** Transform triangle ABC into ABP, and thus C into \( \phi^{-1}(P) \). Then use the usual change of triangle formula. Since ig only involves even powers of the sidelengths, everything goes fine... and the result follows.

**Proposition 17.5.12.** Seen as a Cremona map, the ig transform has the same indeterminacy locus as the antigonal transform. The exceptional locus contains three lines and three conics. A sideline like BC blows-down into the opposite vertex A. Curve \( \gamma_x \) (the same as before) now blows down to \( \Omega_x \), while \( \gamma_y \) blows-down to \( \Omega_x \). Finally, the circumcircle blows-down to its center X(3), while \( X(3) \mapsto \text{ig} X(186) \) is regular and X(265) is the only regular point that maps onto \( H = X(4) \).

**Proof.** This lack of symmetry (a curve that blows down to a regular point, a point of indeterminacy that does not blows-up under the reverse transform) is related to the fact that ig is not involutory.

### 17.6 Isogonality and perspectivity

**Lemma 17.6.1.** Results from Proposition 3.8.11. Let \( T_2 \) be a triangle perspective wrt \( T_1 = ABC \). Then triangle \( T_2 \), perspector \( P \) and perspectrix \( \Delta \) can be written as:

\[
T_2 = \begin{pmatrix}
  u & p & p \\
  q & v & q \\
  r & r & w
\end{pmatrix};
\quad P \simeq p : q : r;
\quad \Delta_2 = \begin{pmatrix}
  1 & 1 & 1 \\
  p - u & q - v & r - w
\end{pmatrix}
\]

Point \( U \simeq u : v : w \) is the perspector of \( T_1 \) and \( T_3 \doteq \text{cross}(T_1, T_2) \). But, in these formulas, the same proportionality factor must be applied to \( (p, q, r) \) and \( (u, v, w) \), so that factor \( k \doteq (u + v + w) / (p + q + r) \) is a projective quantity.

**Proposition 17.6.2.** Assume that vertices of \( T_2 \) are not on the sidelines of \( T_1 \), and perspectivity as in Lemma 17.6.1. Define \( T_2^* \) as the triangle whose vertices are the isogonal images of the \( T_2 \) vertices. Then triangles \( T_1 \) and \( T_2^* \) are perspective, and \( P, U \) are replaced by \( P^* \) and \( U^* \). Moreover, \( T_2 \) and \( T_2^* \) share the same perspectrix if and only if \( U = k P^* \). In this case, the isogon \( T_2^* \) and the crosstri \( T_3 \) are equal.

**Proof.** Computations are easy from the lemma. If \( A_1 \) is on \( BC \) then \( A_2 = A \) and everything degenerates, etc.

**Remark 17.6.3.** This obviously occurs with the 27 Taylor-Marr triangles since, for example, \( A, B_1, C_2 \) are aligned on the same trisector (and cyclically).
Chapter 18

Pencils of cycles in the complex plane

In this chapter we will transpose to the Complex Plane what has already been done for the Triangle Plane at Chapter 14. There are three usual ways for this purpose, each of them using its own basis for the cycles space.

Veronese \[ V_{\text{Ver}}(Z : T : Z) \simeq [ZT, T^2, \overline{Z}T, Z\overline{Z}] \]

Pedoe \[ V_{\text{Ped}}(Z : T : Z) \simeq [ZT, T^2, \overline{Z}T, Z\overline{Z} - T^2] \]

Spherical \[ V_{\text{Sph}}(Z : T : Z) \simeq [2XT, 2YT, \overline{Z}Z - T^2, Z\overline{Z} + T^2] \]

The first choice is the simpler, the better. Dividing by four the size of each element of a $4 \times 4$ matrix divides the whole size by 64. The second choice is for comparison with the illuminating Pedoe (1970).

The third choice is rather an introduction to the stereographic formalism. This allows the most nice and intuitive figures (but there is a price to pay: computations are slower!).

Notation 18.0.1. For the the further use of the reader, we summarize here all the notations that will be introduced throughout this chapter.

<table>
<thead>
<tr>
<th>index</th>
<th>barycent.</th>
<th>z Morley</th>
<th>p Pedoe</th>
<th>s Spherical</th>
</tr>
</thead>
<tbody>
<tr>
<td>Veronese</td>
<td>Ver</td>
<td>Verz</td>
<td>Verp</td>
<td>Vers</td>
</tr>
<tr>
<td>Ver(\infty)</td>
<td>Sirius</td>
<td>Sirius</td>
<td>Sirius</td>
<td>South pole</td>
</tr>
<tr>
<td>Q</td>
<td>mQQ</td>
<td>zQQ</td>
<td>pQQ</td>
<td>sQQ</td>
</tr>
<tr>
<td>Q^{-1}</td>
<td>mQQI</td>
<td>zQQI</td>
<td>pQQI</td>
<td>sQQI</td>
</tr>
<tr>
<td>[G] \mapsto G</td>
<td>mkgram</td>
<td>mkzgram</td>
<td>mkpgram</td>
<td>mksgram</td>
</tr>
<tr>
<td>eq \mapsto V</td>
<td>eq2colu</td>
<td>eq2coluz</td>
<td>eq2colup</td>
<td>eq2colus</td>
</tr>
<tr>
<td>V \mapsto (M, \rho^2)</td>
<td>colu2bar</td>
<td>coluz2mor</td>
<td>colup2mor</td>
<td>colus2mor</td>
</tr>
<tr>
<td>(M, \rho^2) \mapsto V</td>
<td>bar2colu</td>
<td>mor2colu</td>
<td>mor2coluz</td>
<td>mor2colus</td>
</tr>
<tr>
<td>C \mapsto V</td>
<td>mmz2colu</td>
<td>mmp2colu</td>
<td>mms2colu</td>
<td></td>
</tr>
<tr>
<td>h \mapsto \text{action}</td>
<td>mhatz</td>
<td>mhatp</td>
<td>mhats:</td>
<td></td>
</tr>
</tbody>
</table>

The induced objects will be named using the relevant index (z,p,s). For example, the (barycentric) matrix \( \frac{Q}{b} \) will be declined as \( \frac{Q}{z} \), \( \frac{Q}{p} \) and \( \frac{Q}{s} \).

Fact 18.0.2. In the Morley space, the equation

\[ p Z\overline{Z} + (q_1 Z + q_2 \overline{Z}) T + r T^2 = 0 \]
describes an ordinary, visible circle when \( p = 1, q = q_2 = \overline{q_1} \) and \( r^2 \neq q_1 - r > 0 \) (center is \(-q\), radius is \(\rho\)). When \( p = 0\), this describes the union of an ordinary line and the line at infinity (or the line at infinity where each point is counted twice, i.e. the horizon circle).

**Definition 18.0.3.** We will say that four points \( M_j \) are concyclic when
\[
\det_{j=1}^{4} \begin{vmatrix} Ver_z(M_j) \end{vmatrix} = 0
\]
Due to linearity, the \( Ver \) can be replaced (all the four at the same time) by the \( Ver \) or the \( Ver \).

**Definition 18.0.4.** Alternatively, we can say that a **cycle** is a conic that goes through the umbilics, i.e. the points :
\[
\Omega_x = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \quad \Omega_y = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}
\]
We call it a "circle" when the conic is either \( T^2 \) (the horizon circle) or isn’t degenerate, and a "line" when it degenerates into the union of the line at infinity and another line.

**Remark 18.0.5.** When searching the intersection of two circles, we better subtract the normalized equations, and obtain \( T \Delta \) where \( \Delta \) is the "flat" radical axis (first degree equation).

## 18.1 Pencil of cycles in the complex plane

### 18.1.1 Veronese map

**Definition 18.1.1.** The Veronese map used in the triangle plane was defined at (14.4). The Veronese map used in the Morley space is simply:
\[
Ver_z(Z : T : \overline{Z}) \simeq [ZT, T^2, \overline{ZT}, Z\overline{Z}]
\]

**Proposition 18.1.2.** Both umbilics are mapped to \([0,0,0,0]\) (points of indeterminacy) while all other points at infinity are mapped to \([0,0,0,1]\), called Sirius. When using \((Z : T : \overline{Z}) = L_u(x : y : z)\), we have
\[
Ver_z(Z : T : \overline{Z}) \simeq Ver_b(x : y : z) \cdot t L_{vz} \quad \text{where} \quad L_{vz} = \begin{pmatrix} Lu \\ [1,1,1] \end{pmatrix} 1/R^2
\]

**Proof.** The \( Lu \) part is obvious. The \( 1/R^2 \) acknowledges the fact that \((X_3, R)\) is the model of all circles in the triangle plane.

**Maple 18.1.3.** One can check this result by:
\[
(Factor@subs)(les1lon, Ver(retour.vz)).Tr(LLv): Factor(%);
\]

**Remark 18.1.4.** The Veronese map amounts to generate the projective space of all the cycles from four of them, namely the horizon circle \( T^2 \) (i.e. the line at infinity described twice), the fundamental isotropic lines (each of them completed by the line at infinity to obtain \( ZT \) and \( Z\overline{T} \)) and the "factored" circle \( ZZ \).

**Proposition 18.1.5.** The hyperplanes \( Ver_z(M_\tau) \) related to all the points \( M_\tau \) of a same cycle are all going through a same point of \( \mathbb{P}_\mathbb{C}(\mathbb{C}^4) \), called the circle representative, that will be noted \( V_z \).

**Proof.** When the cycle is a circle, then \( M_\tau \), the generic point, can be written as \( \begin{pmatrix} z \\ t \\ \zeta \end{pmatrix} /t + \rho \begin{pmatrix} \tau \\ 0 \\ 1/\tau \end{pmatrix} \), and we only have to check that \( \wedge_3 (Ver_z M_\tau, Ver_z M_{\tau+1}, Ver_z M_{\tau-1}) \) doesn’t depends on
As a result, we have:

\[
V_z \left[ \left( \begin{array}{c} z \\ t \\ \zeta \end{array} \right), \rho^2 \right] = \left( \begin{array}{c} z \zeta - t \rho^2 \\ t - z \\ +t \end{array} \right)
\]

When the cycle is the line \( \Delta \simeq [a, b, c] \) then \( V_z(\Delta) = a : b : c : 0 \).

**Proposition 18.1.6.** The Veronese row-images of the \( \mathbb{P}_C(\mathbb{C}^3) \) points belong to a 3D quadric:

\[
Ver_z(P) \cdot Q^{-1}_z \cdot ^tVer_z(P) = 0 \quad \text{where} \quad Q^{-1}_z = \left( \begin{array}{cccc} 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 \\ -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{array} \right)
\]

And the column-representatives of point-circles, i.e. the \( V_z(P) \simeq Q^{-1}_z \cdot ^tVer_z(P) \), belong to the 3D paraboloid defined by \( Q_z \). For a circle \( (\gamma) \), we have the more precise result:

\[
\rho^2 = -\frac{1}{2} \times \frac{^tV_z(\gamma) \cdot Q^{-1}_z \cdot V_z(\gamma)}{V_z(\gamma) \cdot N_z \cdot V_z(\gamma)}
\]

where \( Q_z = \left( \begin{array}{cccc} 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 \\ -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{array} \right) \) and \( N_z = \left( \begin{array}{cccc} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{array} \right) \).

**Proof.** We have \( Q^{-1}_z = L^{-1}_v \cdot Q^{-1}_b \cdot L^{-1}_v \) and \( Q_z = L_v \cdot Q \cdot ^tL_v \).

**Remark 18.1.7.** We have tried to use a set of factors \(-2\) and \(-1/2\) in order to obtain the squared radius itself in formula 18.3. But our final choice is to keep the value \( d^2 - \rho_1^2 - \rho_2^2 \) as the result of 18.4.

**Maple 18.1.8.** One can check these results by:

(\( LLv \cdot (FActor@subs)(les1lon, mQQ).Tr(\( LLv \)) : \text{FActor(\%)} : \text{zipd}(zQQ, \%) ; \rightarrow \mathbb{R}^2 \)

\( \text{Tr}(1/\( LLv \)).(\text{FActor@subs})(les1lon, mQQI).\text{(1/LLv)} : \text{FActor(\%)} ; \text{zipd}(zQQI, \%) ; \rightarrow 1/\mathbb{R}^2 \)

\( \text{(nor4@mor2colu)(vz,K)} : \text{factor(\( \text{Tr(\%) \cdot zQQ \cdot \%} \))} ; \rightarrow K \)

**Proposition 18.1.9.** Two point-circles are orthogonal wrt the quadric when their centers share one of their two coordinates.

**Proof.** Assuming \( M_j \simeq z_j : t_j : \zeta_j \), we have the more precise result:

\[
\frac{^t \left( V_z[M_1, r_1^2] \right) \cdot Q^{-1}_z \cdot \left( V_z[M_2, r_2^2] \right)}{^t \left( V_z[M_1, r_1^2] \right) \cdot N_z \cdot \left( V_z[M_2, r_2^2] \right)} = \left( \frac{z_2}{t_2} - \frac{z_1}{t_1} \right) \left( \frac{\zeta_2}{t_2} - \frac{\zeta_1}{t_1} \right) - r_1^2 - r_2^2
\]

**Remark 18.1.10.** And then, business as usual, using the adapted matrices.

**Maple 18.1.11.** The Maple package ‘faisceaux’ contains:

\[
\text{constants: } "zQQ" = Q^{-1}_z \quad "zQQI" = Q^{-1}_z
\]

---

pldx : Translation of the Kimberling’s Glossary into barycentrics ---
functions: "Verz" = \( V_{er} \), "mor2colu" = \( V \), "colu2mor", "eqz2colu", "coluz2mm", "mmz2colu", "mkzgram"

**Theorem 18.1.12.** *Common orthogonal cycle.* Let be given three cycles \( \Omega_1, \Omega_2, \Omega_3 \). If they don’t belong to the same pencil, the bundle they generate is exactly the set of all the cycles orthogonal to a fixed cycle \( \Omega_\perp \). We have the formulas:

\[
W = \bigwedge_3 \left( \frac{\mathcal{V}_1}{z}, \frac{\mathcal{V}_2}{z}, \frac{\mathcal{V}_3}{z} \right) \quad \text{(18.1)}
\]

\[
\mathcal{V}_\perp = \left[ \begin{array}{cl} \mathcal{Q}^{-1} & \mathcal{f} \end{array} \right] \quad \text{(18.2)}
\]

center = \( W_1 : W_2 : W_3 \)

squared radius = \( \left( \frac{1}{2} \right)^2 \frac{W \cdot \mathcal{V}_\perp}{(\mathcal{V}_\perp | z)} \) \( \bigwedge_3 \) \( \mathcal{V}_\perp \cdot \mathcal{Q} \cdot \mathcal{V}_\perp \)

**Remark 18.1.13.** These formulas are –and should remain– exactly the same as the barycentric formulas (14.18). Don’t even think of using any trick to "simplify" anything. Only remember that

\[
\mathcal{V}_1 : \mathcal{Q} : \mathcal{V}_2 = (d_1^2 - r_1^2 - r_2^2) \times (\mathcal{V}_1 | z) \mathcal{V}_2 | z \)

**Proposition 18.1.14.** When \( \Delta \) = \( \left( \mathcal{V}_1 \wedge \mathcal{V}_2 \right) \) describes the pencil generated by cycles \( C_1, C_2 \), the orthogonal pencil is described by:

\[
\Delta_\perp \simeq \frac{\mathcal{Q}}{z} \cdot \Delta^* \cdot \frac{\mathcal{Q}}{z} \quad \text{(18.3)}
\]

**Proof.** See the proof of (14.19). \( \blacksquare \)

### 18.1.2 Circular group acting over the cycles’ space

CAVEAT: in this subsection, letters a, b, c, d, a’, b’, c’, d’, k, \( \kappa \) are general complex numbers, while \( a’ = \pi, etc. \) is intended for visible objects.

**Proposition 18.1.15.** We will say that \( H \), a Cremona transform \( H \) acting over \( \mathbb{P}_C (\mathbb{C}^3) \), is an homography when \( H \) is seen as \( h : z \rightarrow (az + b) / (cz + d) \) on the upper sphere \( \mathbb{Z} : T \) and seen as \( \tilde{h} : \zeta \rightarrow (a’\zeta + b') / (c'\zeta + d') \) on the lower sphere \( \mathbb{Z} : T \). As already said, a, b, c, d, a’, b’, c’, d’ \( \in \mathbb{C} \), together with \( a - b - c \neq 0, a' - b' - c' \neq 0 \) are assumed. Therefore, we have:

\[
H : \left( \begin{array}{ccl} Z & T \end{array} \right) \simeq \left( \begin{array}{ccl} (aZ + bT) (c'Z + d'T) & (cZ + dT) (c'Z + d'T) \end{array} \right) \quad \text{(18.4)}
\]

Assuming that \( cc' \neq 0 \), the points of indeterminacy are both umbilics, and the so-called pole: \( P = -d/c : 1 : -d'/c' \). The exceptional locus is the union of \( L_z, P\Omega_x, P\Omega_y \) while \( H (L_z) \) collapses to a single point, the elop \( E \simeq a/c : 1 : a'/c' \) (i.e. the pole of \( H^{-1} \)).

**Proof.** This is nothing but the usual formulas: \( h^{-1}(\infty) = -d/c \) and \( h(\infty) = a/c \). \( \blacksquare \)

**Proposition 18.1.16.** Homography \( H \), as an action over the points of \( \mathbb{P}_C (\mathbb{C}^3) \), induces an action which is linear over the columns of \( \mathbb{P}_C (\mathbb{C}^4) \). Its matrix is:

\[
\bar{H}_z \simeq \frac{1}{|\det|} \left[ \begin{array}{ccccc} +ba' & -ca' & +cb' & -db' \\ -ba' & +aa' & -ab' & +bb' \\ +b'a' & -ac' & +ad' & -bd' \\ -dc' & +ec' & -cd' & +dd' \end{array} \right] \quad \text{(18.5)}
\]

and we have \( \bar{H}_z \cdot \frac{\mathcal{Q}}{z} \cdot \bar{H}_z = \frac{\mathcal{Q}}{z} \) where \( |\det| = \sqrt{(a'd - b)c'(a'd' - b'c')} \) \( \text{(18.6)} \)

As a result, the image of a pencil of cycles is a pencil of cycles, while orthogonality is preserved.
Proof. $H^{-1}$ is obtained by the substitution $a \leftrightarrow -d$; $a' \leftrightarrow -d'$ into (18.7). And then one identifies $V^{-1}_z(H^{-1}(Z : T : Z)) = V^{-1}_z(Z : T : Z)$.

Proposition 18.1.17. Inversion of cycles in a cycle. Let $\Omega_0$ be a fixed cycle, and define the transformation $\sigma$ by

$$\sigma \doteq \text{Id} - 2 V_z(\Omega_0) \cdot \frac{Q}{V_z(\Omega_0)} \cdot \frac{Q}{V_z(\Omega_0)}$$

(18.10)

Then a point circle is mapped onto a point circle, and the corresponding action is the reflection into the circle $\Omega_0$.

Proof. Compute $^{t}\left(\begin{array}{c} \hat{\sigma} \\ \hat{X} \end{array}\right)$ and re-obtain $^{t}\hat{X} \cdot \frac{Q}{\hat{X}}$. Moreover, $\gamma$ is invariant ($\lambda = -1$) and so is any cycle orthogonal to $\gamma$ ($\lambda = +1$). One can also use (14.21).  

Proposition 18.1.18. When the ordinary homography $h$, acting over $\mathbb{P}_c(C^2)$, has exactly two fixed points $z_1, z_2$ then conjugating by $g : z \mapsto (z - z_1)/(z - z_2)$ leads to $(g^{-1} \circ h \circ g) (z) = \mu z$ where $\mu = \frac{\partial h}{\partial z}|_{z_1} \in \mathbb{C} \setminus \{0, 1\}$. As a result, $H$ acts over $\mathbb{P}_c(C^3)$ according to:

$$H \begin{pmatrix} Z \\ T \\ Z \end{pmatrix} \simeq \begin{pmatrix} (t_2 z_1 - k \kappa t_1 z_2) Z + (k \kappa z_2 z_1 z_2 - z_2 z_1) T \\ (t_1 t_2 - t_1 t_2 k \kappa) Z + (k \kappa t_2 z_1 - t_2 z_1) T \\ (t_2 t_1 - k \kappa t_2 t_1) \frac{Q}{\hat{X}} + (k \kappa t_1 t_2 - k \kappa t_1 t_2) \frac{Q}{\hat{X}} \end{pmatrix}$$

$H$ has four fixed points $F_1$ and $\hat{H}_z$ admits a basis of eigencolumns, namely the $\hat{s}_1 \simeq V_z(F_j)$. Using $\mu = k \kappa$ where $\mu = k / k \kappa$, we have:

$$F_1 \simeq \begin{pmatrix} z_1 \\ \frac{z_1}{t_1} \\ \frac{z_1}{t_2} \end{pmatrix}, \begin{pmatrix} \frac{z_1}{t_2} \\ \frac{z_1}{t_1} \\ \frac{z_1}{t_2} \end{pmatrix} \ldots \hat{s}_1 \simeq \begin{pmatrix} -t_2 \xi_2 \\ -\xi_1 \cdot \xi_2 - t_1 \xi_2 - t_1 \xi_2 - t_1 \xi_2 \\ z_2 \xi_2 \\ -\xi_1 \cdot \xi_2 - t_1 \xi_2 - t_1 \xi_2 - t_1 \xi_2 \\ -t_2 \xi_2 \\ -\xi_1 \cdot \xi_2 - t_1 \xi_2 - t_1 \xi_2 - t_1 \xi_2 \\ t_2 \xi_2 \\ t_2 \xi_2 - t_1 \xi_2 - t_1 \xi_2 - t_1 \xi_2 \end{pmatrix}$$

$\hat{s}_1^{-1} \cdot \hat{H}_z \cdot \hat{s}_1 \simeq \begin{pmatrix} k & 0 & 0 \\ 0 & k & 0 \\ 0 & 0 & 1/k \end{pmatrix}$

Proof. Straightforward computation.

Proposition 18.1.19. When the ordinary homography $h$, acting over $\mathbb{P}_c(C^2)$, has exactly one fixed point $z_1$ then conjugating by $g : z \mapsto 1/(z - z_1)$ leads to $(g^{-1} \circ h \circ g) (z) = z + k$ where $k \in \mathbb{C} \setminus \{0\}$). Acting over $\mathbb{P}_c(C^3)$, $H$ has only one fixed point $F \simeq z_1 : t_1 : \xi_1$, while $\hat{H}_z$ can be reduced in Jordan form according to:

$$\hat{s}_1 \simeq \begin{pmatrix} -k \\ k \frac{z_1}{t_1} - \kappa \frac{\xi_1}{t_1} \\ \frac{\kappa}{t_1} \\ 0 \end{pmatrix}$$

$$\hat{s}_1^{-1} \hat{H}_z \hat{s}_1 \simeq \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & -2 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix} \cdot \frac{Q}{\hat{X}} \hat{s}_1 \simeq \begin{pmatrix} -k \xi_1 \\ -\kappa \xi_1 - k \xi_1 \xi_1 \\ -\kappa \xi_1 \xi_1 - k \xi_1 \xi_1 \\ 1 \end{pmatrix}$$

— pldx : Translation of the Kimberling’s Glossary into barycentrics —
Proof. The factorization is easy to check. Let us give some rationales for this choice of the matrix \( \Theta \).
One has cross\_ratio \( (z - k, z + k, z, \infty) = -1 \). By conjugacy, one has also cross\_ratio \( (h^2 M, M, h M, F) = -1 \). Therefore, each cycle \((F, M, h M)\) is globally invariant.

Let \( P \) and \( E \) be the pole and the elop, i.e. \( P = h^{-1}(\infty) \) and \( E = h(\infty) \). From the previous result, the line \((PFE)\) is invariant (our first column). The tangent pencil generated by the circle point \( F \) and line \((PE)\) is therefore invariant. So is its orthogonal pencil, so that \( med[E, F] \) is a good candidate for the third column. And we use \( L_z \) for the last one.

Finally, the expansion of \( \Theta^{-1} \cdot \left[ \begin{array}{c} H_z \\ z \end{array} \right] \cdot \Theta \) is obtained using \( 1/ \left( 1 - X \times \Theta^{-1} \cdot \left[ \begin{array}{c} H_z \\ z \end{array} \right] \right) \). \( \square \)

Exercise 18.1.20. For \( n \in \mathbb{N} \), let \( \gamma_n = h^n(\gamma_0) \) be the iterated images of a cycle \( \gamma_0 \). Describe this circle by \( x : y : z : t \), the coordinates of \( \gamma_0 \) wrt \( \Theta \) as a basis, and find the condition such that \( \gamma_0 \) and \( \gamma_1 \) are tangent. Show that, in this case, cycles \( \gamma_n \) and \( \gamma_{n+1} \) are tangent for all \( n \). A first case is when all the circles are going through \( F \).

Otherwise, let \( N_n \) be the contact point of \( \gamma_n \) and \( \gamma_{n+1} \). Show that all the \( N_n \) belong to a same circle \( q_0 \) while all the circles \( \gamma_n \) are tangent to two fixed circles \( q_1, q_2 \) that are inverse wrt \( q_0 \). Check your results by inversion into \( C_4 \).

Exercise 18.1.21. Explore the following situation. A similitude \( (M, k, \tau) \) acts over the cycles space. And the four proper spaces are:

\[
\begin{align*}
    k : L_\infty ; & \quad \frac{1}{k} : C(M, 0) ; \\
    \tau : \Omega_y \wedge M ; & \quad \frac{1}{\tau} : \Omega_x \wedge M
\end{align*}
\]

18.2 Revisiting the Euler pencil

Remark 18.2.1. Cycles representatives \( \mathcal{V} \) are columns. Thus \( \left( \mathcal{V}_1 \wedge \mathcal{V}_2 \right) \) is a square matrix. The reader can safely ignore the "colu" subscript here. This is only a reminder to the author that representatives of cycles are written in column since the last change (so many years ago).

The Euler pencil has been treated in detail at Section 14.8. Let us examine again this pencil, in the light of the new formalism.

1. We use the Lubin-1 representation, i.e. \( A^2 \simeq \alpha : 1 : 1/\alpha \). The representative of \( \Gamma \) is known to be \( V_{cir} \simeq 0 : -1 : 0 : 1 \). The Euler circle goes through the midpoints. It’s normalized representative is therefore:

\[
\bigwedge_3 (V_{cir} \wedge (A^2 + B^2), \text{etc}) \simeq V_{eul} \simeq \frac{1}{4 \sigma_3} \begin{pmatrix}
    -2 \sigma_2 \\
    \sigma_1 \sigma_2 - \sigma_3 \\
    -2 \sigma_1 \sigma_3 \\
    4 \sigma_3
\end{pmatrix}
\]

2. Applying (18.3), we have

\[
\tau V_{eul} \cdot \left[ \begin{array}{c}
    \mathcal{V}_{cir} \\
    \mathcal{V}_{eul}
\end{array} \right] = \left( \begin{array}{c}
    \sigma_1 \\
    \frac{1}{2} \\
    \frac{2 \sigma_2}{\sigma_3} \\
    \frac{\sigma_1 \sigma_2 - \sigma_3}{4 \sigma_3}
\end{array} \right) \mathcal{V}_{eul} = \frac{-1}{2}
\]

i.e. a confirmation of center \( X(5) \) \( z = \sigma_1/2 \) and radius \( 1/2 \).

3. The line representative of the Euler pencil (generated by circum- and Euler circles) is therefore:

\[
Euler_{colu} \simeq \left( \mathcal{V}_{cir} \wedge \mathcal{V}_{eul} \right) \simeq \begin{pmatrix}
    0 & 2 \sigma_1 \sigma_3 & \sigma_1 \sigma_2 + 3 \sigma_3 & 2 \sigma_1 \sigma_3 \\
    -2 \sigma_1 \sigma_3 & 0 & 2 \sigma_2 & 0 \\
    -\sigma_1 \sigma_2 - 3 \sigma_3 & -2 \sigma_2 & 0 & -2 \sigma_2 \\
    -2 \sigma_1 \sigma_3 & 0 & 2 \sigma_2 & 0
\end{pmatrix}
\]
4. Consider a point \( F \simeq z : t : \zeta \). Its Veronese image is \([zt, t^2, \zeta t, z\zeta]\) and the associated point circle is represented by \( V_F \simeq -t\zeta : z\zeta : -tz^2\). This circle belongs to the Euler pencil when \( t_{V_F} \cdot \text{Euler}^z_{colu} = 0 \). One obtains

\[
F_\pm \simeq \left( \frac{\sigma_1 \sigma_2 + 3 \sigma_3}{\sigma_2} \pm \frac{\sigma_3 W}{\sigma_2} \right)
 \quad \text{where } W^2 = \left( 9 - \frac{\sigma_1 \sigma_2}{\sigma_3} \right) \left( 1 - \frac{\sigma_1 \sigma_2}{\sigma_3} \right)
\]

5. The midpoint of \( F_+, F_- \) is \( X(468) \). This is recognizable to the fact that \( z_{468} = (z_4 + 3z_{186})/4 \) where \( z_4 = \sigma_1 \) is the affix of the orthocenter \( X(4) \) and \( z_{186} = \sigma_3/\sigma_2 \) is the affix of the inverse of \( X(4) \) in the circumcircle, i.e. \( X(186) \).

6. The direction of line \( F_+, F_- \) is \( \sigma_1 \sigma_3 : 0 : \sigma_2 \) i.e. \( X(30) \)... the direction of the Euler line. This was obvious from the beginning.

7. Both factors of quantity \( W^2 \) are real. This can be seen from \( \overline{\sigma_1} = \sigma_2/\sigma_3 \). Since \( |\sigma_1| < 3 \), the first factor is ever positive. The second factor can be split into :

\[
+1 - \frac{\sigma_1 \sigma_2}{\sigma_3} = \frac{(\beta + \gamma)(\alpha + \gamma)(\beta + \alpha)}{\alpha \beta \gamma}
\]

Thus \( W^2 = 1 \) for equilateral triangles (\( \sigma_1 = 0 \)) and vanishes only for right-angled triangles, so that \( W^2 > 0 \) characterizes acute triangles. One can also use:

\[
9 - \frac{\sigma_1 \sigma_2}{\sigma_3} = \frac{32 S_a S_b S_c^2}{a^2b^2c^2} ; \quad 1 - \frac{\sigma_1 \sigma_2}{\sigma_3} = \frac{8 S_a S_c S_b}{a^2b^2c^2}
\]

8. Points \( F_\pm \) are the base points of the Euler pencil, seen as an isotomic pencil. Consider now the isoptic pencil of the cycles that are orthogonal to all of the cycles of the Euler pencil (i.e. the orthic pencil). When \( V_1, V_2 \) are the representatives of two orthogonal cycles, then \( t_{V_1} \cdot V_2 = 0 \). Thus \( t_{V_{colu}} \cdot Q = 0 \) is an hyper-plane that describes the bundle of cycles orthogonal to \( \Omega_{colu} \). And therefore

\[
\text{Orthic}_{colu} = \text{dual} \left( \left( \left( t_{V_a} \cdot Q \right) t_{V_b} \left( t_{V_b} \cdot Q \right) \right) \right)
\]

\[
\simeq \begin{pmatrix}
0 & -2 \sigma_1 \sigma_3 & 0 & 2 \sigma_1 \sigma_3 \\
2 \sigma_1 \sigma_3 & 0 & 2 \sigma_2 & \sigma_1 \sigma_2 + 3 \sigma_3 \\
0 & -2 \sigma_2 & 0 & 2 \sigma_2 \\
-2 \sigma_1 \sigma_3 & -\sigma_1 \sigma_2 - 3 \sigma_3 & -2 \sigma_2 & 0
\end{pmatrix}
\]
9. Using the same method as above, we search the centers of the point-circles that belong to the Orthic pencil. We obtain:

\[
E_{\pm} \simeq \left( \frac{\sigma_1 \sigma_2 + 3 \sigma_3}{\sigma_2} \pm \frac{\sigma_3 W}{\sigma_2} \right) \quad \text{where } W \text{ as above}
\]

10. The midpoint of \( E_{+} \), \( E_{-} \) is \( X(468) \) again, while the direction of \( E_{+}E_{-} \) is \( \sigma_1 \sigma_2 : 0 : -\sigma_2 \), i.e. \( X(511) \).

11. When the triangle is acute, \( W^2 \) is positive, and \( W \) is real. Thus \( \text{conjugate}(W) = W \) so that the \( F_{\pm} \) are visible, while the \( E_{\pm} \) are not. When the triangle is obtuse, \( W^2 \) is negative, and \( W \) is imaginary. Thus \( \text{conjugate}(W) = -W \) so that the \( E_{\pm} \) are visible, while the \( F_{\pm} \) are not...

as can be seen at Figure 18.1.

18.3 Revisiting the Brocard-Lemoine pencil

**Notation** 18.3.1. Here, \( J_a \), etc are the cevians of the incenter \( I_0 \) while \( P_a \), etc are its cocevians. See Section 14.9 for more details.

The Brocard-Lemoine pencils are build from the so-called **Apollonian circles**, whose diameters are the segments \([J_a, P_a]\), etc. They have been treated in detail at Section 14.9. Let us examine again these pencils, in the light of the new formalism.

1. We start from the Lubin-2 representation i.e. \( A^2 \simeq \alpha^2 : 1 : \alpha^{-2} \) and obtain:

\[
J^2_a, P^2_a \simeq \frac{1}{2} \left( \begin{array}{c}
\beta \gamma (\alpha^2 \beta^3 + \alpha^2 \beta \gamma + \alpha^2 \gamma^2 - \beta^2 \gamma^2) \\
(\alpha^2 + \beta \gamma) \beta \gamma \\
\beta^2 - \alpha^2 + \beta \gamma + \gamma^2
\end{array} \right) = \left( \begin{array}{c}
\beta \gamma (\alpha^2 \beta^3 - \alpha^2 \beta \gamma + \alpha^2 \gamma^2 - \beta^2 \gamma^2) \\
(\alpha^2 - \beta \gamma) \beta \gamma \\
\alpha^2 - \beta^2 + \beta \gamma - \gamma^2
\end{array} \right)
\]

2. Now, we take the wedge of the Veronese of \( A, J_a, P_a \). Since \( J_a, P_a \) are Lemoine-conjugates, the result can be expressed in the Lubin-1 frame, leading to:

\[
V_a \simeq \left( V_{\text{Ver}A^2} \right)_z \simeq \frac{1}{2} \left( \begin{array}{c}
\beta + \gamma - 2 \alpha \\
\alpha^2 - \beta \gamma \\
\alpha (2 \beta \gamma - \alpha \beta - \alpha \gamma)
\end{array} \right) \simeq \left[ \begin{array}{c}
L_{V_1}^{-1} \end{array} \right] V_a
\]

3. When computing the line describing the pencil generated by \( V_a, V_b \), one obtains a symmetric expression, proving that \( V_z \) belongs to the pencil.

\[
\text{Lemoine}^z_{\text{colu}} \simeq \left( V_a \wedge V_b \right) \simeq \frac{1}{2} \left[ \begin{array}{c}
L_{V_1} \end{array} \right] \cdot \text{Lemoine}^b_{\text{colu}} \cdot L_{V_1}
\]

\[
\simeq \frac{1}{2} \left( \begin{array}{cccc}
0 & 3 \sigma_1 \sigma_3 - \sigma_2^2 & 0 & \sigma_3^2 - 3 \sigma_1 \sigma_3 \\
\sigma_2^2 - 3 \sigma_1 \sigma_3 & 0 & \sigma_1^2 - 3 \sigma_2 & \sigma_3 \sigma_1 - 9 \sigma_3 \\
0 & 3 \sigma_2 - \sigma_1^2 & 0 & \sigma_1^2 - 3 \sigma_2 \\
3 \sigma_1 \sigma_3 - \sigma_2^2 & 9 \sigma_3 - \sigma_1 \sigma_2 & 3 \sigma_2 - \sigma_1^2 & 0
\end{array} \right)
\]

4. The conjugate pencil is obtained by:

\[
\text{Brocard}^z_{\text{colu}} \simeq \left[ \begin{array}{c}
Q \end{array} \right] \cdot \text{dual} \left( \text{Lemoine}^z_{\text{colu}} \right) \cdot \left[ \begin{array}{c}
Q \end{array} \right] \simeq \frac{1}{2} \left[ \begin{array}{c}
L \end{array} \right] \cdot \text{Brocard}_{\text{colu}} \cdot L
\]

\[
\simeq \frac{1}{2} \left( \begin{array}{cccc}
0 & \sigma_2^2 - 3 \sigma_1 \sigma_3 & \sigma_1 \sigma_2 - 9 \sigma_3 & \sigma_3^2 - 3 \sigma_1 \sigma_3 \\
3 \sigma_1 \sigma_3 - \sigma_2^2 & 0 & \sigma_1^2 - 3 \sigma_2 & 0 \\
9 \sigma_3 - \sigma_1 \sigma_2 & 3 \sigma_2 - \sigma_1^2 & 0 & 3 \sigma_2 - \sigma_1^2 \\
3 \sigma_1 \sigma_3 - \sigma_2^2 & 0 & \sigma_1^2 - 3 \sigma_2 & 0
\end{array} \right)
\]
5. Solving $\text{Ver}^z (Z : T : \overline{Z}) \cdot Q^{-1}$ gives the point-circles that generate the Brocard pencil. One obtains:

$$F_+^z \simeq \left( \frac{\sigma_1 \sigma_2 - 9 \sigma_3}{\sigma_1^2 - 3 \sigma_2}, \frac{\sqrt{3} \sigma_4}{\sigma_1^2 - 3 \sigma_2}, \frac{9 \sigma_3 - \sigma_1 \sigma_2}{3 \sigma_1 \sigma_3 - \sigma_2^2} \right) \left( \frac{\beta \gamma + \alpha J + \alpha \gamma J^2}{1}, \frac{- (\alpha + \gamma J + \beta J^2)}{\beta \gamma + \alpha J + \alpha \beta J^2} \right)$$

where $J \doteq J_+ \doteq (1 + i\sqrt{3}) / 2$ (and thus $J_- \doteq J^2$). One can check that $F_+$ is $X(15)$ while $F_-$ is $X(16)$, i.e. the result already obtained using barycentrics.

6. Solving $\text{Ver}^z (Z : T : \overline{Z}) \cdot Q^{-1} \cdot Lemoine^z_{\text{point}} = \overline{0}$ gives the point-circles that generate the Lemoine pencil. One obtains:

$$E_+^z \simeq \left( \frac{\sigma_1 \sigma_2 - 9 \sigma_3}{\sigma_1^2 - 3 \sigma_2}, \frac{\sqrt{3} \sigma_4}{\sigma_1^2 - 3 \sigma_2}, \frac{9 \sigma_3 - \sigma_1 \sigma_2}{3 \sigma_1 \sigma_3 - \sigma_2^2} \right) \left( \frac{\beta \gamma + \alpha J + \alpha \gamma J^2}{1}, \frac{- (\alpha + \gamma J + \beta J^2)}{\beta \gamma + \alpha J + \alpha \beta J^2} \right)$$

7. Obviously, pairs $F_+$ and $E_+$ share the same midpoint, while their directions are orthogonal. But there is a huge difference between both pairs: points of the $F$ pair are visible, the others are not. Finally, one can check the usual relation: if one pair is noted $z_1 : 1 : \overline{z_1}, z_2 : 1 : \overline{z_2}$, the other one is $z_1 : 1 : \overline{z_2}, z_2 : 1 : \overline{z_1}$.

### 18.3.1 Isodynamic points

**Proposition 18.3.2.** Consider three distinct points $\alpha, \beta, \gamma$ on the unit circle. There are six homographies of $Z : T$ that left invariant the triangle, when seen as a set $\{\alpha, \beta, \gamma\}$ of three vertices. Grouping them with the inversion $\rho : Z/T \mapsto \overline{Z}/T$ leads to a copy of the dihedral group $D_6 = S_3 \times S_2$. One obtains the following table, where $h(\infty)$ is the image of $\infty \simeq 0 : 1 \in (Z : T)$.

<table>
<thead>
<tr>
<th>$h$</th>
<th>$h(\infty)$</th>
<th>$h(b, \infty)$</th>
<th>$h(c, \infty)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$E_1$</td>
<td>$1 : E_1$</td>
<td>$X_3$ circ</td>
<td>$0$</td>
</tr>
<tr>
<td>$\sigma$</td>
<td>$9 s_3 - s_1 s_2 - i s_4$</td>
<td>$B_{r_1, a} \alpha^2 b^2$</td>
<td>$6 s_1 s_3 - 2 s_2^2$</td>
</tr>
<tr>
<td>$\sigma^2$</td>
<td>$9 s_3 - s_1 s_2 + i s_4$</td>
<td>$B_{r_2, a} \alpha^2 c^2$</td>
<td>$6 s_1 s_3 + 2 s_2^2$</td>
</tr>
<tr>
<td>$\tau_{bc}$</td>
<td>$E_{c, a} \alpha s_1 - s_2$</td>
<td>$E_{c, a} \alpha$</td>
<td>$E_{c, a} \alpha$</td>
</tr>
<tr>
<td>$\tau_{ca}$</td>
<td>$E_{c, b} \alpha s_3 - s_1$</td>
<td>$E_{c, b} \alpha$</td>
<td>$E_{c, b} \alpha$</td>
</tr>
<tr>
<td>$\tau_{ab}$</td>
<td>$E_{c, c} \alpha s_2 - s_3$</td>
<td>$E_{c, c} \alpha$</td>
<td>$E_{c, c} \alpha$</td>
</tr>
</tbody>
</table>

where $\sigma$ is $\alpha \mapsto \beta \mapsto \gamma \mapsto \alpha$ and $\rho_{bc}$ is the inversion into the circle through $A, A', X(15), X(16)$, center $-E_a$, i.e. the A-Apollonian circle.

**Proof.** Inversion wrt the A-Apollonian circle takes the values $\alpha, \gamma, \beta$ at $\alpha, \beta, \gamma$.

**Fact 18.3.3.** We have the following facts. Their proofs are left as exercises.

1. Points 4, 5, 6, 7, 8, 9 are on the Brocard 3-6 circle (diameter $[O, K]$). Points 8, 9 are the Brocard points themselves.
2. Points 1, 2, 3, 10, 11, 12 are on the Lemoine axis (tripolar of $X_6$). In fact, 10, 11, 12 are the cocevians $E_j$ of $X_6$.
3. Points $n, n + 6$ are inverse in the circum- circle (and thus aligned with $O$).
4. Points $A, X_6, (4) = E_{a}'$ and $A'$ (circumcevian of $X_6$) are aligned, etc.

— pldx : Translation of the Kimberling’s Glossary into barycentrics ——
5. The fixed points of homographies $\sigma, \sigma^2$ are the isodynamics points $X_{15}, X_{16}$, while the fixed points of homography $\tau_{bc}$ are $A$ and $A''$, the $A$-circum-cevian of $X_b$.

6. The Apollonian circles are at $120^\circ$ from each other (consider the multiplier of an homography such that $\sigma^3 = id$).

**Proposition 18.3.4.** Suppose now that $P_a, P_b, P_c$ are distinct but aligned. Then define $Q_a$, etc by cross_ratio $(P_b, P_c, P_a, Q_a) = -1$, etc. Then circles having the $[P_j, Q_j]$ as diameters define the inversions to be used as $\rho \tau_{bc}$ in the previous proposition. They are called the Apollonian circles of the aligned points (the cyan circles at Figure 18.2).

**Proof.** When the points aren’t aligned, we already have cross_ratio $(B, C, A, A'') = -1$. So that property holds by continuity. 

\[\begin{align*}
\text{January 3, 2024 21:08 published under the GNU Free Documentation License}
\end{align*}\]
Proof. When the cycle is a circle, the generic point $M_\tau$ can be written as \[
\begin{pmatrix} z \\ t \\ \zeta \end{pmatrix} / t + \rho \begin{pmatrix} \tau \\ 0 \\ 1/\tau \end{pmatrix},
\]
and we only have to check that $\bigwedge_3 \left( \text{Ver}_p M_\tau, \text{Ver}_p M_{\tau+1}, \text{Ver}_p M_{\tau-1} \right)$ doesn't depend on $\tau$. As a result, we have:

$$\mathcal{V}_p \left[ \begin{pmatrix} z \\ t \\ \zeta \end{pmatrix}, \rho^2 \right] = \begin{pmatrix} \frac{-\zeta + t^2}{t} - t \rho^2 \\ -z \\ +t \end{pmatrix}.$$ 

\[ \square \]

**Proposition 18.4.4.** The Pedoe row-images of the $\mathbb{P}_C(C^3)$ points belong to a 3D quadric:

$$\text{Ver}_p (P) \cdot \mathcal{Q}_P^{-1} \cdot \text{Ver}_p (P) = 0 \quad \text{where} \quad \mathcal{Q}_P^{-1} = \begin{pmatrix} 0 & 0 & -1 & 0 \\ 0 & +2 & 0 & +1 \\ -1 & 0 & 0 & 0 \\ 0 & +1 & 0 & 0 \end{pmatrix} \quad (18.12)$$

And the columns representative of point-circles, i.e. the $\mathcal{V}_p (P) \simeq \mathcal{Q}_P^{-1} \cdot \text{Ver}_p (P)$, belong to the 3D paraboloid defined by $\mathcal{Q}_P$. For a circle $(\gamma)$, we have the more precise result:

$$\rho^2 = \frac{-1}{2} \times \frac{\mathcal{V}_p (\gamma) \cdot \mathcal{Q}_P \cdot \mathcal{V}_p (\gamma)}{\mathcal{V}_p (\gamma) \cdot \mathcal{N}_P \cdot \mathcal{V}_p (\gamma)} \quad (18.13)$$

where

$$\mathcal{Q}_P = \begin{pmatrix} 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & +1 \\ -1 & 0 & 0 & 0 \\ 0 & +1 & 0 & -2 \end{pmatrix} \quad \text{and} \quad \mathcal{N}_P = \mathcal{N} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}.$$

**Proof.** See the proof of (14.9). \[ \square \]

**Proposition 18.4.5.** Two point-circles are orthogonal wrt the quadric when their centers share one of their two coordinates.

**Proof.** Assuming $M_j \simeq z_j : t_j : \zeta_j$, we have the more precise result:

$$\left( \mathcal{V}_p [M_1, r_1^2] \right) \cdot \mathcal{Q}_P \left( \mathcal{V}_p [M_2, r_2^2] \right) = \left( \frac{z_2}{t_2} - \frac{z_1}{t_1} \right) \left( \frac{\zeta_2}{t_2} - \frac{\zeta_1}{t_1} \right) - r_1^2 - r_2^2 \quad (18.14)$$

\[ \square \]

**Remark 18.4.6.** And then, business as usual, using the adapted matrices.

**Maple 18.4.7.** The Maple package ‘faisceaux’ contains:

constants: "pQQ"= \[ \mathcal{Q}_P \] "pQQI"= \[ \mathcal{Q}_P^{-1} \]

functions: "Verp"= \[ \text{Ver}_p \], "mor2ped_"= \[ \mathcal{V}_p \], "colu2mor", "eqz2colu",

"coluz2mm", "mmz2colu", "mkzgram"
18.4.2 Pedoe version of the homographic action

CAVEAT: letters a, b, c, d, a', b', c', d', k, κ are general complex numbers, while a' = π, etc is intended for visible objects.

**Proposition 18.4.8.** The homography \( H \), defined at (18.7), is an action over the points of \( \mathbb{P}_C(\mathbb{C}^3) \). It induces an action \( \eta_2 \) that is linear over the Pedoe-columns. Its matrix is:

\[
\begin{pmatrix}
\frac{1}{|\text{det}|} \\
\end{pmatrix}
\]

**Proof.** Same as the Veronese proof.

**Proposition 18.4.9.** When the ordinary homography \( h \), acting over \( \mathbb{P}_C(\mathbb{C}^2) \), has exactly two fixed points \( z_1, z_2 \) then a basis of eigencolumns is \( \tilde{\mathcal{S}}_p \),

\[
\tilde{\mathcal{S}}_p \simeq \begin{pmatrix}
-\zeta_2 & -\xi_2 & -\xi_2 & -\zeta_1 \\
n_2 & n_1 & n_1 & n_2 \\
\end{pmatrix}
\]

leading to

\[
\begin{pmatrix}
0 & 0 & 0 & 1 \\
0 & 0 & 1 & 0 \\
1 & 0 & 0 & 0 \\
\end{pmatrix}
\]

When homography \( h \) has exactly one fixed point \( z_1 \) then a basis of triangulation is

\[
\tilde{\mathcal{S}}_p \simeq \begin{pmatrix}
-k & \frac{\kappa z_1}{t_1} & \frac{\kappa z_1}{t_1} & \kappa \\
0 & t_1 & t_1 & 0 \\
0 & 0 & 0 & 0 \\
\end{pmatrix}
\]

**Proof.** Having a basis made of the fixed point and three independent lines is clearly interesting. To understand this specific choice, one can start from cross_ratio \((z - k, z + k, z, \infty) = -1\), and obtain cross_ratio \((h^2M, M, hM, F) = -1\). Therefore, each cycle \((F, M, hM)\) is globally invariant, the line \((PFE)\) among them. Since two of them cannot have any other common point apart \( F \), these cycles form a tangent pencil \( F \), generated by the circle-point \((F)\) and the line \((PFE)\). Since \( F \) is invariant as a pencil, then \( F^{-1} \) is globally invariant. For the expansion of \( \tilde{\mathcal{S}}^{-1} \cdot \tilde{\mathcal{S}} \), consider the generating series \( 1/(1 - X \cdot \tilde{\mathcal{S}}^{-1} \cdot \tilde{\mathcal{S}}\).

**Proposition 18.4.10.** Assume that \( s \), acting over \( \mathbb{P}_C(\mathbb{C}^3) \), is the reflection into cycle \( \gamma \). This induces a linear action over the affixes, described by the matrix:

\[
\sigma \simeq 2 \cdot \begin{pmatrix}
V_p \cdot Q_p \\
V_p \cdot Q_p \\
\end{pmatrix}
\]
Proof. Rewrite (14.21).

**Exercise 18.4.11.** Consider the homographic transposition \( \beta \leftrightarrow \gamma, \alpha \leftrightarrow \infty \). Determine the associated Cremona transform, and its action on the cycles space. Determine a basis of \( \ker (+1) \) and a basis of \( \ker (-1) \). A line and a circle centered somewhere on \( BC \) would be a great choice. Check for the (14.19)(18.5). The images of the four in-ex-circles are the so-called four \textbf{A-mixtilinear circles} tangent to the circumcircle and to the \( AB, AC \) sidelines.

### 18.5 The Spherical formalism

**On 2021-10-05, it has been decided to adopt the South pole point of view. Vae victis!**

#### 18.5.1 The Spherical map

**Definition 18.5.1.** The projective sphere of \( \mathbb{P}_\mathbb{C}(\mathbb{C}^4) \) is defined as the locus of the points \( V \) such that \( v_1^2 - v_2^2 - v_3^2 = 0 \). In other words

\[
V \in S_4 := \left( iV \cdot \begin{bmatrix} \text{Mink} \end{bmatrix} \cdot V = 0 \right)
\]

where \( \begin{bmatrix} \text{Mink} \end{bmatrix} := \begin{pmatrix} -1 & -1 & -1 & +1 \\ +1 & -1 & -1 & -1 \\ -1 & +1 & -1 & -1 \\ -1 & -1 & +1 & -1 \end{pmatrix} \)

**Remark 18.5.2.** When restricted to \( E_3 \), i.e. to the real points where \( v_4 \neq 0 \), this is nothing but the ordinary unit sphere of the elementary geometry.

**Definition 18.5.3.** The Spherical version of the Veronese map uses both the unit visible circle and the unit imaginary circle and is defined by:

\[
Ver(\mathbb{Z}:T:Z) \simeq \left[ 2TX, 2TY, -Z\bar{Z} + T^2, +Z\bar{Z} + T^2 \right]
\]  

where \( 2X = Z + \bar{Z} ; 2Y = -i \left( Z - \bar{Z} \right) \) so that \( X + iY = Z ; X - iY = \bar{Z} \)

**Remark 18.5.4.** These four cycles are orthogonal to each other, since \( d^2 - v_1^2 - v_2^2 \) applied to the last two gives obviously zero. And thus, the associated matrices \( \begin{bmatrix} \mathbb{Q} \end{bmatrix} \) will be diagonal.

**Remark 18.5.5.** Due to the relation:

\[
Ver(\mathbb{Z}:T:Z) = Ver_\tau(\mathbb{Z}:T:Z)
\]

we shall not expect some breaking results compared to Section 18.1. Nevertheless, the Veronese quadric \( v_4 v_2 - v_3 v_1 \), with signature \((+2,-2)\), has been replaced by the spherical quadric \( v_1^2 - v_2^2 - v_3^2 \), with signature \((+1,-3)\).

**Proposition 18.5.6.** Both umbilics are mapped to \([0,0,0,0]\) (points of indeterminacy) while all other points at infinity are mapped to \([0,0,-1,1]\), the so-called South pole. When using \((\mathbb{Z}:T:Z) = Lu(x:y:z)\), we have

\[
Ver(\mathbb{Z}:T:Z) \simeq Ver(x:y:z) \cdot \begin{bmatrix} Z2S \end{bmatrix}
\]

where \( \begin{bmatrix} Z2S \end{bmatrix} = \begin{pmatrix} 1 & 0 & 1 & 0 \\ -i & 0 & i & 1 \\ 0 & 1 & 0 & -1 \\ 0 & 1 & 0 & 1 \end{pmatrix} \),

**Proof.** One has \( \begin{bmatrix} \mathbb{L}v_s \end{bmatrix} = \begin{bmatrix} Z2S \end{bmatrix} \cdot \begin{bmatrix} Lv_s \end{bmatrix} \)

**Proposition 18.5.7.** The hyperplanes \( Ver_\tau(M_r) \) related to the points \( M_r \) of a given cycle are all going through a same point of \( \mathbb{P}_\mathbb{C}(\mathbb{C}^4) \), called the cycle spherical-representative and noted \( \Psi_s \).

— pldx : Translation of the Kimberling’s Glossary into barycentrics ——
Proof. When the cycle is a circle, the \( M_\tau \) can be written as
\[
\begin{pmatrix}
z \\
t \\
\zeta \\
\end{pmatrix}
+ \rho \begin{pmatrix}
\frac{\zeta + z}{t} \\
\frac{i \zeta - iz}{t + t^2 - z \zeta} + t \rho^2 \\
\frac{-i t^2 - z \zeta}{t} + t \rho^2 \\
\end{pmatrix}
\]
and we only have to check that \( \bigwedge_3 \left( \text{Ver}_s (M_\tau), \text{Ver}_s (M_{\tau + 1}), \text{Ver}_s (M_{\tau - 1}) \right) \) doesn’t depend on \( \tau \). As a result, we have:
\[
\begin{pmatrix}
z \\
t \\
\zeta \\
\end{pmatrix}
\pmatrix{z + \zeta \cr i \zeta - iz \cr + t^2 - z \zeta} + t \rho^2 \begin{pmatrix}
0 \\
0 \\
1 \\
\end{pmatrix}
\]

When the cycle is the line \([f, g, h]\), the generic point is \( \frac{1}{f} : \frac{t}{g} : \frac{-1 + t}{h} \). Like above, the wedge of the three \( \text{Ver}_s \) doesn’t depend on \( t \) and we obtain:
\[
\begin{pmatrix}
z \\
t \\
\zeta \\
\end{pmatrix}
\pmatrix{f + h \cr i f - i h \cr g} = \begin{pmatrix}
f + h \\
if - ih \\
g \\
\end{pmatrix}
\]
(18.18)

Remark 18.5.8. One can remark that the North plane, i.e. the plane \( V_4 - V_3 = 0 \), is the locus of the representatives of the non-circles among the cycles (i.e. the completed lines).

**Proposition 18.5.9.** The spherical row-images of the \( \mathbb{P}_C (\mathbb{C}^3) \) points belong to the 3D sphere defined by
\[
\text{Ver}_s (P) \cdot Q_s^{-1} \cdot \text{Ver}_s (P) = 0 \quad \text{where} \quad Q_s^{-1} = \frac{1}{2} \begin{pmatrix}
\text{Mink} \\
\end{pmatrix}
\]
(18.19)

And the columns representative of point-circles, i.e. the \( S_s (P) \simeq Q_s^{-1} \cdot \text{Ver}_s (P) \), belong to the 3D sphere defined by
\[
\begin{pmatrix}
Q_s \\
\end{pmatrix}
\]
where \( Q_s = 2 \begin{pmatrix}
\text{Mink} \\
\end{pmatrix} \)

Remark 18.5.10. Obviously, the two spheres are two copies of \( S_4 \). But one of them is a punctual object, while the other is a tangential one, whose elements are hyper-planes. In a later step (stereographic formalism), we will only consider one sphere (and describe it using the \( \text{Mink} \) matrix).

**Proposition 18.5.11.** Given two circles, we have the formula:
\[
\begin{pmatrix}
\text{Ver}_s [M_1, r_1^2] \\
\text{Ver}_s [M_2, r_2^2] \\
\end{pmatrix}
\pmatrix{Q_s \\
N_s} \begin{pmatrix}
\text{Ver}_s [M_1, r_1^2] \\
\text{Ver}_s [M_2, r_2^2] \\
\end{pmatrix}
= \begin{pmatrix}
z_2 - z_1 \\
t_2 - t_1 \\
\zeta_2 - \zeta_1 \\
\end{pmatrix} - r_1^2 - r_2^2
\]
(18.20)

where \( N_s = \begin{pmatrix}
0 & 0 & 0 & 0 \\
0 & 0 & 0 & +1 \\
0 & 0 & -1 & +1 \\
\end{pmatrix} \)

Proof. Direct computation, or application of (18.4).

**Corollary 18.5.12.** For a circle \( (\gamma) \), we have the radius formula:
\[
\rho^2 = -\frac{1}{2} \times \begin{pmatrix}
\text{Ver}_s (\gamma) \\
\text{Ver}_s (\gamma) \\
\end{pmatrix}
\pmatrix{Q_s \\
N_s} \begin{pmatrix}
\text{Ver}_s (\gamma) \\
\text{Ver}_s (\gamma) \\
\end{pmatrix}
\]
(18.21)
Corollary 18.5.13. Two point-circles are orthogonal wrt the quadric when their centers share one of their two coordinates. And then each center lies on one of the isotropic lines through the other center.

Maple 18.5.14. The Maple package 'faisceaux' contains:

constants: \( \text{sQQ} = \frac{1}{Q} \), \( \text{sQQ} = \frac{1}{Q} \)

functions: \( \text{Vers} = V_{\text{er}}, \text{mor2sphr} = V, \text{sphr2mor}, \text{eqz2sphr}, \text{sphr2mm}, \text{mmz2sphr}, \text{mksgram} \)

18.5.2 Spherical version of the homographic action

CAVEAT: in this section, the letters \( a, b, c, d, k, \alpha, \beta, \gamma, \delta, \kappa \) don’t keep their usual meaning! All of them are general complex numbers, while \( \alpha = \pi \), etc is intended for visible objects.

Proposition 18.5.15. The Cremona-homography \( H \), defined at (18.7), is an action over the points of \( \mathbb{P}_C(C^3) \). It induces an action \( \vec{H}_s \) over the Spherical-columns. This action is linear and its matrix is:

\[
\vec{H}_s \simeq \frac{1}{2|\det|} \times \\
\begin{bmatrix}
-ad' - da' - bc' - cb' & i(da' - ad' + bc' - cb') & ac' + ca' - bd' - db' & ac' + ca' + bd' + db' \\
i(da' - da' + be' - cb') & -ad' - da' + be' + cb' & i(ca' - ac' + bd' - db') & i(ca' - ac' - bd' + db') \\
ab' + ba' - cd' - dc' & i(ab' - ba' + cd' + dc') & -aa' + bb' + cc' - dd' & -aa' - bb' + cc' + dd' \\
ab' + ba' + cd' + dc' & i(ab' - ba' + cd' - dc') & -aa' + bb' - cc' + dd' & -aa' - bb' - cc' - dd'
\end{bmatrix}
\]

moreover \( \vec{H}_s \left| \frac{Q}{s} \right. \vec{H}_s = \left| \frac{Q}{s} \right. \) where \( |\det| = \sqrt{(ad - bc)(a'd' - b'c')} \)

Proof. Same as the Veronese proof. Mind the following fact: as it should be, a point-circle is more as the Kimberling’s Glossary into barycentrics —–

Proposition 18.5.16. When the ordinary homography \( h \), acting over \( \mathbb{P}_C(C^2) \), has exactly two fixed points \( z_1, z_2 \) then a basis of eigencolumns is \( \vec{s}_z \),

\[
\vec{s}_z \simeq \frac{1}{|F_1F_2|} \\
\begin{bmatrix}
-z_2 & -z_2 & -z_2 & -z_2 \\
\frac{z_2}{t_2} & \frac{z_2}{t_2} & \frac{z_2}{t_2} & \frac{z_2}{t_2} \\
\frac{z_2}{t_2} & \frac{z_2}{t_2} & \frac{z_2}{t_2} & \frac{z_2}{t_2} \\
1 & 1 & 1 & 1
\end{bmatrix}
\]

leading to

\[
\det \vec{s}_z = +4i; \quad \vec{s}_z^{-1} \cdot \vec{H} \cdot \vec{s}_z = \begin{bmatrix}
k & 0 & 0 & 0 \\
0 & \kappa & 0 & 0 \\
0 & 0 & 1/\kappa & 0 \\
0 & 0 & 0 & 1/k
\end{bmatrix}; \quad \vec{s}_z \cdot \left| \frac{Q}{s} \right. \vec{s}_z \simeq 4 \begin{bmatrix}
0 & 0 & 0 & 1 \\
0 & 0 & -1 & 0 \\
0 & -1 & 0 & 0 \\
1 & 0 & 0 & 0
\end{bmatrix}
\]

Proof. Direct computation using

\[
a = k \frac{z_2}{t_2} - \frac{z_1}{t_1}; \quad b = \frac{z_1z_2}{t_1t_2} (1 - k); \quad c = k - 1; \quad d = -k \frac{z_1}{t_1} + \frac{z_2}{t_2}, \text{etc}
\]

\[
|\det| = \sqrt{(a'd' - b'c')} (ad - bc) = |F_1F_2|^2 \sqrt{k\kappa}
\]
Proposition 18.5.17. Assume that $s$, acting over $\mathbb{P}_C(\mathbb{C}^3)$, is the reflection into cycle $\gamma$. This induces a linear action over the spherical affixes, described by the matrix:

$$
\begin{bmatrix}
\varphi \\
\end{bmatrix} = \text{Id} - 2 \frac{\mathcal{V}(\gamma) \cdot \mathcal{V}(\gamma) \cdot Q}{s}
$$

(18.23)

Proof. Rewrite (14.21). Moreover, one can check that, in $\mathbb{C}^4$, $\mathcal{V}(\gamma)$ is changed into its opposite, while the representative of any cycle orthogonal to $\gamma$ is unchanged.

Example 18.5.18. The following table describes how to generate the four actions $v_j \mapsto -v_j$:

<table>
<thead>
<tr>
<th>eqn</th>
<th>param</th>
<th>$\mathcal{V}_s$</th>
<th>action</th>
</tr>
</thead>
<tbody>
<tr>
<td>$X$</td>
<td>$z : 1 : -z$</td>
<td>$1 : 0 : 0 : 0$</td>
<td>$v_1 \mapsto -v_1$</td>
</tr>
<tr>
<td>$Y$</td>
<td>$z : 1 : +z$</td>
<td>$0 : 1 : 0 : 0$</td>
<td>$v_2 \mapsto -v_2$</td>
</tr>
<tr>
<td>$-T^2 + Z\overline{Z}$</td>
<td>$+\tau : 1 : 1/\tau$</td>
<td>$0 : 0 : 1 : 0$</td>
<td>$v_3 \mapsto -v_3$</td>
</tr>
<tr>
<td>$+T^2 + Z\overline{Z}$</td>
<td>$-\tau : 1 : 1/\tau$</td>
<td>$0 : 0 : 0 : 1$</td>
<td>$v_4 \mapsto -v_4$</td>
</tr>
</tbody>
</table>

18.6 Stereographic projection

On 2021-10-05, it has been decided to adopt the South pole point of view. Vae victis!

Definition 18.6.1. When $\mathbb{C}$ is identified with the $z = 0$ plane in $\mathbb{E}_3$, the 'tangential' Riemann sphere is defined by its diameter $[S, O]$ where $S = [0, 0, -1]$ is the South pole and $O = [0, 0, 0]$ is the origin. And the 'tangential' stereography is defined by $M$ on the plane, $P_t$ on the tangential sphere, and $S, M, P_t$ aligned.

Remark 18.6.2. This stereography is the projection used by cartographers to draw maps of the circumpolar places. Obviously, both local metrics in the vicinity of $O$ (on the plane and on the sphere) are the same. Nevertheless, having the center of the sphere at the origin is more handy when studying the isometries of the sphere itself.

Definition 18.6.3. The sphere in $\mathbb{E}_3$ whose equator is the trigonometric circle (in the $z = 0$ plane) should be called the equatorial Riemann sphere. But we will rather call it as the Riemann sphere. And the correspondence where $A$ is on the plane, while $P$ is on the sphere and $S, A, P$ are aligned will be called the stereographic projection.

Figure 18.3: Both projections of $A, B$ in the plane.
Proposition 18.6.4. The stereographic correspondence can be computed as

\[
\tilde{\pi} : \left( \begin{array}{c} z \\ t \\ \zeta \end{array} \right) \in \mathbb{P}_c(\mathbb{C}^3) \mapsto \frac{1}{t^2 + \zeta} \left( \begin{array}{c} t(z + \zeta) \\ -i(t(z - \zeta)) \\ +t^2 - z\zeta \end{array} \right) \tag{18.24}
\]

\[
\tilde{\pi}^{-1} : \left( \begin{array}{c} x \\ y \\ r \end{array} \right) \in \mathbb{R}^3 \mapsto \left( \begin{array}{c} x + iy \\ 1 + r \\ x - iy \end{array} \right) \in \mathbb{P}_c(\mathbb{C}^3)
\]

where \( \zeta \) is the conjugate of \( z \) and \( x^2 + y^2 + r^2 = 1 \) is assumed.

Proof. Write \( P = \mu A + (1 - \mu) S \) and obtain \( \mu \) using \( x^2 + y^2 + r^2 = 1 \).

Theorem 18.6.5. The spherical map \( V_{\text{er}} \) introduced at 18.17 is the projective version of the stereographic map (from the South pole) introduced at 18.24. While the map \( M \mapsto V_s(O - M, 0) \) is the projective version of the stereographic map relative to the North pole.

Proof. Simple comparison between both formula. Remember: a theorem is characterized among the propositions by its efficiency, not by the difficulty of the proof.

Example 18.6.6. Let us consider two visible points: \( z_1 = 1 + 2i \), \( z_2 = 3 + i \). We have

\[
F_1 \simeq V_s(z_1) \simeq \left( \begin{array}{c} 2 \\ 4 \\ -4 \\ -6 \end{array} \right); \quad F_4 \simeq V_s(z_2) \simeq \left( \begin{array}{c} 6 \\ 2 \\ -9 \\ -11 \end{array} \right)
\]

One computes the family:

\[
(1 - \mu) F_1 + (1 + \mu) F_4 \mapsto \text{Ponce} \mu = \left. V_s \left( (1 - \mu) z_1 + (1 + \mu) z_2, R^2 = \frac{5}{4} (\mu^2 - 1) \right) \right] = \left. V_s \left( \left[ \begin{array}{c} 4 + 3i \\ 2 \\ 1 \\ -6 + 5i \end{array} \right] + \mu \left[ \begin{array}{c} 2 - i \\ 0 \\ 2 + i \end{array} \right], R^2 = \frac{5}{4} (\mu^2 - 1) \right]
\]

When \( \mu \) is real and \( |\mu| > 1 \) we obtain a set of visible circles.

Let us consider the orthogonal point circles and their representatives:

\[
F_2 \simeq V_s \left[ \left( \begin{array}{c} z_1/t_1 \\ 1 \\ \zeta_2/t_2 \end{array} \right), 0 \right] \simeq \left( \begin{array}{c} 4 + i \\ 3 + 2i \\ -4 - 5i \\ -6 + 5i \end{array} \right); \quad F_3 \simeq V_s \left[ \left( \begin{array}{c} z_2/t_2 \\ 1 \\ \zeta_1/t_1 \end{array} \right), 0 \right] \simeq \left( \begin{array}{c} 4 - i \\ 3 - 2i \\ -4 + 5i \\ -6 + 5i \end{array} \right)
\]

One computes the family:

\[
(1 + i \mu) F_2 + (1 - i \mu) F_3 \mapsto \text{Arcs} \mu = \left. V_s \left( \left[ \begin{array}{c} 4 \\ 3 \\ -2 \\ +5 \end{array} \right] + \mu \left[ \begin{array}{c} 0 \\ 5 \\ 0 \end{array} \right], R^2 = \frac{5}{4} (\mu^2 + 1) \right]
\]

When \( \mu \) is real, we obtain a set of visible circles (visible center, real radius). All of them are orthogonal to \( F_1 \) and \( F_4 \); these circles are going through points \( z_1 \) and \( z_2 \). And therefore, the first pencil is the isoptic (Poncelet) pencil of points \( z_1 \) and \( z_2 \), while the second one is their isoptic pencil.
Proposition 18.7.1. Consider the visible Cremona transform \( \hat{\sigma} \) defined by
\[
\sigma \left( \begin{array}{c} Z \\ T \end{array} \right) \simeq \left( \begin{array}{cc} a & b \\ c & d \end{array} \right) \left( \begin{array}{c} Z \\ T \end{array} \right)
\]
where \( \det \sigma = 1 \)

and suppose that \( \hat{\sigma} \) induces a rotation \( \hat{H} \) in \( \mathbb{E}_3 \), then it exists reals \( A, f, g, h \) such that
\[
\left( \begin{array}{cc} a & b \\ c & d \end{array} \right) \simeq \cos A \left[ \begin{array}{cc} 1 & 0 \\ 0 & 1 \end{array} \right] + \sin A \left[ \begin{array}{cc} -h & -f + ig \\ -f - ig & +h \end{array} \right]
\] (18.25)

Proof. In a rotation of the sphere, the antipodal relation is preserved. This amounts to the invariance by \( v_4 \mapsto -v_4 \), and matrix \( \hat{H} \) as to commute with \( \hat{Q} \), leading to \( \alpha = \frac{\delta d}{a}, \beta = -\frac{\delta c}{a}, \gamma = -\frac{\delta b}{a} \) where \( \delta = a \) can be chosen. This leads to
\[
a = c' - ih' \; ; \; b = g' - if' \; ; \; c = -g' - if' \; ; \; d = c' + ih'
\]

It only remains to put \( c' = \cos A \) and then normalize \( f' : g' : h' \).

Proposition 18.7.2. Conversely, the equation defined at (18.25) induces a rotation of the unit sphere in \( \mathbb{E}_3 \). Seen from the unit vector \( [f, g, h] \), its angle is \( +2A \)... while seen from \( [-f, -g, -h] \), its angle is \( -2A \).

Proof. Substitute (18.25) into (18.22), use:
\[
\sin A^2 = U - \cos A^2 \; ; \; \sin A = \frac{1}{2} \sin B, \; \cos A^2 = \frac{1}{2} \cos B + \frac{U}{2}
\]
and collect in \( U, \sin, \cos \). Restore \( U = 1 \) and obtain:
\[
\begin{align*}
\hat{H}_s &= m_U + \sin 2A m_S - \cos 2A m_S^2 \\
where m_U &= \left[ \begin{array}{cccc} f^2 & fg & fh & 0 \\ fg & g^2 & gh & 0 \\ fh & gh & h^2 & 0 \\ 0 & 0 & 0 & 1 \end{array} \right] ; \; m_S &= \left[ \begin{array}{cccc} 0 & h & -g & 0 \\ -h & 0 & f & 0 \\ g & -f & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array} \right] 
\end{align*}
\]

Remark 18.7.3. One has: \( \chi \left( \begin{array}{cc} \hat{H}_s \end{array} \right) \right) = (X - 1)^2 (X - k) (X - \kappa) \) where \( \kappa = k = \exp iB\pi \), while \( \chi \left( \begin{array}{cc} \hat{h} \end{array} \right) \right) = (X - \mu_1) (X - \mu_2) \) where \( \mu_2 = \mu_1 = \exp iA\pi \). But the multipliers of \( h \) are the ratios of the eigenvalues, giving again \( k \) and \( \kappa \).

Theorem 18.7.4. When rotations are composed in \( \mathbb{E}_3 \), quaternions are multiplied as \( 2 \times 2 \) matrices. This amounts to use the Hamilton's rule: \( + j \cdot \overline{k} = i = -\overline{k} \cdot j \), etc.
18.8 Stereographic formalism

Here, we will use the stereographic projection from the South pole, and apply the Theorem 18.6.5 to obtain a visual formalism using:

**Definition 18.8.1.** We define the apex of a point, or of a cycle, as the columns:

\[
\mathcal{A}(M) \doteq \mathcal{V}_s(M)
\]

\[
\mathcal{A}(\mathcal{C}) \doteq \mathcal{M}_s \cdot \mathcal{V}(\mathcal{C})
\]

**Proposition 18.8.2.** The apex of a point \(Z = X + iY\) in the \(xOy\) plane is the stereographic projection of this point. All apexes are columns and live in the same copy of \(\mathbb{P}_\mathbb{C}(\mathbb{C}^4)\). Moreover, the apex of a point is the same as the apex of the point circle centered at this point.

**Proof.** From former results, \(\mathcal{A}(M) \cdot \mathcal{M}_s \cdot \mathcal{V}(M) = 0\), so that \(\mathcal{A}(M) \in \mathbb{S}_4\). One can check easily that \(M = X : Y : 0 : \mathbf{T}\) and the apex \(\mathcal{A}(M)\) are aligned with \(\mathbf{S}\), proving the stereographic property. Moreover

\[
\mathcal{A}(M) \doteq \mathcal{V}_s(M) \simeq \begin{pmatrix}
(z + \zeta)t \\
-i(z - \zeta)t \\
t^2 - z\zeta \\
t^2 + z\zeta
\end{pmatrix} \simeq \mathcal{M}_s \cdot \mathcal{V}(M, 0) \doteq \mathcal{A}(M, 0)
\]

This is the rationale for using the same name for both objects.

**Example 18.8.3.** Consider the points

\[M_j = 0.22 + 0.84i; 1.34 - 0.28i; 0.22 - 0.44i\]

They define a circle with center \(B = 0.7 + 0.2i\) and radius \(\rho = 0.8\). Computing the apexes, we obtain:

\[
A_B, A_0, A_1, A_2 \simeq \begin{pmatrix}
1.40 \\
0.40 \\
0.47 \\
1.53
\end{pmatrix},
\begin{pmatrix}
0.440 \\
1.680 \\
0.246 \\
1.754
\end{pmatrix},
\begin{pmatrix}
+2.680 \\
-0.560 \\
-0.874 \\
+2.874
\end{pmatrix},
\begin{pmatrix}
+0.440 \\
-0.880 \\
+0.758 \\
+1.242
\end{pmatrix}
\]

and therefore,

\[
U \doteq \mathcal{A}(\mathcal{C}(M_j)) = \mathcal{M}_s \cdot \mathcal{N}_j \cdot (A_j) = \begin{pmatrix}
1.40 \\
0.40 \\
0.47 \\
1.53
\end{pmatrix} + \begin{pmatrix}
0 \\
0 \\
+0.64 \\
-0.64
\end{pmatrix}
\]

**Proposition 18.8.4.** The apex of a line lies in the South plane, and we have:

\[
\mathcal{A}([f, g, h]) \doteq \begin{pmatrix}
(f + h) \\
i(f - h) \\
+g \\
-g
\end{pmatrix}
\]

(18.26)

**Proposition 18.8.5.** When using apexes, the orthogonality formula between two circles is now:

\[
\frac{\mathcal{A}(M_1, r_1^2)}{\mathcal{A}(M_2, r_2^2)} \cdot \mathcal{M}_s \cdot \mathcal{A}(M_2, r_2^2) = \begin{pmatrix}
\frac{z_2}{t_2} - \frac{z_1}{t_1} \\
\frac{\zeta_2}{t_2} - \frac{\zeta_1}{t_1} \\
0 \\
0
\end{pmatrix} - r_1^2 - r_2^2
\]

(18.27)

where \(\mathcal{N}_k = \frac{1}{2} \begin{pmatrix}
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
0 & 0 & 1 & 1 \\
0 & 0 & 1 & 1
\end{pmatrix}\)

so that \(M \in \mathcal{C}\) can be restated as \(\mathcal{A}(M) \perp \mathcal{A}(\mathcal{C})\).
Proof. In (18.21) the normalization has to be changed due to the product by $\text{Mink}$.

Remark 18.8.6. Sending the plane $A_4 = 0$ at infinity "discards" the circles orthogonal to $ZT^2 = 0$ and the quadric appears as a sphere in the remaining space $E_3$. On the contrary, sending the South plane $A_4 + A_3 = 0$ at infinity "discards" the (completed) straight lines and allows for a normalization of the circle-representatives.

**Proposition 18.8.7.** Define the shadow of a cycle $C$ as the locus of the apexes of the points of $C$. The shadow of a cycle is a circle on the sphere $S_4$. Its center belongs to line $[O,U]$ and, in fact, is the inverse of $U$ wrt the sphere. When $C$ is a circle, the line $[S,B]$ describes the pencil of all circles centered at $B$, and therefore goes through $A(B)$ and $U = A(C)$ (see Figure 18.4).

**Proposition 18.8.8.** Let $A_j$, $j = 1,..,4$ be the apexes of four cycles $C_j$. Describe the pencil generated by $C_1, C_2$ using the matrix $\Delta_{12} = (A_1 \wedge A_2)$, and the pencil generated by $C_3, C_4$ using the matrix $\Delta_{34} = (A_3 \wedge A_4)$. When each pencil is orthogonal to the other, then

$$\Delta_{34} = \text{Mink} \cdot \Delta_{12} \cdot \text{Mink}$$

Using electrical notation (see Definition 8.1.5), if $\Delta_{12} = (\hat{E}, \hat{B})$ then $\Delta_{34} = (\hat{B}, -\hat{E})$.

**Proof.** Cut $\Delta_{12}$ by the four base hyperplanes. Among the four expressions obtained, at most two are $0:0:0:0$. Do the same with the proposed matrix. And check that all the 16 orthogonality relations are fulfilled (this computation is rather easy since most results are obviously 0, while the others involve $E_xB_x + E_yB_y + E_zB_z$.

**18.9 Comparison with Cartesian and Artinian metrics**

To be written.

For all metrics, $\text{OrtO} = (\text{OrtO}^\dagger)^*$ while trace $((\text{OrtO})) = 1$. 

January 3, 2024 21:08 published under the GNU Free Documentation License
18. Pencils of cycles in the complex plane

Figure 18.4: Stereographic projection and apexes

(a) From circle to shadow

(b) From shadow to apex

(c) Back from apex to center

Figure 18.4: Stereographic projection and apexes
Chapter 19

Hyperbolic geometry

In a first step, hyperbolic geometry was created to prove that a geometry can be build which satisfies all of the usual axioms except from the Euclidean one about the parallel lines. As a result, it becomes proven that the Euclidean axiom is independent from the other axioms. Cannon et al. (1997) Arlan Ramsay (1995)

Notation 19.0.1. For the further use of the reader, we summarize here all the notations that will be introduced throughout this chapter.

- Prefixes "C-", "P-" and "K-" are used to distinguish the usual cartesian/!complex objects from their Poincare or Klein counterparts.

- Letter $O$ is ever the common origin to all spaces. The $C$-unit circle, i.e. $\gamma_C(O,1)$, is noted $\Gamma$. The same circle, when used as the P- or the K- horizon circle, is noted $\partial\mathbb{H}$, while $\mathbb{H}$ denotes the open disk limited by $\partial\mathbb{H}$.

- A line through points $A, B$ is noted $AB$, a circle centered at $A$ and going through $M$ is noted $(A;M)$. When enforcing the brand of a specific object seems useful, notations $\Delta_X (A,B)$ or $\gamma_X (A;M)$ are used.

- Coordinates $A_P \simeq z : t : \zeta$ of a P-point are the usual $C$-coordinates, while a widehat denote the $C$-reflection of a point into $\Gamma$, i.e. $\hat{A} \simeq t/\zeta : 1 : t/z$.

- Coordinates $A_K \simeq k : 1 : \kappa$ are used for a K-point. From $W = \sqrt{1-k \kappa} = (1 - z \zeta) / (1 + z \zeta) \in \mathbb{R}$ the K-points are not supposed to cross the boundary $\Gamma$.

19.1 The Poincaré plane

Remark 19.1.1. Our strategy is as follows. We start from the $C$-objects which occur in the vicinity of $O$, and then we convey everything to the vicinity of the generic point $A_P$. Therefore, the P-lines through $O$ are the $C$-diameters of $\Gamma$, while the P-circles $(O;M)$ are identified with the $C$-circles $(O;M)$. Moreover, the space is assumed to be locally Euclidian at $O$, i.e.

$$(ds_O)^2 = \text{Cte}^2 (dz \, d\zeta)$$

Definition 19.1.2. The Poincaré model is what is obtained by using homographies as conveyors.

Theorem 19.1.3. The Poincaré conveyor is the Cremona homography defined by:

$$\tau_A : O \mapsto A_P \simeq \begin{pmatrix} z \\ t \\ \zeta \end{pmatrix} ; \begin{pmatrix} Z \\ T \\ \overline{Z} \end{pmatrix} \mapsto \begin{pmatrix} \frac{t \overline{Z} + \zeta \overline{T}}{z \overline{Z} + t \overline{T}} \\ 1 \\ \frac{t \overline{Z} + \zeta \overline{T}}{z \overline{Z} + t \overline{T}} \end{pmatrix}$$
Proof. Conveyor $\tau_A$ has to keep globally invariant both the horizon circle $\partial \mathbb{H}$ and the line $OA$. Therefore points $OA \cap \Gamma$, i.e. $\pm \sqrt{1/\zeta} : 1 : \pm \sqrt{1/\zeta}$, are the fixed points. Knowing 3 points and their images, we conclude using the cross-ratio. 

Remark 19.1.4. Seen as a Cremona transform, $\tau_A$ has two more fixed points (not visible) $\pm \sqrt{1/\zeta} : 1 : \mp \sqrt{1/\zeta}$, while the points of indeterminacy are the umbilics $\Omega_x \simeq 0 : 0 : 1 \Omega_y \simeq 1 : 0 : 0$ together with $1/\zeta : -1/t : 1/z$ (the reflection of A into the unit imaginary circle). Moreover, $\tau_A^{-1}$ is obtained by $t \mapsto -t$.

Theorem 19.1.5. The $P$-metric near the $P$-point $A_P$ is induced by the $P$-metric near $O$ and we have:

$$(ds_p)^2 (A) = C t^2 \frac{dz d\zeta}{(1 - z\zeta)^2} \quad (19.1)$$

Proof. Use the $\tau_A^{-1}$ conveyor to carry back $A + dA \simeq z + dz : 1 : \zeta + d\zeta$ and then use the $P$-metric at $O$.

Proposition 19.1.6. The $P$-line through points $A, B$ is the visible part of the $C$-circle through $A, B, \hat{A}, \hat{B}$. When $A$ is fixed, the supports of the $P$-lines through $A$ are the members of the $C$-isotomic $(A, A)$ pencil.

Proof. Since $\tau_A$ is conformal, we have $\Delta_P (A, B) \perp \Gamma$ and therefore $\hat{A}$ belongs to the $C$-circle which supports the $P$-line.

Proposition 19.1.7. The $P$-circles $\gamma_P (A, \rho)$ are the members of the $C$-isotomic $(A_P, \hat{A}_P)$ pencil, orthogonal to the former $C$-pencil of the $P$-lines through $A_P$.

When $M_1 \simeq z_1 : t_1 : \zeta_1$, then $\gamma_P (A, M_1) = \gamma_C (U, \rho_C)$ where

$$U \simeq \begin{bmatrix}
    t z (t_1^2 - z_1 \zeta_1) \\
    t_1 (t_1^2 t_1 - t z \zeta_1 - t \zeta z_1 + z \zeta t_1) \\
    t \zeta (t_1^2 - z_1 \zeta_1)
\end{bmatrix}$$

$$\rho_C^2 = \frac{(\zeta_1 t - t_1 \zeta_1)(z_1 t - z_1 t_1)(t_1 t_1 - z_1 \zeta_1)(t_1 - t_1 \zeta)}{t_1^2 (t_1^2 t_1 - t z \zeta_1 - t \zeta z_1 + z \zeta t_1)^2}$$

$$\rho_C = \frac{\text{distance}(M, A) \text{distance}(M, \hat{A})}{2 \text{distance}(M, \text{med}(A, \hat{A}))}$$

Proof. Adjust the value of $\rho$ in $V (\gamma) \simeq p V (A, 0) + (1 - p) V \left(\hat{A}, 0\right)$ so that $\gamma$ goes through $M_1$ and obtain

$$\gamma_P \simeq \begin{bmatrix}
    t \zeta (z_1 z_1 - t_1^2) \\
    - (z \zeta + t^2) z_1 \zeta_1 + t t_1 (z \zeta_1 + \zeta z_1) \\
    z t (z_1 z_1 - t_1^2) + (z \zeta + t^2) t_1^2 - t t_1 (z \zeta_1 + \zeta z_1)
\end{bmatrix}$$

Proposition 19.1.8. As points on the $C$-line $\hat{A} \hat{A}$, the center $U$ and the diametrical points $u_1, u_2 = \gamma_C \cap \hat{A} \hat{A}$ are given by the following Geogebra commands

- $U =$barycenter($\{A, A\}$, $\{\text{distance}(M, \hat{A})^2, -\text{distance}(M, A)^2\}$)

- $u_1 =$barycenter($\{A, A\}$, $\{\text{distance}(M, \hat{A}), -\text{distance}(M, A)\}$)

- $u_2 =$barycenter($\{A, A\}$, $\{\text{distance}(M, \hat{A}), -\text{distance}(M, A)\}$)

Proof. Simple substitution.

Remark 19.1.9. Spoiler: the line $\text{med}(A_P, \hat{A}_P)$ is the $\Gamma$ polar of the later defined $A_K$. 

Published under the GNU Free Documentation License
Proposition 19.1.10. The P-symmetry \( \sigma_A : M \mapsto M' \) where \( M' \) belongs together to the P-line \( AM \) and to the P-circle \( (A;M) \) is the following homography:

\[
Z \mapsto (z \zeta + 1) Z - \frac{2z z \zeta}{2Z \zeta - (z \zeta + 1)} ; \quad \begin{pmatrix} Z \\ T \\ \overline{Z} \end{pmatrix} \mapsto \begin{pmatrix} \frac{(z \zeta + t^2) Z - 2tz T}{2t \zeta Z - (z \zeta + t^2) T} \\ \frac{1}{2t \zeta Z - (z \zeta + t^2) T} \end{pmatrix}.
\]

Proof. One has \( \sigma_A = \tau_A \circ \sigma_O \circ \tau_A^{-1} \) where \( \sigma_O = Z : T \mapsto -Z : T : -\overline{Z} \). As a result, \( \hat{A} \) is also invariant and we have cross ratio \( \frac{C(A, \hat{A}, M, M')}{-1} \).

Construction 19.1.11. Construct the P-circle when a P-diameter \([M,N]\) is known

1. Draw the C-circle \( (B) = (M,N,\hat{M}, \hat{N}) \). This is the P-line \( MN \).
2. Draw the C-tangents \( MC \) and \( NC \) to the C-circle \((B)\) and obtain \( C \).
3. The required P-circle \( \gamma \) is the C-circle \((C;M)\).
4. Add point \( Q \) on \( \gamma \). The C-tangent to \( \gamma \) at \( Q \) cut the C-mediatrix of \([Q, \hat{Q}]\) at some point \( B_Q \). Then the C-circle \((B_Q;Q)\) is a P-diameter of \( \gamma \).
5. The intersection of both diameters gives \( A(\text{visible}) \) and \( \hat{A} \) (not visible).

Construction 19.1.12. Construct the midpoint \( J \) of the P-segment \([O,A]\).

1. Draw the C-circle \( \delta \) having \([A,O]\) for diameter.
2. Add a point \( R \) on this circle. Draw the C-tangent at \( R \). Cut by the C-mediatrix of \([R, \hat{R}]\) and obtain \( B_R \).
3. Then the C-circle \((B_R;R)\) is another diameter of \( \delta \), obtaining \( J \).

19.2 The Klein plane

Definition 19.2.1. The Klein model is what is obtained by using collineations as conveyors.

Theorem 19.2.2. The Klein conveyor is the collineation defined by:

\[
\vartheta_A : O \mapsto A \simeq \begin{pmatrix} k \\ 1 \end{pmatrix} ; \quad \begin{pmatrix} Z \\ T \\ \overline{Z} \end{pmatrix} \mapsto \begin{pmatrix} \frac{1 + W}{2} & k & \frac{1 - W}{2} \\ \kappa/2 & 1 & \kappa/2 \\ \kappa \left(\frac{1 - W}{2}\right) & \kappa & \frac{1 + W}{2} \end{pmatrix} \cdot \begin{pmatrix} Z \\ T \\ \overline{Z} \end{pmatrix},
\]

where \( W = \sqrt{1 - \kappa} \).

Proof. Matrix \( \vartheta_A \) fulfills its requirements. Moreover, this matrix diagonalizes as

\[
\begin{pmatrix} 1 + W_a \\ 1 - W_a \\ \sqrt{1 - W_a^2} \end{pmatrix} ; \quad \begin{pmatrix} k & k & -k \\ W_a & -W_a & 0 \\ \kappa & \kappa & \kappa \end{pmatrix} \text{ where } W_a = \sqrt{\kappa}.
\]

The proper columns are related to the unavoidable fixed points, namely the intersections \( OA \cap \Gamma \) and the \( \Gamma \)-pole of the \( OA \) line, while one has also \( \lambda_1 \lambda_2 = \lambda_3^2 \). As a result, we have not only the existence, but also the unicity of \( \vartheta_A \). Moreover, here again, \( \vartheta_A^{-1} \) is nothing but \( \vartheta_{-A} \).
Theorem 19.2.3. The K-metric near the K-point \( A_K \simeq k : 1 : \kappa \) is:

\[
(d s_K)^2 (A_K) = \frac{\text{Cte}^2}{4} \left( \frac{d k d \kappa}{1 - k \kappa} + \frac{(\kappa d k + k d \kappa)^2}{4(1 - k \kappa)^2} \right)
\]

See Theorem 19.2.3 for the fact that Cte = 2.

Proof. Use the \( \vartheta_A^{-1} \) conveyor to carry back \( A_K + dA \simeq k + dk : 1 + k \kappa \) near the origin and then use the K-metric at \( O \).

Remark 19.2.4. The K-line through \( A, B \) is nothing but the \( C \) line through the same points. Circles centered at \( A_K \) will be studied later.

Proposition 19.2.5. The K-symmetry \( s_A : M \mapsto M' \) centered at \( A_K \) is the involutive collineation defined by:

\[
s_A \simeq \frac{1}{1 - k \kappa} \begin{bmatrix}
-1 & 2k & -k^2 \\
-k & 1 + k \kappa & -k \\
-k^2 & 2\kappa & -1
\end{bmatrix}
\]

Proof. This comes from \( s_A = \vartheta_A \circ \vartheta_A \circ \vartheta_A^{-1} \). Alternate proof: diagonalize and obtain:

\[
\begin{pmatrix}
+1 \\
-1 \\
-1
\end{pmatrix} \circ \begin{bmatrix}
k & \kappa^{-1} & -k \\
1 & 1 & 0 \\
\kappa & k^{-1} & \kappa
\end{bmatrix}
\]

The proper columns are related to the unavoidable fixed points, namely \( A_K \) itself together with \( A_K \) and the \( \Gamma \)-pole of \( OA \). As a result \( s_A \) is an homology.

19.3 From a model to the other

Proposition 19.3.1. Consider \( \Delta_P \) and \( \Delta_K \), the P- and K-lines sharing the same \( C \)-turns \( \alpha, \beta \) as points at horizon \( \partial \mathbb{H} \). When written in \( \mathbb{P}_C(\mathbb{C}^1) \), the projective space of the \( C \)-cycles, the maps \( \Phi \circ \Phi : \Delta_P \mapsto \Delta_K \) and \( \Phi \circ \Phi : \Delta_K \mapsto \Delta_P \) (Klein to Poincaré and converse) are:

\[
\begin{align*}
V (\Phi (\Phi (\Delta_K))) &= V (\Delta_K) + V (\text{horz}) \\
V (\Phi (\Phi (\Delta_P))) &= V (\Delta_P) - V (\text{horz})
\end{align*}
\]

(when normalizing the \( C \)-circles by \( U_4 = 1 \) and the \( C \)-lines by \( U_2 = 2 \)).

Proof. We have \( \Gamma \simeq 0 : -1 : 0 : 1 \) while \( V (\Delta_P) \simeq * : 1 : * : 1 \) and \( V (\Delta_K) \simeq * : * : * : 0 \).

Proposition 19.3.2. The map \( \Phi \circ \Phi \) transforms the pencil of all the P-lines through a given \( A_P \) into the pencil of all the K-lines through a same point \( A_K \). And one has the punctual maps:

\[
\begin{align*}
\Phi \circ \Phi (A_P) &= \Phi (A_K) = \begin{pmatrix} 2z t \\ t^2 + z \zeta \\ 2 \zeta t \end{pmatrix} = A_K \\
\Phi \circ \Phi (A_K) &= \Phi (A_P) = \begin{pmatrix} 1 - \sqrt{1 - k \kappa} \zeta \\ 1 \\ 1 - \sqrt{1 - k \kappa} k \end{pmatrix} = A_P
\end{align*}
\]

Proof. Straightforward computation. Moreover: \( k = \frac{2z}{1 + z \zeta} = (z + \bar{z})/2 \).

Proposition 19.3.3. The Poincaré ↔ Klein maps can be illustrated by involving the 3D sphere \( S \) having \( \Gamma \) as equatorial circle. Let \( N \) be the North pole of \( S \). Start from a point \( A_P \) in the P-plane. Draw the \( C \)-line \( NA \). It cuts the sphere at a point \( Q \) (the stereographic projection). And then draw the vertical of \( Q \), intersecting the equatorial plane at \( A_K \).
Proof. Assume \( t = 1 \) and describe \( Q \) as a point in \( \mathbb{C}^2 \times \mathbb{R} \). Then
\[
\begin{pmatrix}
z_Q \\
t_Q
\end{pmatrix} = \frac{1}{1 + z \zeta} \begin{pmatrix} 2 z \\ 1 - z \zeta \end{pmatrix} = \left( \frac{k}{\sqrt{1 - k \kappa}} \right)
\]
The first sign is the stereographic formula, the second one is (19.3).

Figure 19.1: The Poincaré to Klein transform

**Proposition 19.3.4.** The value of the Cte at Theorem 19.1.5 and Theorem 19.2.3 is \( \text{Cte} = 2 \).

**Proof.** At Figure 19.1, one sees that \( A_P \approx O \) implies \( dS = 2dA_P \) together with \( dS = dA_K \).

**Remark 19.3.5.** The symmetry wrt a point in \( \mathbb{H} \) induces an involutive map on \( \partial \mathbb{H} \)
\[
\alpha \mapsto \beta = \frac{(t^2 + z \zeta) \alpha - 2 z t}{2 \zeta t \alpha - (t^2 + z \zeta)} = \frac{\alpha - k}{\alpha \kappa - 1}
\]
while the symmetry wrt the \( \mathbb{H} \)-line \( (\gamma, \delta) \) induces the map
\[
\alpha \mapsto \beta = \frac{(\gamma + \delta) \alpha - 2 \gamma \delta}{2 \alpha - (\gamma + \delta)}
\]

**Remark 19.3.6.** One has the following relation
\[
\text{cross\_ratio} (a, b, u, v) \times \text{cross\_ratio} (b, c, u, v) = \frac{(u - a)(v - b)}{(u - b)(v - a)} \times \frac{(u - b)(v - c)}{(u - c)(v - b)} = \text{cross\_ratio} (a, c, u, v)
\]
and therefore the function \( \ln(\text{cross\_ratio}) \) is additive.

**Proposition 19.3.7.** Consider the \( \mathbb{C} \)-circles corresponding to the \( P \)-lines \((\alpha, \beta)\) and \((\gamma, \delta)\). And then define the four points
\[
U_+^a = \left( \begin{array}{c}
\alpha \beta - \delta \gamma \pm W (\beta - \gamma)(\alpha - \delta) \\
\alpha \beta - \delta - \gamma \\
\beta \delta - \alpha \gamma - \beta \gamma - \beta \delta \gamma
\end{array} \right)
\]
where \( W = \sqrt{\frac{(\beta - \delta)(\alpha - \gamma)}{(\beta - \gamma)(\alpha - \delta)}} \)

Then the points \( U_+^a, U_-^a \) are \( \Gamma \)-inverse of each other and belong to the given \( P \)-lines, while the points \( U_+^b, U_-^b \) belong to \( \partial \mathbb{H} \) and define a \( P \)-line orthogonal to the given \( P \)-lines.

When \( W^2 < 0 \), points \((\alpha, \beta)\) and \((\gamma, \delta)\) are tangled, points \( U_+^a, U_-^a \) are \( \mathbb{C} \)-visible and one of them belongs to \( \mathbb{H} \). When \( W^2 > 0 \), points \((\alpha, \beta)\) and \((\gamma, \delta)\) are untangled, points \( U_+^b, U_-^b \) are \( \mathbb{C} \)-visible and define a (visible) \( \mathbb{H} \) line.

**Proof.** Direct computation. Remark: when the given lines are orthogonal to each other, then \( W^2 = -1 \).
19.4 More about hyperbolic distance

Proposition 19.4.1. In the K-plane, the distance between the K-points $M_1, M_2$ is

$$\frac{1}{2} \ln \text{cross} \_ \text{ratio} \ (M_1, M_2, M_3, M_4)$$

(19.4)

where $M_3, M_4 \in \Gamma$ and $M_4, M_1, M_2, M_3$ are K-aligned in this order.

Proof. Use $M_1, M_2, M_3, M_4 = x_1, x_2, -1, +1$ on the x-axis. Substitute $k = \kappa = x$ and $dk = d\kappa = dx$ into

$$(ds_K)^2 = \frac{C\kappa}{4} \left( \frac{1}{1 - \kappa} + \frac{1}{\kappa + 1} \right)$$

and obtain $ds = \frac{C\kappa}{4} \ln \left( \frac{x_2 + 1}{x_2 - 1} \right)$. Integrating, we obtain

$$\delta = \frac{C\kappa}{4} \ln \left( \frac{x_2 + 1}{x_2 - 1} \right)$$

i.e. the required cross-ratio formula. And this applies to all pair of points since the cross-ratio is invariant under any conveyor. \qed

Proposition 19.4.2. In the P-plane, the distance between the P-points $M_1, M_2$ is

$$\ln \text{cross} \_ \text{ratio} \ (M_1, M_2, M_3, M_4)$$

(19.5)

where $M_3, M_4 \in \Gamma$ and $M_4, M_1, M_2, M_3$ are P-aligned in this order.

Proof. Let $C, D \simeq \gamma : 1 : \gamma^{-1}, \delta : 1 : \delta^{-1} \in \Gamma$. They define the circle $\gamma \simeq \left[ \frac{-2}{\gamma + \delta}, 1, -\frac{2}{\gamma + \delta}, 1 \right]$. Therefore the points $A_P, B_P$ on $\Delta_P (C, D)$ can be written as

$$A_P, B_P \simeq \left( \frac{2 \gamma \delta}{\delta + \gamma}, \frac{1}{2}, \frac{i(\delta - \gamma)}{\delta + \gamma}, \frac{\alpha}{1 + \alpha} \right), \text{etc}$$

where $\alpha, \beta$ are some turns. And a straightforward computation leads to:

$$\text{cross} \_ \text{ratio} \ (A_K, B_K, C, D) = \left( \frac{(i\alpha + \gamma)(i\beta - \delta)}{(i\beta + \gamma)(i\alpha - \delta)} \right)^2 = \text{cross} \_ \text{ratio} \ (A_P, B_P, C, D)^2$$ \qed

Theorem 19.4.3. There are algebraic expressions for the cosh of the hyperbolic distances:

$$\cosh (d_P(A_P, B_P)) = 1 + \frac{2\delta_{AB}}{p_A p_B}$$

$$\cosh (d_K(A_K, B_K)) = \frac{\delta_{AB}^2 - p_A - p_B}{2\sqrt{p_A p_B}} = \frac{(t_2 t_1 - \frac{1}{2} (z_1 \zeta_2 + z_2 \zeta_1))}{\sqrt{t_2^2 - z_2 \zeta_2 \sqrt{t_1^2 - \zeta_1 z_1}}}

(19.6)$$

Proof. For the P formula: obtain $M_3, M_4$ from $\Delta_P \cap \Gamma$. This involves the radical

$$W = \sqrt{(t_1 t_2 - \zeta_1 z_2)(t_1 t_2 - z_1 \zeta_2)(t_1 z_2 - t_2 z_1)(t_1 \zeta_2 - \zeta_1 t_2)}$$

Then substitute into (19.5) and take the cosh. For the K formula, the same method can be used, but using $\Psi$to$\Phi$ is another possibility. \qed

Corollary 19.4.4. In the Poincaré model, we have the additional formula:

$$\cosh^2 (d_P(A_P, B_P)) / 2 = \frac{(t_2 t_1 - z_1 \zeta_2)(t_2 t_1 - \zeta_1 z_2)}{(t_1^2 - z_1 \zeta_1)(t_2^2 - z_2 \zeta_2)} = 1 + \frac{\delta_{AB}^2}{p_A p_B}$$

$$\sinh^2 (d_P(A_P, B_P)) / 2 = \frac{(|z_2| - |z_1|)(|z_2| - z_1 t_2)}{(t_1^2 - z_1 \zeta_1)(t_2^2 - z_2 \zeta_2)} = \frac{\delta_{AB}^2}{p_A p_B}$$

$$\tanh (d_P(A_P, B_P)) / 2 = \frac{|z_1 - z_2|}{|1 - z_1 \zeta_2|} = \frac{\delta_{AB}}{\sqrt{\delta_{AB}^2 + p_A p_B}}$$
Remark 19.4.5. Another way to write (19.6) is
\[
\cosh (d_K (M, N)) = \frac{1 - (M | N)}{\sqrt{(1 - (M | M))(1 - (N | N))}}
\]

**Proposition 19.4.6. Hyperbolic Pythagoras Theorem.** When \(a, b\) are the right-angled sides and \(c\) is the third side, one has
\[
\cosh c = \cosh a \cosh b
\]

**Proof.** Consider the triangle \(0, x, iy\) and use either the Klein or the Poincaré formulas. \(\Box\)

**Proposition 19.4.7.** The \(\mathbb{C}\)-homology whose center \(A_K\) and axis \(\Delta\) are \(\Gamma\)-polar of each other describes the \(K\)-symmetry wrt \(A_K \simeq k : 1 : \kappa\) when this point is inside the Klein disk \(\mathbb{H}\) but describes the \(K\)-symmetry wrt \(\Delta\) when \(A_K\) is in the outer world (and \(\Delta_K\) is a real \(K\)-line).

**Proof.** The matrix \(s_A\) of this homology is given at (19.2). We have:
\[
\cosh^2 (d_K (A_K, M_j)) = \frac{(\kappa Z + kZ - 2T)^2}{4(T^2 - Z\overline{Z})(1 - \kappa k)}
\]
\[
\cosh^2 (d_K (U_K, M_j)) = 1 + \frac{(\kappa Z + kZ - 2T)^2}{4(T^2 - Z\overline{Z})(\kappa k - 1)}
\]
where \(M_2 = s_A \cdot M_1\) and \(U_K = \Delta \cap M_1M_2\). \(\Box\)

**Proposition 19.4.8.** In the \(P\)-plane, the previous proposition becomes:
\[
M_2, M'_2 \simeq \begin{pmatrix}
(t^2 + \zeta z) Z - (2t z) T \\
(2t \zeta) Z - (t^2 + \zeta z) T \\
(2t z) \overline{Z} - (t^2 + \zeta z) T
\end{pmatrix} \begin{pmatrix}
(2t z) \overline{Z} - (t^2 + \zeta z) T \\
(t^2 + \zeta z) Z - (2t \zeta) T \\
(2t \zeta) Z - (t^2 + \zeta z) T
\end{pmatrix}
\]
where \(M_2\) is the \(P\)-symmetric of \(M_1\) wrt \(A_P \simeq z : t : \zeta\) when \(A_P \in \mathbb{H}\), while \(M_3\) is the \(P\)-symmetric of \(M_1\) wrt the \(P\)-line which is \(\mathbb{C}\)-centered at \(A_P\) when \(A_P \notin \mathbb{H}\).

**Proof.** Substitute \(k : 1 : \kappa\) with \(\mathfrak{HtoR}(z : t : \zeta)\), etc, compute and then goes back using \(\mathfrak{HtoP}\). Remark: \(M_2, M'_2\) are \(\mathbb{C}\)-inverse wrt \(\Gamma\). \(\Box\)

**Exercise 19.4.9.** When the radius \(\rho\) varies, the \(K\)-circles \(\gamma_K (\mathfrak{A}, \rho)\) form a family of \(\mathbb{C}\)-ellipses. Find the locus of their \(\mathbb{C}\)-foci.

**Exercise 19.4.10.** Consider the \(P\)-points \(A, B\). Determine the parameter \(R\) corresponding to the \(K\)-circles \((\mathfrak{A}, \mathfrak{B})\).

### 19.5 Hyperbolic triangle

(list of results)

1. The \(K\)-line is equal to the \(\mathbb{C}\)-line, while the \(P\)-line is \(C(B, \hat{B}, C, \hat{C})\)

2. The point \(O_{BC} \simeq \text{Polar} (\text{klin} A, \text{horz}) = \text{Center} (\text{plin} A)\) is outside of \(\mathbb{H}\).
19.5.2 Line-bisectors

1. The $P$-formula are:

$$\nu \left( \mu_p \right) \simeq p_B V(A_1) - p_C V(B_1)$$

$$\omega = \left( p_2 \frac{z_1}{t_3} - p_3 \frac{z_3}{t_2} \right) / \left( p_2 - p_3 \right) ; \quad \rho^2 = \omega \omega - 1$$

2. The $K$-formula are (where lines are normalized using $\Delta_2 = 1$)

$$\text{med}_A \simeq \sqrt{p_B \text{Polar}(B)} - \sqrt{p_C \text{Polar}(C)}$$

3. The three $\mathbb{H}$-line-bisectors are ever collinear. As a result, the $K$-line-bisectors are ever going through some point $O_K \in \mathbb{C}$. When this point belongs to $\mathbb{H}$, there exists an $\mathbb{H}$-circle going through $A, B, C$ (and centered at $O_3$). Otherwise, $O_3$ is the pole of an $\mathbb{H}$-line and this line is the common perpendicular to the three line-bisectors.

Exercise 19.5.1. Let $B,C$ be given in $\mathbb{H}$. Find all the $A \in \mathbb{H}$ such that the $\mathbb{H}$-circle $(A,B,C)$ exist. Hint: use the $K$-model together with $B,C = \pm k ; \ A = x + iy$ and obtain the horicycles:

$$(k^2 y^2 + k^2 - x^2 - 2 y^2) \pm 2 y (k^2 - 1)$$

as boundaries. One obtains also

$$(2 k^2 y^2 - k^2 + x^2 - y^2) \pm 2 i x y (k^2 - 1)$$

Any opinion on this extra-locus?

19.5.3 Altitudes

1. $paltA = C \left( A, \hat{A}, \text{Reflect}(A, \text{pin}A) \right)$

2. $kaltA = \Delta(A, O_{BC})$

3. the three $\mathbb{C}$-lines $kaltX$ are concurrent (inside $\mathbb{H}$ or outside !)

4. The three $\mathbb{H}$-altitudes are ever collinear. As a result, the $K$-altitudes are ever going through some point $H_K \in \mathbb{C}$. When this point belongs to $\mathbb{H}$, the triangle admits an $\mathbb{H}$-orthocenter. Otherwise, $H_3$ is the pole of an $\mathbb{H}$-line and this line is the common perpendicular to the three altitudes.

Exercise 19.5.2. Let $B,C$ be given in $\mathbb{H}$. Find all the $A \in \mathbb{H}$ such that $(A,B,C)$ admits an $\mathbb{H}$-orthocenter. Hint: use the $K$-model together with $B,C = \pm k ; \ A = x + iy$ and obtain

$$(k^2 y^2 + k^2 - x^2 - 2 y^2) \pm 2 y (k^2 - 1)$$

as boundary.

19.5.4 Trigonometry

Proposition 19.5.3. When triangle $A,B,C$ is rectangular in $C$ then:

$$\sin A = \frac{\sinh a}{\sinh c} ; \ \cos A = \frac{\tanh b}{\tanh c} ; \ \tan A = \frac{\tanh a}{\sinh b}$$

$$\cosh(b) = \frac{\cos B}{\sin A} , \ \cosh(c) = \cosh(a) \cosh(b) = \frac{\cos A \cos B}{\sin A \sin B}$$

Proof. Consider the standard rectangular triangle $z_A = k, z_B = iK, z_C = 0$ and use the usual formula for the angle between two circles. \[\square\]
19.6 The Upper-half plane

Taking a point \( P \) on \( \Gamma \) and reflecting everything from the \( P \)-plane wrt the circle \( \gamma_{C}(P, 2) \) leads to another model of the hyperbolic plane, where U-circles are C-circles and U-lines are half-C-circles, ending at the horizon line.

**Exercise 19.6.1** (Soland’s porism). (2018) Take \( n \) points \( a_{j} \) on circle \( \Gamma \), with the intent to construct a sequence of \( n \) circles, sequentially tangent to each other and internally tangent to \( \Gamma \). When \( n \) is odd, there is exactly one solution. When \( n \) is even, the \( n \)-th point is determined by the others. And then, one of the radiuses can be chosen at will. 

Hint: at Figure 19.2, \( P \) is taken at \( a_{0} \) and cir\( P \), the inversion circle, is tangent to \( \Gamma \). This leads to \( a_{j} \mapsto A_{j} \). Find a relation between \( d_{12} = d_{C}(A_{1}, A_{2}) \) and \( r_{1}, r_{2} \), the C-radiuses of the C-circles \( O_{1}, O_{2} \).

![Figure 19.2: The Poincaré to Upper-half transform](image)

19.7 Teaching tensors to a computer

Informally, tensors are friendly multi-dimensional arrays implementing the "two" Einstein's rules: 

(0_1) in \( X_{k}^{j} \), there aren't exponents, but only indices, the \( j \) being "down" while the \( k \) is "up" 

(0_2) indices are implying ranges and variables 

(1): repeated index (one up, one down): \( X_{jk}^{j} \) means \( \sum_{k \in \text{range}(k)} X_{jk}^{k} \) 

(2): comma: \( X_{j,k} \) means \( \partial X_{j}/\partial x_{k} \) (assuming that index \( k \) implies variable \( x \))

**Notation 19.7.1.** In the end, we will use four set of variables, two external and two internal. Using an index will imply which variable is indexed. For example, \( \mathcal{R}_{\mu}^{\nu} \) (where \( \mathcal{R} \) means Jacobian) is to be read as \( \frac{\partial u^{\nu}}{\partial x^{\mu}} \) while \( \mathcal{R}_{\mu}^{\nu} \) is to be read as \( \frac{\partial x^{\mu}}{\partial u^{\nu}} \). Everything will be introduced in details but, for the reader’s convenience, these associations are summarized here, in Table 19.1.

<table>
<thead>
<tr>
<th>index</th>
<th>range</th>
<th>tag</th>
<th>variables</th>
<th>comment</th>
<th>here</th>
</tr>
</thead>
<tbody>
<tr>
<td>( m, n, p, q, r )</td>
<td>1.3</td>
<td>x</td>
<td>( x, y, z )</td>
<td>external</td>
<td>cartesian</td>
</tr>
<tr>
<td>( a, b, c, d, e )</td>
<td>1.3</td>
<td>w</td>
<td>( u, v, w )</td>
<td>extended</td>
<td></td>
</tr>
<tr>
<td>( \mu, \nu, \phi, \psi, \rho )</td>
<td>1.2</td>
<td>u</td>
<td>( u, v )</td>
<td>1° internal</td>
<td>stereographic</td>
</tr>
<tr>
<td>( \sigma, \tau, \epsilon, \omega, \kappa )</td>
<td>1.2</td>
<td>t</td>
<td>( t, s )</td>
<td>2° internal</td>
<td>spheric</td>
</tr>
</tbody>
</table>

Table 19.1: Indices and the associated variables
Definition 19.7.2. We define a tensor as a triple \([\text{index\_list,updo\_list,Maple\_rtable}]\). The \text{index\_list} is a Maple list of unassigned names, where repetitions are allowed. The \text{updo\_list} is a list of booleans, coded 1 for "up" and 0 for "down". When a name is repeated in the \text{index\_list}, only two occurrences are allowed, one being tagged as "up" and the other as "down".

Remark 19.7.3. Another method would have been to build a stratospheric theory using the marvelous operator \(\otimes\) (read it as \(\otimes\)). But we need something else, i.e. a practical computing tool. Using Mathematica, Sage or any other formal computing tool instead of Maple is probably possible... mind the details, they are where the devil lives.

Fact 19.7.4. When \(X\) is a Maple\_rtable, the command \text{ArrDim}, defined by \text{macro(ArrDim=ArrayTools[Dimensions])} returns a list of Maple ranges, i.e. something like \(1..3,1..2,1..4\) and then \(X[\text{tut1,tut2,tut3}]\) returns some value, assuming that variables \text{tut1,tut2,tut3} contain integer numbers inside the right ranges. Here, the \text{tut} \(j\) are freshly build Maple variables, and not the names given in the \text{index\_list}.

Maple 19.7.5. In order to implement Einstein’s (1) or (2), the naming conventions must be explicitly stated. When assuming Table 19.1 the following procedure will inform the computer of our choices:

```maple
1: \text{setvars} := \text{proc (var)} : \text{global glodex}
2: \text{if member(var, [m, n, p, q, r])} \text{then} \text{glodex} := [m, n, p, q, r] : \text{return} [x, y, z]
3: \text{else if member(var, [a, b, c, d, e])} \text{then} \text{glodex} := [a, b, c, d, e] : \text{return} [u, v, w]
4: \text{else if member(var, \{\mu, \nu, \phi, \psi, \rho\})} \text{then} \text{glodex} := [\mu, \nu, \phi, \psi, \rho] : \text{return} [u, v]
5: \text{else} \text{glodex} := [\sigma, \tau, e, \omega, \kappa] \text{return} [s, t]
6: \text{end if}
```

Listing 19.1: \text{setvars} tells our naming conventions to the computer

Maple 19.7.6. Applying some process to each element of the internal array is done by procedure Alg. 19.2. As examples, \text{action} can be \text{factor} or \(U \rightarrow \text{simplify}(U,\text{symbolic})\).

```maple
1: \text{atens} := \text{proc (qui, action)} ; \text{local tt1, tt2, tt3}
2: \text{tt1, tt2, tt3} := \text{op(qui)}
3: \text{return} [\text{tt1, tt2, map(action, tt3)}]
```

Listing 19.2: The atens procedure (A means Action)

Maple 19.7.7. While the reader is supposed to decipher \(X^j_i\) as \(\partial X^j_i/\partial u^m\), this has to be explained to a computer by procedure Alg. 19.3.

Require: qui depends on the var indexed variables

```maple
1: \text{dtens} := \text{proc (qui, var, updo)}
2: \text{local laproc, tt1, tt2, tt3, nn, lesdex, ttq, vars}
3: \text{tt1, tt2, tt3} := \text{op(qui)} ; \text{vars} := \text{setvars(var)}
4: \text{nn} := \text{nops(tt1)} ; \text{lesdex} := \text{seq(Catenate(tut,j), j = 1..nn)}
5: \text{(lesdexx, ttq)} \leftrightarrow \text{diff(tt3[lesdex], vars[ttq])}
6: \text{laproc} := \text{subs(lesdexx = lesdex, \%)}
7: \text{tt1} := [\text{op(tt1)}, \text{var}] ; \text{tt2} := [\text{op(tt2)}, \text{updo}]
8: \text{return} [\text{tt1, tt2, table(op(ArrDim(tt3)), 1..nops(vars), eval(laproc))}]
```

Listing 19.3: The dtens procedure (D means Derivation)

Maple 19.7.8. While the reader is supposed to decipher \(a^m b^n\) as another tensor build using \((m, n) \mapsto a^m \times b^n\), this has to be explained to a computer by procedure Alg. 19.4.
19. Hyperbolic geometry

Example 19.7.10. \( x^m y^n \) dw means \( \left( \frac{\partial x}{\partial u} \frac{du}{dv} + \frac{\partial x}{\partial v} \frac{dv}{du} \right) \cdot j = 1.3 \). In our formalism, a sequence DCR is used to construct this 1-index tensor. From a complexity point of view, this is far from being optimal. But, in what we are doing, the underlying arrays are not sufficiently large to require a careful optimization.

Maple 19.7.11. Adding two tensors require that indices lists are exactly the same (and the up/down list also). Reordering of indices is done by 19.6, while the addition itself is done by 19.7.

19.8 The sphere: dealing with an example

19.8.1 External and internal coordinates

Fact 19.8.1. When describing a "surface", the simplest way to proceed is embedding the surface into some larger space, leading to something like

\[(x, y, z) \in (E) \text{ means } x^2 + y^2 + z^2 - 1 = 0\]

— pldx : Translation of the Kimberling’s Glossary into barycentrics —
describes a pair of bijections

Example 19.8.2. One can check that equations \([u, v, w] = W\) and \([x, y, z] = \hat{X}\) where :

\[
W^m = \left[ a, [1], \left[ \frac{x}{1 + z}, \frac{y}{1 + z}, \frac{x^2 + y^2 + z^2 - 1}{1 + z} \right] \right]
\]

\[
\hat{X}^m = \left[ m, [1], \left[ \frac{u (w + 2)}{1 + u^2 + v^2}, \frac{v (w + 2)}{1 + u^2 + v^2}, \frac{1 + w - u^2 - v^2}{1 + u^2 + v^2} \right] \right]
\]

describes a pair of bijections \((x, y, z) \mapsto (u, v, w) \mapsto (x, y, z)\). They are differentiable quite everywhere. When making \(w = 0\), we are selecting the points of the sphere \(x^2 + y^2 + z^2 = 1\).

Let us define \(U\) and \(X\) by:

\[
U^\mu = \left[ \mu, [1], \left[ \frac{x}{1 + z}, \frac{y}{1 + z} \right] \right]
\]

\[
X^\mu = \left[ m, [1], \left[ \frac{2 u}{u^2 + v^2 + 1}, \frac{2 v}{u^2 + v^2 + 1}, \frac{1 - u^2 - v^2}{1 + u^2 + v^2} \right] \right]
\]

Then the pairs \(U = (u, w)\) are a system of internal coordinates for the sphere, while the triples \((x, y, z)\) are a system of external coordinates for the same surface, bound by the so called implicit equation of the sphere.

We will also introduce another set of internal coordinates \(T^\sigma = [t, s]\) by equations

\[
T^\sigma = \left[ \sigma, [1], \left[ \arcsin \left( \frac{1 - u^2 - v^2}{1 + u^2 + v^2} \right), \arctan \left( \frac{u}{1 + u^2 + v^2}, \frac{v}{1 + u^2 + v^2} \right) \right] \right]
\]

and their converses:

\[
U^\mu = \left[ \mu, [1], \left[ \cos (s) \cos (t), \sin (s) \cos (t) \right] \right]
\]

\[
X^\mu = \left[ [m], [1], \left[ \cos (s) \cos (t), \sin (s) \cos (t), \sin (t) \right] \right]
\]
19.8.2 Jacobians

Notation 19.8.3. More than ever, notations of Table 19.1 are used.

Fact 19.8.4. The external description of the tangent plane at \( M \) is

\[
\frac{\partial (x^2 + y^2 + z^2 - 1)}{\partial (x; y; z)} \cdot [dx, dy, dz] = 0
\]

At a regular point, i.e. almost everywhere, the \([dx, dy, dz]\) variations of the external coordinates \([x, y, z]\) are linearly correlated with the variations of any other set of coordinates. In fact, this is the very existence of such an invertible transform which decides if changing from a set of coordinates to another set is allowed or not. The matrices expressing these transformations are called the Jacobians and noted \( \mathbf{\mathcal{R}}^m_n \) (where \( \mathbf{\mathcal{R}} \) indicates Jacobian, \( \mu \) indicate new \( u, v \) and \( m \) indicates old \( x, y, z \). In other words: \( d\nu = \mathbf{\mathcal{R}}^m_n \cdot d\mu \).

19.8.2.1 Internal versus another internal

Proposition 19.8.5. The \( 2 \times 2 \) tensors \( \mathbf{\mathcal{R}}^\nu_{\sigma} \mathbf{1}^\tau_{\nu} \mathbf{1}^\tau_{\nu} \) and \( \mathbf{\mathcal{R}}^\nu_{\sigma} \mathbf{1}^\tau_{\nu} \mathbf{1}^\tau_{\nu} \), are respectively equal to and \( \mathbf{1}^\tau_{\sigma} \) and \( \mathbf{1}^\tau_{\nu} \).

Proof. Both tensors describe respectively \([ds, dt]\) and \([du, dv]\) wrt themselves, so that obvious is obvious. But, in order to check the procedures given above, let us compute explicitly the Jacobians of \((u, v)\) versus \((s, t)\) and conversely. And then rewrite them using the other set of variables. This leads to:

\[
\frac{\partial (u, v)}{\partial (s, t)} = \mathbf{\mathcal{R}}^\nu_{\sigma} = \frac{\partial (s, t)}{\partial (u, v)} = \mathbf{\mathcal{R}}^\nu_{\sigma} \sigma, [1, 0], \begin{bmatrix} -\cos (s) & -\cos (t) \sin (s) \\ 1 + \sin (t) & 1 + \sin (t) \\ -\sin (s) & \cos (t) \cos (s) \\ 1 + \sin (t) & 1 + \sin (t) \end{bmatrix}
\]

\[
\frac{\partial (s, t)}{\partial (u, v)} = \mathbf{\mathcal{R}}^\nu_{\sigma} \sigma, [1, 0], \begin{bmatrix} -u (1 + u^2 + v^2) & -v \\ -2\sqrt{u^2 + v^2} & -v (1 + u^2 + v^2) \\ -u (1 + u^2 + v^2) & +u \\ 2\sqrt{u^2 + v^2} & 1 + z \\ (1 + z) \sqrt{1 - z^2} & 1 + z \\ -y & +x \\ -y & +x \end{bmatrix}
\]

\[
\frac{\partial (s, t)}{\partial (u, v)} = \mathbf{\mathcal{R}}^\nu_{\sigma} \nu, [1, 0], \begin{bmatrix} -2 \frac{u}{1 + u^2 + v^2} \sqrt{u^2 + v^2} & -2 \frac{v}{1 + u^2 + v^2} \sqrt{u^2 + v^2} \\ -u (1 + u^2 + v^2) & +u \\ u^2 + v^2 & u^2 + v^2 \end{bmatrix}
\]

\[
\frac{\partial (s, t)}{\partial (u, v)} = \mathbf{\mathcal{R}}^\nu_{\sigma} \sigma, [1, 0], \begin{bmatrix} -\cos (s) (1 + \sin t) & -\sin (s) (1 + \sin t) \\ -\sin (s) (1 + \sin t) & \cos (s) (1 + \sin t) \end{bmatrix}
\]

\[
\frac{\partial (s, t)}{\partial (u, v)} = \mathbf{\mathcal{R}}^\nu_{\sigma} \nu, [1, 0], \begin{bmatrix} -x (1 + z) & -y (1 + z) \\ -y & +x \\ -y & +x \\ (1 + z) \sqrt{1 - z^2} & 1 + z \end{bmatrix}
\]

Therefore, \( \mathbf{\mathcal{R}}^\nu_{\sigma} \mathbf{1}^\tau_{\nu} \mathbf{1}^\tau_{\nu} \) can be computed using any of \( u, t, x \) as algebraic basis, leading to

\[
\text{seq(rten}(ctens(\text{Jac}(\sigma, \mu, X), \text{Jac}(\nu, \tau, X), \text{kron}(\mu, \nu)), \mu, \nu), X=[u, t, x]);
\]

The same remarks apply to

\[
\text{seq}(rten}(ctens(\text{Jac}(\sigma, \mu, X), \text{Jac}(\nu, \tau, X), \text{kron}(\tau, \sigma)), \sigma, \tau), X=[u, t, x]);
\]

— pldx : Translation of the Kimberling’s Glossary into barycentrics —
19.8.2.2 internal versus external

Exercise 19.8.6. Show that the \((u, v)\) versus \((x, y, z)\) Jacobians are respectively:

\[
\mathcal{N}^u_{m, n} = \begin{bmatrix}
[m, m], [1, 0], \\
\frac{1}{1 + z} & 0 & \frac{-x}{1 + z} \\
0 & 1 & \frac{-y}{1 + z}
\end{bmatrix}
\]

\[
= \begin{bmatrix}
[m, m], [1, 0], \\
\frac{1}{2} (1 + u^2 + v^2) & 1 & 0 \\
0 & 1 & -v
\end{bmatrix}
\]

\[
\mathcal{N}^m_{m, \mu} = \begin{bmatrix}
[m, \mu], [1, 0], \\
\frac{2}{(1 + u^2 + v^2)^2} & 1 - u^2 + v^2 & -2uv \\
-2uv & 1 + u^2 - v^2 & -2u
\end{bmatrix}
\]

\[
= \begin{bmatrix}
[m, \mu], [1, 0], \\
1 - x^2 + z & -yx & 0 \\
-yx & 1 - y^2 + z & 0 \\
x(1 + z) & -y(1 + z) & 0
\end{bmatrix}
\]

Moreover, compute \(\mathcal{N}^u_{m, \mu} \mathcal{N}^{m, n}_{\mu, \nu}\) and \(\mathcal{N}^m_{m, \mu} \mathcal{N}^{m, n}_{\mu, \nu}\).

Exercise 19.8.7. Show that the \((t, s)\) versus \((x, y, z)\) Jacobians are respectively:

\[
\mathcal{N}^x_{m, n} = \begin{bmatrix}
[m, m], [1, 0], \\
0 & 0 & \frac{1}{\sqrt{1 - z^2}}
\end{bmatrix}
\]

\[
= \begin{bmatrix}
[m, m], [1, 0], \\
0 & 0 & \frac{1}{\cos t}
\end{bmatrix}
\]

\[
= \begin{bmatrix}
[m, m], [1, 0], \\
0 & 0 & \frac{1}{\sqrt{2(u^2 + v^2)}}
\end{bmatrix}
\]

\[
\mathcal{N}^m_{m, \sigma} = \begin{bmatrix}
[m, \sigma], [1, 0], \\
-\cos s \sin t & -\sin s \cos t & 0
\end{bmatrix}
\]

\[
= \begin{bmatrix}
[m, \sigma], [1, 0], \\
-\sin s \sin t & \cos s \cos t & 0
\end{bmatrix}
\]

\[
= \begin{bmatrix}
[m, \sigma], [1, 0], \\
-\cos t & 0 & 0
\end{bmatrix}
\]

\[
\mathcal{N}^\sigma_{m, n} = \begin{bmatrix}
[m, \sigma], [1, 0], \\
-\frac{u(1 - u^2 - v^2)}{\sqrt{u^2 + v^2}} & -2v & -2u
\end{bmatrix}
\]

\[
= \begin{bmatrix}
[m, \sigma], [1, 0], \\
\frac{-v(1 - u^2 - v^2)}{\sqrt{u^2 + v^2}} & 1 + u^2 + v^2 & 1 + u^2 + v^2
\end{bmatrix}
\]

Moreover, compute \(\mathcal{N}^x_{m, \mu} \mathcal{N}^{m, n}_{\mu, \nu}\) and \(\mathcal{N}^x_{m, \tau} \mathcal{N}^{m, n}_{\tau, \nu}\).

19.8.3 More about the projectors

Proposition 19.8.8. The \(2 \times 2\) tensor \(\mathcal{N}^x_{m, \mu} \mathcal{N}^{m, n}_{\mu, \nu}\) equals \(1^x_4\), while the \(3 \times 3\) tensor \(\mathcal{N}^x_{m, \tau} \mathcal{N}^{m, n}_{\tau, \nu}\) describes a projector onto the tangent plane.
Proof. The first tensor describes \([dt, ds]\) wrt itself and therefore is the Kronecker tensor. The second assertion comes from the fact that the normal vector necessarily belongs to orthogonal of colspan \((\mathcal{N}_n)\). This can be checked using 
\[
pt_x, pt_u, pt_t := \text{seq(rtens(ctens(Jac(m,sigma,X),Jac(tau,n,X),kron(sigma,tau)), sigma,tau)[3], X=[x,u,t]):}
\]
which produces a matrix having \(X (X-1)^2\) as characteristic polynomial:

\[
U_t = \begin{bmatrix}
[m,n], [1,0], 1 \frac{1}{x+yz} \begin{bmatrix}
y^2 +yx -xz 
yx +x^2 -yz 
0 0 x^2+y^2
\end{bmatrix} = \frac{1}{u} \frac{1}{u^2 + v^2} \begin{bmatrix}
v^2 -uv u^2 \frac{u}{2} (u^2 + v^2 - 1)
-uv u^2 \frac{u}{2} (u^2 + v^2 - 1)
0 0 1
\end{bmatrix}
\end{bmatrix}
\]

And now, we can consider matrices \(P,Q\) where:

\[
P = \begin{bmatrix}
x y 
0
\end{bmatrix} \frac{M^m}{\sigma} = \begin{bmatrix}
x -zx -y 
y -yz +x 
0 1 -z^2 0
\end{bmatrix} \cdot \text{diag} \left( 1, \frac{1}{W}, 1 \right)
\]

\[
Q = \begin{bmatrix}
1 1 -z^2 
0 0
\end{bmatrix} \frac{M^n}{\sigma} = \begin{bmatrix}
x y z 
0 0 W
\end{bmatrix}
\]

Then \(PQ = 1\) while \(Q \cdot U_t \cdot P = \text{diag}(0,1,1)\).

19.8.4 The metric tensor

Proposition 19.8.9. For each set of variables, the metric tensor \(\mathfrak{g}_{mn}\) is defined by

\[
ds^2 = dx^m \mathfrak{g}_{mn} dx^n
\]

This is indeed a tensor, since a change of variable results into

\[
\mathfrak{g}_{\mu\nu} = N^m_{\mu} N^n_{\nu} \mathfrak{g}_{mn} \tag{19.7}
\]

Proof. Substitute \(dx^n = \mathcal{N}^n_{\nu} du^\nu\), etc and obtain an equality which must hold for all the \(\overrightarrow{du}\)

Example 19.8.10. Here are the values of the two metric tensors \(\mathfrak{g}_{\mu\nu}\) and \(\mathfrak{g}_{\sigma\tau}\).

\[
\mathfrak{g}_{\mu\nu} = \begin{bmatrix} \mu, \nu, [0,0], \frac{2}{1+u^2+v^2} \end{bmatrix}^2 \begin{bmatrix} 1 & 0 
0 & 1 
\end{bmatrix}
\]

\[
\mathfrak{g}_{\sigma\tau} = \begin{bmatrix} \sigma, \tau, [0,0], 1 \end{bmatrix} \text{cos}^2 \begin{bmatrix} 1 
0 
\end{bmatrix}
\]

\[
\mathfrak{g}_{\sigma\tau} = \begin{bmatrix} \sigma, \tau, [0,0], \frac{2(u^2+v^2)}{1+u^2+v^2} \end{bmatrix}^2 \begin{bmatrix} 1 
0 
0 & 1 -z^2 
\end{bmatrix}
\]

- pldx : Translation of the Kimberling’s Glossary into barycentrics -
19.9 Christoffel symbols

Notation 19.9.1. More than ever, notations of Table 19.1 are in use.

19.9.1 Covariance

Definition 19.9.2. God said to Abraham: "down is covariant, up is contravariant. Don’t discuss any more about that".

Remark 19.9.3. Before the divine decree, some theologians were arguing that

\[ \nabla \equiv [\bar{e}_1^1, \bar{e}_2^2] \cdot (dx^1_1; dx^2) = dx^1 \bar{e}_1^1 + dx^2 \bar{e}_2^2 \]

provides a model for covariant/contravariant. If \(\nabla\) is supposed to be a real, observable thing then the \(dx^1\) have to be divided by a factor 2 when the \(\bar{e}_j^i\) are multiplied by the same factor 2. Indeed, observable doesn’t means that all observers obtain the same figures for their measurements, but that each observer can compute and predict which figures another specified observer will obtain for her measurements.

But the devil is in the details. The symbol \(dx^1\) means the first element in the set of all the \((dx^i)_{i \in J}\), while \(dx^j\) means all of them... i.e. another copy of \((dx^i)_{i \in J}\).

Remark 19.9.4. In linear algebra, \(x = P \cdot x’\) implies \(x = P^{-1} \cdot x\) where \(P\) is the change of basis matrix. And this is because basis \(= P \cdot \text{basis}\). Here, in tensor calculus, you only have to write:

\[ \mathbf{e}_{\mu \nu} = \mathbf{e}_{mn} \mathbf{e}_{\mu n} \mathbf{e}_{\nu m} ; \mathbf{e}_{mn} = \mathbf{e}_{\mu n} \mathbf{e}_{\nu m} \]

and the pairing of indices tells you which is the one to use from \(\mathbf{e}_{mn}\) or \(\mathbf{e}_{m n}\).

19.9.2 Taking the steepest line as an example

Definition 19.9.5. Vector \(\vec{k} \equiv k_1, k_2, k_3 \equiv \{k_1, k_2, k_3\}\) is a "true" vector in the \(\mathbb{C}^3\) space. Its tensor representation is \([m], [1], [k_1, k_2, k_3]\). Then the tensor:

\[(\wedge \vec{k})^m_n \equiv \left[ [m, n], [1, 0], \begin{bmatrix} 0 & k_3 & -k_2 \\ -k_3 & 0 & k_1 \\ k_2 & -k_1 & 0 \end{bmatrix} \right] \]

implements the rule \((\wedge \vec{k})^m_n (\nabla)^n = (\nabla \wedge \vec{k})\).

Fact 19.9.6. When seeing the usual sphere \((E)\) as embedded into \(\mathbb{C}^3\), a normal vector to \((E)\) at \(M = (x, y, z)\) is \(\vec{n} = \vec{e}_3^\perp = \{0, 0, 1\}\). Then an horizontal vector at \(M\) and a North pointing vector at \(M\) are respectively \(B^m\) and \(A^m\) where

\[ B^m \equiv (\wedge \vec{k})^m_n (\bar{e}_3^2)^n \equiv [m], [1], [-y, x, 0] \]
\[ A^m \equiv (\wedge \vec{k})^m_n A^n \equiv [m], [1], [xz, yz, -x^2, -y^2] \]

On the other hand, the horizontal line and the steepest line at \(M\) are respectively \(\mathbf{e}_{mn} B^m\) and \(\mathbf{e}_{mn} A^m\).

Fact 19.9.7. This requires that \(A^m\) and \(B^m\) aren’t null, i.e. that \((x, y, z) \neq (0, 0, \pm 1)\). Moreover, these vectors belong to the tangent plane. Therefore, it makes sense to compute the "local coordinates" of these vectors using the formula already in use for the \(d\), namely \(d^m \equiv d^m \mathbf{e}_{m n}\). We have:

\[ B^\mu \equiv B^m \mathbf{e}_{m n} = [\mu], [1], [-v, +u] \quad ; \quad A^\mu \equiv A^m \mathbf{e}_{m n} = [\mu], [1], [+u, +v] \]
\[ B^\sigma \equiv B^m \mathbf{e}_{m n} = [\sigma], [1], [0, 1] \quad ; \quad A^\sigma \equiv A^m \mathbf{e}_{m n} = [\sigma], [1], [-\cos t, 0] \]

And we can check that \(B_\sigma = B^\mu \mathbf{e}_{\mu \sigma}\), etc, as required for a covariant tensor while we can check that \(A^\mu = A^\sigma \mathbf{e}_{\sigma \mu}\), etc, as required for a contravariant tensor.\(^1\)

\(^1\)If you are affected by a dyslexia disorder more severe than 5%, then never ever use "contravariant" and "like"
19.9.3 Computing the Christoffels

A theoretical definition of the Christoffel symbols is delayed to a later subsection. Indeed, we want to discuss how to correctly define these objects, and indicate some wrong ways of doing. Therefore, we will introduce these symbols by taking their most important property... as a provisional definition, which can be used to check every equality given in this document.

**Definition 19.9.8.** Quantities $\Gamma_{\psi|\mu\nu}$ and $\Gamma^{\psi}_{\mu\nu}$ (where $\mu, \nu, \phi, \psi$ are internal indices) are respectively called the Christoffels of first kind and second kind. They are not tensors, but will be used to form other tensors by contraction. They are computed from the metric, using

$$
\Gamma_{\psi|\mu\nu} = \frac{1}{2} (\mathcal{G}_{\psi\nu,\mu} + \mathcal{G}_{\psi\mu,\nu} - \mathcal{G}_{\mu\nu,\psi})
$$

$$
\Gamma^{\psi}_{\mu\nu} = (\Gamma_{\psi|\mu\nu} \mathcal{G}^{\phi\psi} \iff (\Gamma_{\psi|\mu\nu} = \mathcal{G}_{\phi\psi} \Gamma^{\phi}_{\mu\nu})
$$

**Maple 19.9.9.** The Alg. 19.8 algorithm implements the computation of the Christoffel symbols.

Require: $\sigma, \tau, v, t$ must be compatible with $\omega$

1: chdotens := proc ($\omega, \sigma, \tau, t$) := global glodex, part1
2: local setglodex, lesvars, part1, fu1, gu2, qq, eq, tmp1, tmp2
3: lesvars := setvars($\omega$); setglodex := convert(glodex, set)
4: if \{\{, $\sigma, \tau)$ \} is a variable \} then Error("wrong indexes") end if
5: if not member(t, lesvars) then Error("wrong variable") end if
6: qqq := setglodex minus \{, $\omega, \sigma, \tau$\}; qq := qqq[1] – un test a lieu
7: part1 := dtens(G($\sigma, \tau, t$, qq)); fu1 := mkfu(part1)
8: gu2 := $\omega, \sigma, \tau$ \(\mapsto\) eval(((fu1($\omega, \sigma, t$) $+$ fu1($\sigma, \tau, t$) $-$ fu1($\tau, \sigma, t$))/2
9: tmp1 := [[], $\sigma, \tau\], [0, 0, 0], rtable(1..2, 1..2, 1..2, gu2)]
10: tmp2 := ctenst(uGG(qq, $\omega, t$), tmp1)
11: return tmp1, rtens(tmp2, $\omega$

Ensure: in $\Gamma_{\omega|\sigma\tau}$ and $\Gamma^{\sigma}_{\sigma\tau}$ the first index is the "derivative" index

**Exercise 19.9.10.** Consider the usual unit sphere $x^2 + y^2 + z^2 = 1$ and introduce the usual geographic coordinates $(t, s)$ by the equations

$$[x, y, z] = [\cos t \cos s, \cos t \sin s, \sin t],$$

where the latitude $t$ ranges from $-\pi/2$ (aka 90° South) to $+\pi/2$ (aka 90° North). Everybody acts like that, except from a small minority of snobs. Therefore, $\cos t \geq 0$ is ever assumed. Show that the corresponding Christoffels are:

$$
\begin{array}{c|cc|cc}
\hline
[t, s] & 111 & 112 & 211 & 212 \\
121 & 122 & 221 & 222 \\
\hline
\Gamma_{\omega|\sigma\tau} & 0 & 0 & 0 & -\cos t \sin t \\
& \cos t \sin t & -\cos t \sin t & 0 & \cos t \sin t \\
& -\cos t \sin t & -\cos t \sin t & 0 & \cos t \sin t \\
\hline
\Gamma^{\sigma}_{\sigma\tau} & 0 & 0 & 0 & -\tan t \\
& \cos t \sin t & -\cos t \sin t & 0 & -\cos t \sin t \\
& -\cos t \sin t & -\cos t \sin t & 0 & -\cos t \sin t \\
\end{array}
$$

**Exercise 19.9.11.** Consider now the usual stereographic coordinates $(u, v)$ defined by:

$$[x, y, z] = \left[\frac{2u}{1 + u^2 + v^2}, \frac{2v}{1 + u^2 + v^2}, \frac{1 - u^2 - v^2}{1 + u^2 + v^2}\right]$$

and show that the corresponding Christoffels are:

in the same sentence. Better use: $A$ is contravariant, while $dz$ is also contravariant. Don’t create a mess like "the north end of a compass is attracted to the south magnetic pole of the earth, which lies close to the geographic north pole".
\[
\begin{array}{|c|c|c|c|c|c|c|}
\hline
[u, v] & K & 111 & 112 & 211 & 212 \\
\hline
\multicolumn{2}{|c|}{\Gamma_{\psi|\mu\nu}} & \frac{8}{(u^2 + v^2 + 1)^3} & -u & -v & +v & -u \\
\multicolumn{2}{|c|}{} & -v & +u & -u & -v \\
\hline
\multicolumn{2}{|c|}{\Gamma_{\phi|\mu\nu}} & \frac{2}{u^2 + v^2 + 1} & -u & -v & v & -u \\
\multicolumn{2}{|c|}{} & -v & +u & -u & -v \\
\hline
\end{array}
\]

### 19.9.4 Defining the Christoffels

**Definition 19.9.12.** Consider two tensors \( X_Z^m, Y_Z^m \), where \( Z \) is a set of internal indices in the same upper or lower places for both tensors and \( m \) is an external index (here \( m \in 1..3 \) is assumed). We will say that \( X \) and \( Y \) are equal up to their normal components, and note

\[
X_Z^m = Y_Z^m + \mathcal{O}(\overrightarrow{n})
\]

when the property

\[
X_{z_1}^1, X_{z_2}^2, X_{z_3}^3 - Y_{z_1}^1, Y_{z_2}^2, Y_{z_3}^3 \in \mathbb{C} \overrightarrow{n} \subset \mathbb{C}^3.
\]

holds for all the instanciations \( z \) of \( Z \).

**Proposition 19.9.13.** Equivalently, tensors \( X_Z^m, Y_Z^m \) are equal up to their normal components when

\[
(X_Z^m - Y_Z^m) \Theta_{mn} \Theta_{\nu} = (0)_{Z\nu} \quad \text{where } \nu \not\in Z
\]

**Proof.** When \( d\nu \) varies, the 3D-vectors \( \overrightarrow{\Theta_{\nu}^1}, d\nu, \overrightarrow{\Theta_{\nu}^2}, d\nu, \overrightarrow{\Theta_{\nu}^3}, d\nu \) span the whole tangent plane. Thus an \( \mathcal{O}(\overrightarrow{n}) \) vector is orthogonal to all of them, and conversely. \( \square \)

**Definition 19.9.14.** The Christoffel symbols are defined by

\[
\frac{\partial^2 x^m}{\partial u^\mu \partial u^\nu} = \Gamma_{\phi|\mu\nu} \frac{\partial x^m}{\partial u^\phi} + \mathcal{O}(\overrightarrow{n}) ; \Gamma_{\phi|\mu\nu} = \Theta_{\phi\psi} \Gamma_{\phi|\mu\nu}
\]

Equivalently, we have:

\[
\Theta_{mn} \left( \frac{\partial^2 x^m}{\partial u^\mu \partial u^\nu} - \Gamma_{\phi|\mu\nu} \frac{\partial x^m}{\partial u^\phi} \right) \frac{\partial x^m}{\partial u^\nu} = (0)_{\psi\mu\nu}
\]

(19.8)

\[
\Gamma_{\phi|\mu\nu} = \Theta_{\psi\phi} \Gamma_{\phi|\mu\nu} = \Theta_{mn} \Theta_{\mu\nu} \Theta_{\psi\phi}
\]

**Remark 19.9.15.** In cyril@ERE (2016), \( \Gamma_{\phi|\mu\nu} \) is defined by \( \Gamma_{\phi|\mu\nu} = \frac{\partial u^\phi}{\partial x^m} \frac{\partial^2 x^m}{\partial u^\mu \partial u^\nu} \) but this formula is wrong when we simply replace each quantity by its (correct) definition. On the contrary, we have to define the \( \Gamma_{\phi|\mu\nu} \) by the already given formula

\[
x^m_{\mu\nu} = x^m_{\phi,\mu} \Gamma_{\phi|\mu\nu} + \mathcal{O}(\overrightarrow{n})
\]

**Theorem 19.9.16.** When the Christoffels are defined by (19.8), then the provisional definition

\[
\Gamma_{\psi|\mu\nu} = \frac{1}{2} \left( \Theta_{\psi\phi,\mu} + \Theta_{\psi\phi,\nu} - \Theta_{\psi\mu,\nu} \right) \quad (19.9)
\]

\[
\Gamma_{\phi|\mu\nu} = \Gamma_{\psi|\mu\nu} \Theta_{\psi\phi} \iff \Gamma_{\psi|\mu\nu} = \Theta_{\psi\phi} \Gamma_{\phi|\mu\nu}
\]

becomes a theorem.
Proof. Start from \( g_{\mu \nu} = \eta_{mn} \frac{\partial x^m}{\partial u^\mu} \frac{\partial x^n}{\partial u^\nu} \) and obtain the following relations:

\[
\frac{\partial g_{\mu \nu}}{\partial u^\sigma} = \frac{\partial}{\partial u^\sigma} \left( \eta_{mn} \frac{\partial x^m}{\partial u^\mu} \frac{\partial x^n}{\partial u^\nu} \right)
\]

Then call \( \mathcal{F}(\mu, \nu, \phi) \) this formula, and compute \( \mathcal{F}(\psi, \nu, \mu) + \mathcal{F}(\nu, \psi, \mu) - \mathcal{F}(\nu, \mu, \psi) \).

**Proposition 19.9.17.** The Christoffels transform according to the rules:

\[
\Gamma^\sigma_{\tau \rho} = \Gamma^\sigma_{\mu \rho} \Gamma^\mu_{\tau \sigma} \Gamma^\nu_{\tau \nu} + \Gamma^\nu_{\sigma \tau} \Gamma^\mu_{\nu \tau} + \Gamma^\nu_{\nu \tau} \Gamma^\mu_{\nu \sigma}
\]

In other words, the "linear tensorial" term which depends on the up/down nature of the indices is completed by a second term containing some second order derivatives.

**Proof.** The formula \( \Gamma_{\omega | \sigma \tau} = \mathcal{G}_{mn} x^m_{\sigma \tau} x^\omega_n \) can be extended into:

\[
\Gamma_{\omega | \sigma \tau} = \mathcal{G}_{mn} \frac{\partial}{\partial t^\sigma} \left( \frac{\partial x^m}{\partial u^\nu} \frac{\partial u^\nu}{\partial t^\omega} \right)
\]

\[
= \frac{\partial}{\partial t^\sigma} \left( \frac{\partial x^m}{\partial u^\nu} \frac{\partial u^\nu}{\partial t^\omega} \frac{\partial u^\nu}{\partial t^\sigma} \right) + \frac{\partial}{\partial t^\sigma} \left( \frac{\partial x^m}{\partial u^\nu} \frac{\partial u^\nu}{\partial t^\omega} \frac{\partial u^\nu}{\partial t^\tau} \right)
\]

\[
= \mathcal{G}_{mn} \left( \frac{\partial^2 x^m}{\partial u^\nu \partial t^\omega} \frac{\partial x^m}{\partial u^\nu} \frac{\partial u^\nu}{\partial t^\tau} \right)
\]

\[
= \mathcal{G}_{mn} \left( \frac{\partial^2 x^m}{\partial u^\nu \partial t^\omega} \frac{\partial x^m}{\partial u^\nu} \frac{\partial u^\nu}{\partial t^\tau} \right)
\]

\[
= \Gamma_{\psi | \mu \nu} \Gamma^\mu_{\nu \sigma} \Gamma^\mu_{\nu \tau} + \Gamma^\nu_{\nu \tau} \Gamma^\mu_{\nu \sigma} \mathcal{G}_{\nu \rho} \Gamma^\rho_{\omega}
\]

**19.9.5 Moving**

From the external point of view, the most probable result when moving, even a little bit, from a point \( M \in (E) \) is to left the surface, evading into the external world. Therefore we need a more elaborated concept of "moving".

**Definition 19.9.18.** We distinguish three different notions of "variation".

1. \( \Delta x \) denotes a non elaborated difference, according to the model \( \Delta x = x_2 - x_1 \);

2. \( \delta x \) denotes an "infinitesimal variation", obeying to the informal rules \( \delta x \neq 0 \) together with \( (\delta x)^2 = 0 \)...

3. \( dx \) denotes a "long range variable", not submitted to the \( (dx)^2 = 0 \) rule, but bound to some specific tangent plane.

**Remark 19.9.19.** Let \( B \) be a covariant vector, depicted at a point \( M \) by the pair of tensor equations: \( B_\mu = x^m_\mu B_m \) and \( B_m = u^\mu_m B_\mu \). Then we move it from the map in use at point \( M \) to the map in use at point \( M + \delta M \). In an ordinary cartesian frame, we simply have \( \delta B_m = 0 \). In another frame, we have:

\[
\delta B_\mu = \delta (x^m_\mu B_m) = \delta (x^m_\mu) B_m + x^m_\mu \delta B_m = (x^m_\mu v^\mu_m) B_\psi \delta u^\nu
\]

— pdlx : Translation of the Kimberling’s Glossary into barycentrics —
Definition 19.9.20. The variation due to a parallel transport of quantity $B_\mu$ is defined by
\[
\delta B_\mu = +B_\psi \Gamma^\psi_{\mu
u} du^\nu
\]
\[
\delta A^\nu = -A^\psi \Gamma^\psi_{\nu\mu} du^\nu
\]
and this is extended to any tensor (a corrective term per index, with the right sign). Therefore this variation doesn't depend on the embedding chosen to introduce the surface, but depends only on its (intrinsic) metric $\mathcal{G}$.

**Proof.** For a contravariant vector $A^\nu$, the constraint $\delta (B_\sigma A^\nu) = 0$ leads to
\[
\delta (B_\mu A^\nu) = A^\nu \delta (B_\mu) + B_\mu \delta (A^\nu) = A^\nu B_\nu \Gamma^\psi_{\nu\mu} du^\nu - A^\psi B_\mu \Gamma^\psi_{\psi\nu} du^\nu \]

Definition 19.9.21. The variation $dB_\sigma = B_\sigma \tau du^\tau$ of a vector is seen as the sum of two terms. One of them is due to the parallel transport, i.e. the change of the local map due to the displacement. And the difference can be seen as "the true variation" of the vector. This term is called the **covariant derivative** of $B_\sigma$ and noted with a semi-colon. In other words:
\[
B_{\sigma\tau} = \Gamma^\tau_{\sigma\psi} B_\psi ; A^\tau_\sigma = A^\tau_\sigma + \Gamma^\tau_{\sigma\psi} B_\psi
\]

**Proposition 19.10.22.** The covariant derivative of a tensor is another tensor.

**Proof.** Starting from the definition, we have:
\[
B_{\sigma\tau} = B_{\sigma\tau} - \Gamma^\tau_{\sigma\psi} B_\psi = \left( t^\mu_{\sigma\tau} B_\mu \right)_\tau - (\Gamma^\psi_{\mu\nu} t^\mu_{\sigma\tau} + t^\mu_{\tau\sigma} u^\nu_\phi + t^\mu_{\tau\sigma} u^\nu_\psi) t^\nu_\phi B_\psi \\
= t^\mu_{\sigma\tau} B_\mu + t^\mu_{\tau\sigma} B_{\mu\nu} t^\tau_\psi - \Gamma^\psi_{\mu\nu} t^\mu_{\sigma\tau} B_\psi (u^\nu_\phi t^\nu_\psi) - t^\mu_{\tau\sigma} B_\psi (u^\nu_\phi t^\nu_\psi) \\
= \left( t^\mu_{\sigma\tau} B_\mu - t^\mu_{\tau\sigma} B_{\mu\nu} - \Gamma^\psi_{\mu\nu} B_\psi \right) + t^\mu_{\tau\sigma} B_{\mu\nu} + \Gamma^\psi_{\mu\nu} B_\psi (u^\nu_\phi t^\nu_\psi) \\
= \left( t^\mu_{\sigma\tau} B_\mu - t^\mu_{\tau\sigma} B_{\mu\nu} - \Gamma^\psi_{\mu\nu} B_\psi \right) + t^\mu_{\tau\sigma} B_{\mu\nu}
\]

**19.10 Curvature**

**Definition 19.10.1. Explicit coordinates.** Consider the surface $(E) = \{(x, y, z) | z = F(x, y)\}$, where the $[x, y, z]$ are living in an euclidean 3D space. Then we can use $[u, v] \equiv [x, y]$ as internal coordinates.

**Notation 19.10.2.** Using indices $\mu, \nu, \phi, \psi, \rho$ in association with variables $u, v$ and indices $m, n, p, q, r$ in association with variables $x, y, z$, we have:
\[
\mathfrak{N}^k_{\mu
u} = \begin{bmatrix} [k, \mu], [1, 0], & 1 & 0 \\ 0 & 1 \\ F_u & F_v \end{bmatrix} ; \mathfrak{G}_{\mu\nu} = \begin{bmatrix} [\mu, \nu], [0, 0], & F_u^2 + 1 & F_u F_v \\ F_u F_v & F_v^2 + 1 \end{bmatrix}
\]
where the $F_u, F_v$ are the Frenet symbols (in any case, we will never use $i, j, u, v$ as tensorial indices).

**Fact 19.10.3.** Let $\mathbf{N}$ be a normal vector to the tangent plane and $N^q$ an associated tensor. We have:
\[
N^q = [[q], [1], [-F_u, -F_v, 1]]
\]

**Remark 19.10.4.** We have the matrix product:
\[
\begin{bmatrix} 0 & 0 & F_{uu} \\ 0 & 0 & F_{uv} \\ 0 & 0 & F_{vv} \end{bmatrix} = \frac{1}{1 + F_u^2 + F_v^2} \begin{bmatrix} -F_u F_{uu} & -F_u F_{uv} & F_{uu} \\ -F_u F_{uv} & -F_u F_{uv} & F_{uv} \\ -F_u F_{vv} & -F_u F_{vv} & F_{vv} \end{bmatrix} \cdot \begin{bmatrix} 1 & 0 & -F_u \\ 0 & 1 & -F_v \\ F_u & F_v & 1 \end{bmatrix}
\]

where the coefficients are
\[
\begin{bmatrix} \Gamma_1^1 & \Gamma_1^2 & L \\ \Gamma_2^1 & \Gamma_2^2 & M \\ \Gamma_2^2 & \Gamma_2^2 & N \end{bmatrix}
\]
Proposition 19.10.5. When cutting \((E)\) by a moving plane \(P_0\) containing \(n\), we obtain a curve \(\gamma_0\) and we examine its curvature \(\kappa_0\). This quantity is obtained by dividing twice the normal increment by the squared tangential increment. When the moving point \(r\) on the curve depends on a single parameter, we have the formula:

\[
\kappa = \frac{\vec{r} \wedge \vec{\dot{r}}}{\|\vec{r}\|^2} = \frac{\vec{r} \wedge \dot{\vec{r}}}{\|\vec{r}\|^3}
\]

where the fluxions are taken wrt the parameter of the curve.

Proof. Well known formula.

Proposition 19.10.6. The curvature is given by the quotient of two quadratic forms:

\[
\kappa_0 = \frac{1}{\sqrt{\det \Theta_{\mu\nu}}} \frac{\mathcal{I}(dx, dy)}{\mathcal{I}(dx, dy)}
\]

where \(\mathcal{I}(dx, dy)\) is the metric form (noted \(\Theta_{\mu\nu}\) here) and gives the squared tangential increment, while the normal increment is measured using:

\[
\mathcal{I}(dx, dy) = 2(\dot{\vec{r}} \cdot \vec{n}) = (dx, dy) \begin{pmatrix} F_{xx} & F_{xy} \\ F_{xy} & F_{yy} \end{pmatrix} \begin{pmatrix} dx \\ dy \end{pmatrix}
\]

Proof. We have \(|n|^2 = 1 + F_u^2 + F_v^2 = \det \Theta_{\mu\nu}\), while \(x \wedge P \frac{\vec{r}}{|\vec{r}|} = x \cdot \frac{n}{|n|}\) holds for any vector \(x\) in the tangent plane.

Exercise 19.10.7. Use \(F(u, v) = \sqrt{R^2 - u^2 - v^2}\) and obtain

\[
\vec{n} = \left[ \frac{u}{R^2 - u^2 - v^2}, 1 \right] ; \quad |\vec{n}| = \frac{R^2}{z^2} ; \quad \mathcal{I} = \frac{1}{z^2} \begin{pmatrix} R^2 - v^2 & uv \\ uv & R^2 - u^2 \end{pmatrix} ; \quad \mathcal{I}^* = -z \mathcal{I}
\]

Finally, \(\kappa_0 = -1/R\) (\(\vec{n}\) is the outer normal... and the center is "inside").

Lemma 19.10.8. When \(\mathcal{I}\) and \(\mathcal{I}\) are two quadratic forms (and \(\mathcal{I}\) is definite), then

\[
\max \left( \frac{\mathcal{I}(x, y)}{\mathcal{I}(x, y)} \right) \ast \min \left( \frac{\mathcal{I}(x, y)}{\mathcal{I}(x, y)} \right) = \frac{\det \mathcal{I}}{\det \mathcal{I}}
\]

Proof. Let \(\mathcal{I} = \left[ \frac{I}{I} \right]\) and \(\mathcal{I} = \left[ \frac{I}{I} \right]\).

(1) brute force method: differentiate \(\kappa \pm \frac{L_{u^2} + 2 M_{uv} + N_{v^2}}{E_{u^2} + 2 F_{uv} + G_{v^2}}\) and solve in \(u\) and \(v\). Then substitute and simplify \(\kappa_{\min} \ast \kappa_{\max}\)

(2) educated method: diagonalize the matrix \(\mathcal{I}^*/\mathcal{I}\). This allows a simultaneous reduction of the quadratic forms, so that \(\kappa_{\min}\) and \(\kappa_{\max}\) are the eigenvalues of matrix \(\mathcal{I}^*/\mathcal{I}\).

Lemma 19.10.9. The following quantity is a tensor, named the Riemann-Christoffel curvature tensor.

\[
B_{ijkl} = \Gamma^r_{ik} \Gamma^s_{jl} - \Gamma^r_{il} \Gamma^s_{jk} + \Gamma^r_{j,k} + \Gamma^r_{j,l} - \Gamma^r_{k,l}
\]

This tensor is anti-symmetrical wrt \(k, l\), and doesn’t contain any third order derivative of \(F\).

Proof. Tensorial property comes from the Christoffel formula. Moreover, we have:

\[
B_{ijkl}^i = \frac{(F_{uu} F_{uv} - F_{u,v}^2)}{(1 + F_{u}^2 + F_{v}^2)^2} \begin{bmatrix} 1121 & -F_u F_v \\ 2221 & +F_u F_v \\ 1112 & +F_u F_v \\ 2212 & -F_u F_v \end{bmatrix} = \begin{bmatrix} 2121 & + (F_u^2 + 1) \\ 1221 & - (F_u^2 + 1) \\ 2112 & - (F_v^2 + 1) \\ 1212 & - (F_u^2 + 1) \end{bmatrix}
\]

— pldx : Translation of the Kimberling's Glossary into barycentrics —
\[ B_{ijkl} = \left( \frac{F_{uu}F_{vv} - F_{uv}^2}{1 + F_u^2 + F_v^2} \right) \begin{bmatrix} 2121 & -1 \\ 1221 & +1 \\ 2112 & +1 \\ 1212 & -1 \end{bmatrix} \] (other components being null).

**Theorem 19.10.10.** The product of the extremal curvatures, i.e.

\[ K = \kappa_{\text{min}} \kappa_{\text{max}} = \frac{F_{uu}F_{vv} - F_{uv}^2}{(1 + F_u^2 + F_v^2)^2} \]

doesn’t depend on the chosen explicit parametrization.

**Proof.** Contract \( B_{ijkl} \) on \( i, l \) and obtain the Ricci tensor:

\[ R_{jk} = B_{ijkl} \begin{bmatrix} \mu, \nu, [0, 0], \left( \frac{F_{uu}F_{vv} - F_{uv}^2}{(F_u^2 + F_v^2 + 1)^2} \right) \left( - (F_u^2 + 1) - F_u F_v \right) \\
\left( - F_u F_v ight) \end{bmatrix} \]

Contract again and obtain the scalar curvature

\[ R = g^{ik} R_{jk} = (-2) \frac{F_{uu}F_{vv} - F_{uv}^2}{(1 + F_u^2 + F_v^2)^2} = -2K \]

Being tensorial, this quantity is therefore an invariant of the surface \((E)\).

**Remark 19.10.11.** Formula \( B_{ijkl} \) is top and foremost about general Riemann spaces, with dimensions greater than 2. When dealing with surfaces, due to the skew-symmetry of \( B_{ijkl} \) wrt pairs of indices \((i, j)\) et \((k, l)\), \( B_{1212} \) is the only independent component of the Riemann-Christoffel tensor. And then, the curvature formula simplifies into:

\[ K = B_{1212}/\det g_{ij} \]

### 19.11 Back to Poincaré and Klein

**Klein**

\[ \kappa_{\mu} = \begin{bmatrix} \mu, [1], \left[ \frac{2z}{1 + \zeta z}, \frac{2\zeta}{1 + \zeta z} \right] \end{bmatrix} \]

**Poincaré**

\[ \kappa_{\sigma} = \begin{bmatrix} \sigma, [1], \left[ \frac{1 - W}{\kappa}, \frac{1 - W}{k} \right] \end{bmatrix} \]

\[ W = \frac{1 - \zeta z}{1 + \zeta z} = \sqrt{1 - k\kappa} \]

As a mnemonic, \( k, \kappa \) are related to Klein, while \( z, \zeta \) are related to Poincaré.

**Exercise 19.11.1.** Compute the Jacobians and obtain:

\[ \mathcal{N}_{\mu} = \begin{bmatrix} \sigma, [1], \left[ \frac{1 + z\zeta}{2(1 - z\zeta)}, \frac{1}{\zeta^2}, 1 \right] \end{bmatrix} \]

\[ \mathcal{N}_{\sigma} = \begin{bmatrix} \mu, [1], \left[ \frac{2}{(1 + z\zeta)^2}, 1 - \zeta^2, 1 \right] \end{bmatrix} = \begin{bmatrix} \mu, [1], \left[ \frac{1}{2} \left[ (1 + W)^2, -k^2 \right], \frac{(1 + W)^2}{(1 - k\kappa)} \right] \end{bmatrix} \]

Check they are inverse of each other.

**Exercise 19.11.2.** Formulate the metrics in tensor form, and obtain:

\[ \mathcal{G}_{\sigma\tau} = \begin{bmatrix} \sigma, [1], \left[ \frac{1}{z}, \frac{2}{(1 - z\zeta)^2}, 0, 1 \right] \end{bmatrix} \]

\[ = \begin{bmatrix} \sigma, [1], \left[ \frac{1}{\kappa}, \frac{1}{2(1 - k\kappa)}, \frac{1}{1 - z\zeta} \right] \end{bmatrix} \]

January 3, 2024 21:08 published under the GNU Free Documentation License
\[ \Theta_{\mu \nu} = \frac{1}{z} \left[ \frac{(1 + z \zeta)^2}{2(1 - z \zeta)^4} \begin{bmatrix} 2 \zeta^2 & 1 + z^2 \zeta^2 \\ 1 + z^2 \zeta^2 & 2z^2 \end{bmatrix} \right] \]

\[ = \frac{k}{\kappa} \left[ \frac{1}{4(1 - \kappa \kappa)^2} \begin{bmatrix} \kappa^2 & 2 - \kappa \kappa \\ 2 - \kappa \kappa & \kappa^2 \end{bmatrix} \right] \]

Check the validity of the transformation formula (19.7).

**Exercise 19.11.3.** Compute the Christoffels and obtain:

\[
\begin{array}{cc|cc}
111 & 112 & 211 & 212 \\
121 & 122 & 221 & 222 \\
\hline
\Gamma^\epsilon_{\sigma \tau} & \frac{2 \zeta}{1 - z \zeta} & 0 & 0 \\
& 0 & 0 & \frac{2z}{1 - z \zeta} \\
\Gamma^\phi_{\mu \nu} & \frac{\kappa}{1 - \kappa \kappa} & \frac{k}{2(1 - \kappa \kappa)} & 0 & \frac{\kappa}{k} \\
& 0 & \frac{2(1 - \kappa \kappa)}{k} & \frac{\kappa}{k} \\
& \frac{2(1 - \kappa \kappa)}{k} & 0 & \frac{1}{1 - \kappa \kappa} \\
\end{array}
\]

Check the validity of the transformation formula (19.10)

**Exercise 19.11.4.** Compute the Riemann-Christoffel tensors and obtain:

\[
\begin{bmatrix}
1121 + 2 \\
2221 - 2 \\
1112 - 2 \\
2212 + 2
\end{bmatrix} : \frac{1}{(1 - z \zeta)^2} \\
\begin{bmatrix}
2121 - \kappa^2 \\
1221 + \kappa^2 \\
2112 + \kappa^2 \\
1212 - \kappa^2
\end{bmatrix} : \frac{1}{4(1 - \kappa \kappa)^2} \\
\begin{bmatrix}
1121 - \kappa \kappa + 2 \\
2221 + \kappa \kappa - 2 \\
2112 + \kappa \kappa - 2 \\
1212 - \kappa \kappa + 2
\end{bmatrix}
\]

Check the validity of the tensor transformation formula.

**Proposition 19.11.5.** The Gauss curvature of both the Poincaré and the Klein hyperbolic planes is \(-1\).

**Proof.** Obvious from the former results.
Chapter 20

About cubics

For a catalog with sketches, visit Gibert (2004-2024), especially Ehrmann and Gibert (2005). Some notations used there:

<table>
<thead>
<tr>
<th>I</th>
<th>G</th>
<th>O</th>
<th>H</th>
<th>N</th>
<th>K</th>
<th>L</th>
</tr>
</thead>
<tbody>
<tr>
<td>X(1)</td>
<td>X(2)</td>
<td>X(3)</td>
<td>X(4)</td>
<td>X(5)</td>
<td>X(6)</td>
<td>X(20)</td>
</tr>
</tbody>
</table>

20.1 Characterisation of a cubic

**Definition 20.1.1.** A cubic is a curve defined by an homogeneous polynomial of degree 3. Using barycentrics, or trilinears or Morley affixes is irrelevant, the degree is the same. Notation : \( \mathcal{K} \).

**Proposition 20.1.2.** A cubic is defined by nine general points.

*Proof.* There are ten coefficients, defined up to a global proportionality factor. We found them by computing

\[
\det_{j=10} [x^3, x^2y, xy^2, y^3, y^2z, yz^2, z^3, z^2x, zx^2, xyz]
\]

applied to the nine given points and the generic point.

**Example 20.1.3. Pivotal isocubics.** Let \( F, U \) be two points not on the sidelines. Then \( p\mathcal{K}(\#F, U) \) is the cubic that goes through the nine points: \( ABC, A_UB_UC_U \) (the cevians of \( U \)) and \( FA_FB_FC \) (the anticevians of \( F \)). This important class will be studied in details at Section 20.4.

**Proposition 20.1.4 (Cayley Bacharach).** When two cubics \( \mathcal{K}_1, \mathcal{K}_2 \) haven’t a line or a conic in common, they cut into exactly nine points. But, even when they are distincts, these nine points aren’t "general points" with respect to the previous proposition. More precisely the family \( \mathcal{F} \) of the cubics that are going through eight of these points is exactly \( \lambda \mathcal{K}_1 + \mu \mathcal{K}_2 \), and all of these cubics are going through the last point.

*Proof.* The first part is Bezout theorem. The second one is obvious: the nine points are not characterizing a cubic, since \( \mathcal{F} \) contains at least two cubics. The last part, i.e. that \( \mathcal{F} \) doesn’t contain any other cubics is proven in great details in Eisenbud et al. (1996).

20.1.1 More about the folium

Some properties of the Descartes curve have already be given at Section 12.2.

**Exercise 20.1.5.** Using the parametrization \( M_p \approx 6p : 6p^2 : 1 + p^3 \), give the condition for three points of the folium be aligned. Deduce the parameter of the tangential, i.e. the point where the tangent at \( M_p \) cuts again the curve. And then, determine the ‘conjugate’ of \( M_p \), i.e. the point having the same tangential.

**Exercise 20.1.6.** Using the same parametrization, find the ninth point of eight distincts points on the Folium (i.e. illustrate the Cayley Bacharach property).
20.1.2 Pascal’s theorem

**Theorem 20.1.7** (Pascal’s theorem). Let $A, C', B, A', C, B'$ be six points on a conic. Define $A'' = BC' \cap CB'$, etc. Then $A'', B'', C''$ are on a same line.

**Proof.** Draw the magenta, orange and green cubics of Figure 20.1, i.e. the cubics by $A, B, C, A', B', C', A'', B''$ and $M_j$ where $M_1 = (A + B') / 2$, $M_2 = (A + C') / 2$, $M_3 = (A'' + B'') / 2$ and conclude by Cayley Bacharach.

20.2 Group structure of a cubic

**Definition 20.2.1.** Suppose that the cubic $K$ is not singular (no cusp, no nodes). When $A \neq B$ are on the cubic, notation $A @ B$ will be used to design the third point where the line $(AB)$ cuts again the cubic. In the same vein, $A @ A$ will denote the tangential of $A$, i.e. the point where the tangent at $A$ cuts again $K$.

**Proposition 20.2.2.** Operation $@$ is commutative, but not associative. By convention, $A @ B @ C$ is to be understood as $(A @ B) @ C$. And we have
1. $P @ Q @ P = Q$
2. $P @ Q = R @ Q$ if and only if $P = R$
3. $P @ Q = R$ if and only if $R @ Q = P$

**Definition 20.2.3.** Chose a special point $O \in K$ and note $+$ the operation

$(A, B) \mapsto A + B \doteq A @ B @ O$

**Proposition 20.2.4.** (1) Operation $+$ is commutative
(2) $O$ is the neutral point, i.e. $P + O = P$;
(3) Defining $N$ by $N \doteq O @ O = O_1$, then $-N = N_1$;

**Proof.** (1) Obvious. (2) $P + O = (P @ O) @ O = P$ since $P, O, Q \doteq P @ O$ are the three intersections of some line with the cubic, while $Q @ O$ is "not $Q$ nor $O"$ on this line
(3) One has $N + N_1 \doteq N @ N_1 @ O = N @ O = O_2 @ O = O$.

**Theorem 20.2.5.** Operation $+$ is associative, and therefore $(K, +)$ is a group.
Figure 20.2: Using Cayley-Bacharach to prove the cubic associativity

Proof. See Figure 20.2 and Durège (1871, p. 135). Since the operation is commutative, what is to be proved can be written as:

\[(\forall P, Q, R \in K)(E = F) \quad \text{where} \quad E \equiv (Q + R) @ P ; \quad F \equiv (Q + P) @ R\]

1. Let us introduce two degenerate cubics: \(K_2\) as the union of the three magenta lines

\[L_1 = [P, Q + R, E], L_2 = [Q, R, Q@R], L_3 = [O, Q@P, Q + P]\]

and \(K_3\) as the union of the three blue lines

\[N_1 = [R, Q + P, F], N_2 = [Q, P, Q@P], N_3 = [O, Q@R, Q + R]\]

2. So we see that cubic \(K\) intersects the other two at

\[K \cap K_2 = O, P, Q, R, Q@R, Q@P, Q + R, Q + P, E\]

\[K \cap K_3 = O, P, Q, R, Q@R, Q@P, Q + R, Q + P, F\]

Since \(K_3\) goes by eight of the nine \(K \cap K_2\) points, the third cubic must go also through \(E\).

3. Suppose that \(E\) lies on \(N_3 = [O, Q@R, Q + R]\). Since \(L_1\) already goes through \(Q + R\), this would induce \(L_1 = N_3\) and therefore \(P \in (O, Q + R)\). Suppose that \(E\) lies on \(N_2 = [Q, P, Q@P]\). Since \(L_1\) already goes through \(P\), this would induce \(L_1 = N_2\) and therefore \(P \in (Q, Q + R)\).

4. It remains \(E \in N_1\), enforcing either \(E = R\) or \(E = Q + P\) or \(E = F\). The first two possibilities couldn’t be the general case, since \(E = P@((Q + R))\) depends on the three points. It remains only \(E = F\) as the general case, implying \(O@E = O@F\) as required.

5. Remark: the cyan cubic goes through the eight points and the random point \(A\). One can see that it goes also through the ninth point \(Q + R@P\)

\[\square\]

Theorem 20.2.6. 3\(k\) points \(P_i \in K\) are on a curve of order \(k\) if and only if \(\sum P_i = kN\).

Proof. When \(k = 1\), \(P@Q@O@R@O = N = O@O\) leads to \(P@Q@O@R = O\), and to \(P@Q@O = R@O\), implying \(P@Q = R\).

When \(k = 2\), define \(X = P@Q, Y = R@S, Z = T@U\) and prove the equivalence: the \(P_j\) are on a conic with \(XYZ\) aligned. The property results since \(P + Q + X = N\), etc. \(\square\)
20.3 Isocubics

Proposition 20.3.1. Define an isocubic \( K \) with pole \( P \) as a cubic which is invariant wrt the \( P \) isoconjugacy. Then \( K \) is either a "pivotal isocubic" (see Section 20.4 for more details) with equation:

\[
pK(P,U) \equiv pK(#F,U) \triangleq (h^2 y^2 - g^2 z^2) wx + (f^2 z^2 - h^2 x^2) vy + (g^2 x^2 - f^2 y^2) wz
\]  

(20.2)

or a "non pivotal isocubic" (see Section 20.5 for more details) with equation:

\[
nK(P,U,k) \triangleq ux (r y^2 + q z^2) + vy (p z^2 + r x^2) + wz (q x^2 + p y^2) + k xyz
\]  

(20.3)

Proof. Direct inspection from \( K(X_p) = \lambda K(X) \). It can be seen that terms like \( xy^2 \) and \( xz^2 \) are to be paired, and that terms like \( x^3 \) are to be avoided. As a corollary, such a cubic goes through the vertices \( A, B, C \).

Proposition 20.3.2. When the pole \( P \) is fixed, and \( X_p \) is defined by (17.2), i.e. by \( x^* = pqz \), the property "\( K \) is an isocubic" is characterized by the exact formula:

\[
K(X_p)/K(X) = \mp (pqr xyz)
\]

Sign "-" characterizes a \( pK \) cubic, sign "+" characterizes a \( nK \) cubic. Each class form a projective space, whose dimensions are respectively 3 and 4.

Definition 20.3.3. The triangular cubic is the union of the three sidelines: \( nK_0 = (BC) \cup (CA) \cup (AB) \). The standard equations of this cubic are defined by:

\[
nK_0(X) \triangleq \frac{xyz}{z} (\overline{Z} \gamma \beta - T\beta - T\gamma + Z) (\overline{Z} \alpha \gamma - T\alpha - T\gamma + Z) (\overline{Z} \alpha \beta - T\alpha - T\beta + Z) \div s_3
\]

(20.4)

Proposition 20.3.4. When substituting the isogonal formulas (17.3) into the equation of a \( \not=X(1) \) pivotal cubic and using (20.4), we have the following equalities:

\[
pK(\text{isog}(X)) = -pK(X) \times nK_0(X)
\]

\[
nK(\text{isog}(X)) = +nK(X) \times nK_0(X)
\]

that are exact identities, not up to a proportionality factor. Once again, the set of the \( X(6) \)-isocubics splits into two projective subspaces, the \( pK \) one (dimension 3) and the \( nK \) one (dimension 4).

Example 20.3.5. The cubic "circumcircle union infinity" is a \( nK \) cubic, characterized by:

\[
\left[ nK, \begin{pmatrix} a \\ b \\ c \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}, a^2 + b^2 + c^2 \right] \triangleq \frac{1}{b} (x + y + z)(a^2yz + b^2zx + c^2yx) \triangleq T(\overline{Z} \overline{Z} - T^2)
\]

20.4 Pivotal isocubics \( pK(P,U) \)

Definition 20.4.1. Pivotal isocubics. Let \( F, U \) be two points not on the sidelines and define \( pK(#F,U) \) as the cubic that goes through the nine points: \( ABC, A_U B_U C_U \) (the cevians of \( F \)) and \( F_A F_B F_C \) (the anticevians of \( F \)). Then \( U \) and \( F_0 = F \) are also on the cubic. Its equation is:

\[
pK(P,U) \triangleq pK(#F,U) \triangleq (h^2 y^2 - g^2 z^2) wx + (f^2 z^2 - h^2 x^2) vy + (g^2 x^2 - f^2 y^2) wz
\]

(20.5)

Remark 20.4.2. Part of the time, isocubics are defined using the pole of the conjugacy, i.e. \( P \simeq p : q : r \simeq f^2 : g^2 : h^2 \).
Table 20.1: Some well-known cubics

<table>
<thead>
<tr>
<th>$pK$</th>
<th>name</th>
<th>$F$</th>
<th>$U,U^*$</th>
<th>$E,E^*$</th>
<th>$D,D^*$</th>
<th>some other points on the cubic</th>
</tr>
</thead>
<tbody>
<tr>
<td>$pK_A$</td>
<td>vertex</td>
<td>20.4.20</td>
<td>1</td>
<td>A</td>
<td></td>
<td></td>
</tr>
<tr>
<td>ZU(1)</td>
<td>20.4.21</td>
<td>1</td>
<td>X(1)</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>$E_1, E_3$</td>
<td>hidden</td>
<td>20.4.23</td>
<td>1</td>
<td>$\Omega$</td>
<td></td>
<td></td>
</tr>
<tr>
<td>K002 Thomson</td>
<td>20.4.29</td>
<td>1</td>
<td>2.6</td>
<td>3.4</td>
<td>9.57</td>
<td>223, 282, 1073, 1249</td>
</tr>
<tr>
<td>K003 McKay</td>
<td>20.4.24</td>
<td>1</td>
<td>3.4</td>
<td>1075,?</td>
<td>1745,3362</td>
<td></td>
</tr>
<tr>
<td>K006</td>
<td>20.4.21</td>
<td>1</td>
<td>4.3</td>
<td>155,254</td>
<td>46,90</td>
<td>371, 372, 485, 486, 487, 488</td>
</tr>
<tr>
<td>K005</td>
<td>Darboux 20.4.5</td>
<td>1</td>
<td>5.54</td>
<td>2120,2121</td>
<td>3460,3461</td>
<td>3, 4, 17, 18, 61, 62, 195, 627, 628</td>
</tr>
<tr>
<td>K004 Darboux</td>
<td>20.4.45</td>
<td>1</td>
<td>20.64</td>
<td>2130,2131</td>
<td>3182,3347</td>
<td>3, 4, 84, 1490, 1498</td>
</tr>
<tr>
<td>K001 Neuberg</td>
<td>20.4.25</td>
<td>1</td>
<td>30.74</td>
<td>2132,2133</td>
<td>3464,?</td>
<td>3, 4, 13, 14, 15, 16, 399, 484, ...</td>
</tr>
<tr>
<td>K035</td>
<td>20.4.29</td>
<td>1</td>
<td>98.511</td>
<td>?,?</td>
<td>1756,?</td>
<td>1687, 1688, 2009, 2010</td>
</tr>
<tr>
<td>K007 Lucas</td>
<td>20.4.5</td>
<td>2</td>
<td>X(69)</td>
<td>4, 7, 8, 20, 189, 253, 329... (15)</td>
<td></td>
<td></td>
</tr>
<tr>
<td>K170</td>
<td>20.4.9</td>
<td>2</td>
<td>X(4)</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>K155 EAC2 20.4.55</td>
<td>2</td>
<td>$\sqrt{31}$</td>
<td>238</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>K060</td>
<td>20.4.7</td>
<td>2</td>
<td>$\sqrt{1989}$</td>
<td>265</td>
<td></td>
<td></td>
</tr>
<tr>
<td>K020</td>
<td>20.4.5</td>
<td>2</td>
<td>X(2394) upto X(2419)</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>K162</td>
<td>20.5.18</td>
<td>6</td>
<td>$\sqrt{1989}$</td>
<td>265</td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

Table 20.1: Some well-known cubics

$F = \sqrt{P}$ (central fixed point), $U, U^*, E = cevadiv(U,U^*)$, $E^*, D = cevadiv(U,\sqrt{P})$

**Definition 20.4.3.** A Kimberling ZU cubic is a $pK(X_6,U)$, giving a special place to isogonal conjugacy. Some examples are given in Table 20.1.

**Remark 20.4.4.** Only 8 ZU cubics have a reflection center: the Darboux cubic (center= $X_3$), the four degenerate cubics that are union of the three bisectors through an incenter, and three other (Maple length $= 135712$ using RootOf, $[4948, 5345, 4215]$ using alias).

**Theorem 20.4.5.** Isocubic property. When point $X$ is on a sideline of triangle $ABC$, then $X^\#_F$ is undefined (geometrically), while the formula gives the third vertex. Otherwise, $X^\#_F$ belongs to $pK(#F,U)$ if and only if $X$ belongs to $pK(#F,U)$. And then $U, X, X^\#_F$ are collinear. For this reason, point $U$ is called the pivot of the cubic. Alternate formulation: $pK(#F,U)$ is the locus of the $X$ such that $U, X, X^\#_F$ are collinear (to be taken 'Cremona more', i.e. allowing indeterminacies).

**Proof.** Compute the determinant of the 10 rows (20.1) relative to the nine points given in the definition and the variable point $X = x : y : z$. And remark that this quantity is proportional to

---

**pldx : Translation of the Kimberling’s Glossary into barycentrics** ---
\[ \det \left( U, X, X P^# \right). \]

**Theorem 20.4.6. Cevadiv property.** When \( X \) is on the sidelines of the cevian triangle of \( U \), then \( Y = \text{cevadiv}(U, X) \) is undefined geometrically, while the formula gives 0 : 0 : 0. Otherwise, this \( Y \) belongs to \( pK(#F, U) \) if and only if \( X \) belongs to \( pK(#F, U) \). And then \( U P^# : X \), cevadiv \((U, X)\) are collinear. This fact is underlined by the name isopivot given to the point \( U P^# \).

**Proof.** Compute \( pK(#F, U) \) (cevadiv \((U, X)\)) and obtain \( pK(#F, U) \) (\( X \)) times the incidence relations, i.e. the rows of Adjoint \((\text{cevian}(U)) \cdot X \). And remark that this quantity is proportional to \( \det \left( U P^#, X, \text{cevadiv}(U, X) \right) \).

**Proposition 20.4.7. The 22 points property.** The \( pK(#F, U) \) cubic goes through

1. \( U, A_U B_U C_U \) and their conjugates \( U P^# \), \( ABC \) (8)
2. \( F_0 F_A F_B F_C \) (cf Theorem 20.4.5) (4)
3. cevadiv \((U, U P^#)\), the four cevadiv \((U, F_j)\) and their isoconjugates (10)

**Proof.** Follows directly from the two theorems.

**Example 20.4.8.** When \( U \) is one of the fixed points of the isoconjugacy (i.e. \( P = U * U \)), the \( pK(P, U) \) cubic degenerates into the lines through the remaining three fixed points.

**Example 20.4.9.** K170 is \( pK(X_2, X_4) \). Equation \( \sum x \left( y^2 - z^2 \right) / S_a = 0 \). On Figure 20.3, one can see the following alignments (general properties, applicable to any \( pK \)):

1. Fixed points : \( F_0, F_a, A \) are collinear, and cyclically for the other fixed points and the other vertices;
2. From \( U : U, X, X P^# \) are collinear (e.g. \( E \) and \( E^* \) are aligned with \( U \)). Therefore, each line from \( U \) to a fixed point is tangent to the cubic at this fixed point; in the same vein, point \( U_b = UB \cap AC \) is on the cubic and viewing \( B \) as \((U_b)^# \) makes sense, but not viewing \( U_b \) as \( B P^# \) (this object would be "quite all points on line \( AC^* \)."
3. From \( U P^# : U P^#, X \), cevadiv \((U, X)\) are collinear (e.g. \( F \) and \( D \) are aligned with \( U P^# \)). Therefore, each line from \( U P^# \) to a vertex or to \( U \) is tangent to the cubic at this point.

**Definition 20.4.10.** The \( PK^#_P(X) \) point is the intersection of the trilinear polars of points \( X \) and \( X P^# \). Using barycentrics and \( F = f : g : h \), we have:

\[ PK^#_P(X) = f^2 x \left( g^2 z^2 - h^2 y^2 \right) : g^2 y \left( h^2 x^2 - f^2 z \right) : h^2 z \left( f^2 y^2 - x^2 g^2 \right) \]

**Proof.** Direct computation.

**Remark 20.4.11.** This transform was introduced by Ehrmann and Gibert (2005) as \( PK_P \), i.e. putting forwards the pole of the conjugacy rather than its fixed points, and turned into Definition 23.4.1.

**Example 20.4.12.** Using \( F = X(1) \), i.e. \( P = X(6) \), we have \( PK(X(I)) = X(J) \) for these \((I, J)\):

\[
\begin{array}{cccccccccccc}
I & 2 & 3 & 4 & 5 & 6 & 9 & 19 & 31 & 44 & 54 & 57 & 63 \\
J & 512 & 647 & 647 & 2081 & 512 & 663 & 810 & 2084 & 3251 & 2081 & 663 & 810
\end{array}
\]

**Proposition 20.4.13.** Points of indeterminacy of \( PK^#_P \) are the fixed points \( F_j \) and their diagonal vertices i.e. \( A, B, C \). The exceptional curves are the six lines through two of the fixed points. Otherwise, each point \( P = PK^#_P(X) \) characterizes a pair \( \{X, X P^#\} \) (so that \( PK^# \) is not a Cremona transform \( I \)).
Proof. Solving $P = 0 : 0 : 0$ gives the first result, and solving the Jacobian gives the second. Otherwise, set $P \simeq p : q : r$ and obtain:

$$
\begin{bmatrix}
2h^2qf^2prg^2 \\
-rg^2 (+g^2h^2p^2 - f^2g^2r^2 + f^2h^2q^2) - rg^2W \\
-qh^2 (+g^2h^2p^2 + f^2g^2r^2 - f^2h^2q^2) + qh^2W
\end{bmatrix}
$$

where $W = 4i f^2g^2h^2 S \left( \frac{p}{f}, \frac{q}{g}, \frac{r}{h} \right)$ and $S$ is the usual Heron formula for the area (7.5).

Proposition 20.4.14. When $PK^\#_F(X)$ belongs to the tripolar line of $U^\#_F$, then point $X$ belongs to the cubic $pK(P, U)$. This amounts to say that $X$ is on $pK$ if and only if the tripolars of $X$, $X^\#_U$, $U^\#_F$ are concurrent.

Proof. Direct computation. In Kimberling (1998, p. 240) the corresponding cubic is noted $Z(U, Y)$ where $P = Y = \infty$.

Proposition 20.4.15. When point $U = u : v : w$ is at infinity, the $pK(P, U)$ cubic can be rewritten as:

$$
\left( \frac{p}{x} + \frac{q}{y} + \frac{r}{z} \right) (xp + y\sigma + z\tau) - (x + y + z) \left( \frac{a^2p}{x} + \frac{b^2\sigma}{y} + \frac{c^2\tau}{z} \right) = 0
$$

where $[\rho, \sigma, \tau]$ is any line whose direction is $U$. In other words $u = \sigma - \tau$, etc.
Proof. Direct examination. We can check that \( pK(P, U) \) contains the intersections of both conics — vertices \( A, B, C \) and \( U_3 \), the points at infinity of \( C_{\text{cir}}(P) \), the intersections of line \( [\rho, \sigma, \tau] \) with the associated conic, and \( U \) itself!

### 20.4.1 Another description

1. Consider \( pK(#F, P) \) and use \( F_j \approx \pm f : \pm g : \pm h \) (isoconjugacy), \( P \approx p : q : r \) (pivot). The name \( U \approx u : v : w \) will be used for the generic point of the cubic. Let \( CP(F, P) \) be the diagonal conic that goes through \( P \) and the four \( F_j \). In fact, this conic goes also through the 3 associates of \( P \) since

\[
CP(F, P) = \begin{bmatrix}
g^2r^2 - h^2q^2 & 0 & 0 \\
0 & h^2p^2 - f^2r^2 & 0 \\
0 & 0 & f^2q^2 - g^2p^2
\end{bmatrix}
\]

2. Cut this conic by line \( PU \) and obtain point \( N \). Acting that way, we obtain a symmetric expression. Cut rather by line \( UU_3^\# \) and say that \( P \) is the other solution. We obtain:

\[
N = \begin{bmatrix}
-f^2pu^2 + 2f^2quv - g^2pu^2 \\
f^2quv - 2g^2puv + g^2qu^2 \\
2w(f^2qu - g^2pu) + r(g^2u^2 - f^2v^2)
\end{bmatrix}
\]

Now, define \( N(U) \) by this formula. Then \( N(U) = N(U_3^\#) \) is equivalent to \( U \in pK... \) except when \( P \) is one of the \( F_j \).

3. Cut the tangent at \( N \) to \( CF(F, P) \) with line \( PP_3^\# \) and obtain the point \( M \)

\[
M = \begin{bmatrix}
qw - rv \\
r^2q^2 - r^2h^2 \\
p^2h^2 - r^2f^2 \\
q^2f^2 - p^2g^2
\end{bmatrix};
\]

\[
t = -\frac{(pw - ru)q}{(qw - rv)p}
\]

4. For all \( U \), point \( M \) is aligned with \( P, P_3^\# \). When \( U \) is on the cubic,

- \( N = \text{cevadiv} (M, P) \).
- \( M \) is aligned with \( N, N_3^\# \).
- \( N_3^\# \) is the intersection of tangents at \( U \) and \( U_3^\# \) to the cubic.
- \( M_3^\# \) is the intersection of the line \( P, U, U_3^\# \) and the fixed circumconic through \( P, P_3^\# \).

5. Line \( \text{cevadiv} (P, U); \text{cevadiv} (P, U_3^\#) \)

Conversely, the locus of such \( U \) is the union of \( pK(#F, P) \) et \( nK(#F, Q, -1 : p^2q^2r^2) \).

### 20.4.2 Group structure (pivotal cubics)

**Proposition 20.4.16.** When using the pivot \( U \) as the neutral point, then

1. \( N \cong U_t = U^* = [f^2uv : g^2uw : h^2uw] \)
2. \( X, Y, Z \) are collinear if and only if \( X + Y + Z = N \). (generic property)
3. \( X \circ X^* = U ; X \circ U = X^* ; X^* \circ U = X ; X + X^* = N \) (isocubic property)
4. \( X + Y = (X \circ Y)^* ; X \circ Y = (X + Y)^* \)
5. \( -X = X \circ N = U/X \) (cevadiv property)
6. On the cubic, \( A^* = A_U \) (the cevian of \( U \)) ; \( \text{grad}(A) = [0, -h^2v, +g^2w] \); \( A_t = N \)
7. \( A + A = B + B = C + C = U ; A + B + C = U \)

**Proof.** (7) \( A + A = A@A@U = A_4@U = U_4@U = U \)
\( B + C = B@C@U = U_A@U = A; \)

**Proposition 20.4.17.** Four points are said to form a tangential quadruple when they have the same tangential.

1. The four \( F_i \) form such a quadruple, the tangential being \( U \).
2. The four \( A, B, C, U \) form such a quadruple, the tangential being \( N \).
3. When \( (X_1, X_2, X_3, X_4) \) is a tangential quadruple, then \( (X_1^*, X_2^*, X_3^*, X_4^*) \) is such a quadruple.
4. Every tangential quadruple is of the form \( (X, X + A, X + B, X + C) \).

**Proof.** (3) From the tangential definition, and the isocubic property
\[
N = T + 2P^* = S + 2P = S + 2Q
\]
\[
N = P + P^* = Q + Q^* \quad \text{then}
\]
\[
T + 2Q^* = N - 2P^* + 2Q^* = N - 2(N - P) + 2(N - Q) = N + 2P - 2Q
\]
\[
= N + (N - S) - (N - S) = N
\]
so that \( T \) is the tangential of \( Q^* \) too.

(4) \( (P + A)_1 = N - 2(P + A) = N - 2P - 2A = N - 2P = S. \)

**Exercise 20.4.18.** Find the tangential common to \( A_U, B_U, C_U, U^* \).

### 20.4.3 ABCIJKL cubics: the Lubin(2) point of view

**Notation** 20.4.19. All visible curves are normalised by \( C = \text{con} \).

**Proposition 20.4.20. Cubics PKA.** The set of all cubics that go through points \( ABCIJKL \) is a projective space. Its dimension is 3. A generating family is given by the three cubics \( pK_A = (BC) \cup (AI) \cup (KL) \), etc. Pivot of \( pK_A \) is \( A \). Its fully factored equation requires Lubin(2), but Lubin(1) is sufficient to use det \( (X, X^*, A) = 0 \) where \( X^* \) is given by the isogonal conjugacy formula. One has:

\[
pK_A = \frac{1}{s_3^2} (-Z \beta^2 \gamma^2 + T \beta^2 + T \gamma^2 - Z) (-Z \alpha^2 \beta \gamma + T \alpha^2 + T \beta \gamma - Z) (Z \alpha^2 \beta \gamma + T \alpha^2 - T \beta \gamma - Z)
\]

**Proof.** Equation of \( pK(F,U) \) is det \( (X, X^* \cup F, U) = 0 \), leading to dimension 3, and allowing to check that pivot of \( pK_A \) is the vertex \( A \). After that, we can go back to Lubin(1) since isogonal conjugacy doesn’t require to identify which is the incenter among the incenters.

**Example 20.4.21.** The Kimberling \( Z(X(1)) \) cubic, i.e. \( (IJ) \cup (IK) \cup (IL) \), is obtained as:

\[
Z(X(1)) = \frac{\alpha (\beta + \gamma)}{(\alpha - \beta) (\alpha - \gamma) pK_A + \frac{\beta (\alpha + \gamma)}{(\beta - \gamma) (\beta - \alpha) pK_B + \frac{\gamma (\alpha + \beta)}{(\gamma - \alpha) (\gamma - \beta) pK_C}}
\]

**Proposition 20.4.22.** The Kiepert RH construction (see Proposition 13.21.1 for more details) can be summarized as follows. Let \( K = \cot \phi \) be a fixed real and define circularly the points \( A'B'C' \) by:

\[
\cot \left( \frac{BC}{BA} \right) + K = 0 ; \cot \left( \frac{CB}{CA} \right) + K = 0
\]

These 3 points and the 7 \( ABCIJKL \) are on a same cubic, together with \( X(3), X(4) \) and the \( A'B'C' \) points related with the other orientation. Naming this cubic \( pK_{(K)} \), we have:

\[
pK_{(K)} = pK_{Darboux} + K^2 \times pK_{Thomson} = (K^2 + 1) pK_{Neuberg} + (1 - 3K^2) pK_{McKay}
\]

(See below for more details on these four cubics). The pivot of \( pK_{(K)} \) is \( (3K^2 zX(2) - zX(20)) \div (3K^2 - 1) \).

--- pldx : Translation of the Kimberling’s Glossary into barycentrics ---
Proof. Let us compute the points \( A', B', C' \) and obtain:

\[
A' \approx \beta + \gamma + i (\gamma - \beta)/K : 2 : (\beta + \gamma + i (\gamma - \beta)/K) : \beta \gamma, \text{ etc}
\]

Then substitute \( A' \) and \( B' \) into \( \sum_i x_j pK_j = 0 \). This system has non zero solutions, proving the result. The circumcenter \( X(3) \) (the \( T^3 \) coefficient), the orthocenter (isogonality) and the \( A'B'C' \) points related to the other orientation (\( pK(K) \) depends only on \( K^2 \)). \( \square \)

### 20.4.4 Using a more handy basis

**Proposition 20.4.23. The hidden IJKL cubics.** Let \( pK_{\Omega_x} \) and \( pK_{\Omega_y} \) be the isogonal cubics whose pivots are the umbilics \( \Omega_x \approx 0 : 0 : 1 \) and \( \Omega_y \approx 1 : 0 : 0 \). Then

\[
pK_{\Omega_x} = \frac{1}{\sigma_3} Z^3 + T \left( \frac{1}{\sigma_3} Z^2 - 2 \frac{Z_2 + \sigma_2 Z}{\sigma_3} + T^3 \sigma_1 \right)
\]

\[
pK_{\Omega_y} = \frac{1}{\sigma_3} Z^3 + T \left( \frac{1}{\sigma_3} Z^2 + \frac{\sigma_1}{\sigma_3} Z + 2 \frac{Z}{\sigma_3} + \frac{\sigma_2}{\sigma_3} T^3 \right)
\]

They are conjugate of each other, but not self conjugate (hidden curves). Their nine intersections are \( ABCIJKL \) (stable by isogonal conjugacy) and the two umbilics (stable as a pair).

**Proof.** When \( M \in \mathbb{P}_C (C^3) \), and isog \( (M) \) is taken from \( \{17.3\} \), then

\[
E \doteq (T^2 - Z \bar{Z}) M - T \text{ isog} (M) = E_1 : E_2
\]

The nine points property comes from (1) at an umbilic, column \( E \) vanishes since both multipliers are zero; (2) at a vertex, the circle vanishes and isog \( (X) \) is \( 0 : 0 : 0 \); (3) at a fixed point of the conjugacy, column \( E \) is proportional to \( M \) but \( E_2 \) is ever 0 and the \( M \) aren’t at infinity. Obviously, this result can also be checked in Lubin(2) by substitution. Or even by factoring the \( T \) resultant of both equations. \( \square \)

**Proposition 20.4.24. K003, the McKay Cubic, \( pK(6, 3) \).** A visible cubic is described by the expression \( (Z pK_{\Omega_x} - Z pK_{\Omega_y}) / T \). And then, pole \( = X(6) \), pivot \( = X(3) \) and equation:

\[
pK_{McKay} = \frac{1}{\sigma_3} Z^3 + \sigma_3 \bar{Z}^3 + T \left( \sigma_1 Z^2 - \sigma_2 \bar{Z}^2 \right) + T^2 \left( -\sigma_2 Z + \sigma_1 \bar{Z} \right)
\]

\( K003 \) goes through \( X(3) \) and \( X(4) \). Its points at infinity are \( \Theta : 0 : 1/\Theta \) where \( \Theta^3 = \sigma_3 \). These points are the directions of the Morley triangles. The asymptotes are:

\[
\left[ \begin{array}{cc}
3 \frac{\Theta^2}{\sigma_3} & \sigma_2 \\
\sigma_1 \Theta^2 & \sigma_3 \\
-3 \frac{\sigma_4}{\Theta^2} & \end{array} \right]
\]

where \( \Theta \) ranges over the three cubic roots, using \( j \Theta \) or \( j^2 \Theta \). They concur at \( X(2) \).

**Proof.** Coeff of \( T^3 \) is 0. Thus \( X(3) \) and then \( X(4) \). Asymptotes are obtained from the gradient. Pivot can be guessed as the intersection of two well chosen lines \( MM^* \), for example for two of the points at infinity. \( \square \)

**Proposition 20.4.25. K001, the Neuberg cubic, \( pK(6, 30) \).** A visible cubic is described by expression \( (\sigma_2/\sigma_3) pK_{\Omega_x} - \sigma_1 pK_{\Omega_y} \). And then pole \( = X(6) \), pivot \( = X(30) \) and equation:

\[
pK_{Neuberg} = \frac{\sigma_2}{\sigma_3} Z^2 \bar{Z} - \sigma_1 Z \bar{Z}^3 + T \left( \frac{\sigma_1}{\sigma_3} Z^2 + \sigma_2 \bar{Z}^2 \right) + T^2 \left( \frac{\sigma_1^2 - 2 \sigma_2}{\sigma_3} Z - \frac{\sigma_2^2 - 2 \sigma_1 \sigma_3}{\sigma_3} \bar{Z} \right)
\]

This is a circular curve. The third point at infinity is \( \sigma_1 \sigma_3 : 0 : \sigma_2 \), i.e. \( X(30) \), the direction of the Euler line. The asymptotes are:

\[
\left[ \begin{array}{ccc}
0 & -\sigma_1 & \sigma_2 \\
-\sigma_1 & \sigma_2 & 0 \\
\sigma_2^2 \sigma_1 & -\sigma_1^3 \sigma_3 + \sigma_3^3 & -\sigma_2 \sigma_1^2 \sigma_3
\end{array} \right]
\]

Intersecting the first two asymptotes, we obtain a singular focus at \( \sigma_2^2 : \sigma_1 \sigma_2 : \sigma_2^2 \), i.e. point \( X(110) \). Moreover, the cubic goes through \( X(3) \) and therefore through \( X(4) \). Visible asymptote goes through \( X(74) \), the isogonal of \( X(30) \).
Proof. X(30) is the pivot because it is the only common point to the curve and line \( MM' \) when \( M \) is an umbilic. An asymptote is the gradient evaluated at the corresponding point at infinity. □

**Corollary 20.4.26.** The Neuberg cubic is the locus of points \( X \) such that the isogonal line \( XX' \) is parallel to the Euler line. See (Gibert, 2004–2024, 2005). Its (barycentric) equation is:

\[
\sum_{\text{cyclic}} x \left( y^2 c^2 - z^2 b^2 \right) \left( 2 a^4 - (b^2 + c^2) a^2 - (b^2 - c^2)^2 \right) = 0
\]

(20.6)

and can be rewritten as:

\[
\left( \frac{a^2}{x} + \frac{b^2}{y} + \frac{c^2}{z} \right) (x p + y \sigma + z \tau) - (x + y + z) \left( \frac{a^2 p}{x} + \frac{b^2 \sigma}{y} + \frac{c^2 \tau}{z} \right) = 0
\]

where \([p, \sigma, \tau]\) is the Euler line – remember: tripole = \( X(648) \). In other words, \( K_{001} = \text{circumcircle} \times \text{Euler} - \text{infinity} \times \text{Jerabek} \).

Proof. See Proposition 20.4.15. We can check that \( K_{001} \) contains the intersections of \( \Gamma \) and Jerabek — vertices \( A, B, C \) and \( X(74) \) —, the points at infinity of the circumcircle — the umbilics —, the intersections of Euler line and Jerabek hyperbola — \( X(3) \) and \( X(4) \) — and \( X(30) \) itself. □

**Proposition 20.4.27.** **Shortest cubic**, \( pK(6, 523) \). The expression \((1/\sigma_1) pK_{\Gamma x} + (\sigma_3/\sigma_2) pK_{\Gamma y}\) gives a visible cubic, with pole = \( X(6) \), pivot = \( X(523) \) and equation:

\[
pK_{\text{shortest}} = \frac{1}{\sigma_1} Z^2 \bar{Z} + \frac{\sigma_3}{\sigma_2} Z \bar{Z}^2 + T \left( \frac{1}{\sigma_2} Z^2 + \frac{\sigma_1}{\sigma_1} \bar{Z}^2 \right) - \left( \frac{\sigma_1^2 + 2 \sigma_2}{\sigma_2 \sigma_1} Z + \frac{\sigma_2^2 + 2 \sigma_1}{\sigma_2 \sigma_1} \bar{Z} \right) + 2 T^3
\]

The third point at infinity is \(-\sigma_1 \sigma_3 : 0 : \sigma_2\), i.e. \( X(523) \), the orthodir of the Euler line. This is also the pivot. The umbilical asymptotes concur at \( X(110) \), giving a singular focus. Barycentric equation of this curve is:

\[
(b^2 - c^2) x (b^2 z^2 - c^2 y^2) + (c^2 - a^2) y (c^2 x^2 - a^2 z^2) + (a^2 - b^2) z (a^2 y^2 - b^2 x^2)
\]

The ETC database provides \( X(21381) \). Thus we have 3 vertices, 4 inexccenters, 3 points at infinity, and 2 others, i.e. 15 points.

**Remark 20.4.28.** The "shortest cubic" has been introduced as the circular \( pK \) cubic whose expression requires the shortest number of characters. Its main property is to provide an handy basis when used together with the McKay and the Neuberg cubics.

**Proposition 20.4.29.** **K002, the Thomson cubic**, \( pK(6, 2) \). The expression \((1/\sigma_1) pK_{\Gamma x} - (1/\sigma_2) pK_{\Gamma y} / T\) gives a visible cubic, with pole = \( X(6) \), pivot = \( X(2) \) and equation:

\[
pK_{\text{Thomson}} = \frac{3}{\sigma_3} Z^3 + \frac{\sigma_2}{\sigma_3} Z \bar{Z}^2 - \sigma_1 Z \bar{Z}^3 + T \left( -\frac{4 \sigma_1}{\sigma_3} Z^2 + 4 \sigma_2 \bar{Z}^2 \right) + T^2 \left( \frac{\sigma_2 + \sigma_1}{\sigma_3} Z - \frac{\sigma_2^2 + \sigma_1}{\sigma_3} \bar{Z} \right)
\]

This cubic is the locus of points \( X \) whose trilinear polar is parallel to their polar line in the circumcircle. This is the \( K = \infty \) cubic of the Kieper RH construct.

**Proposition 20.4.30.** **K004, the Darboux cubic**, \( pK(6, 20) \). The expression \((1/\sigma_1) pK_{\Gamma x} - (1/\sigma_2) pK_{\Gamma y} / T\) gives a visible cubic, with pole = \( X(6) \), pivot = \( X(20) \) and equation:

\[
pK_{\text{Darboux}} = \frac{Z^3}{\sigma_3} + \frac{\sigma_2}{\sigma_3} Z \bar{Z}^2 - \sigma_1 Z \bar{Z}^3 + T^2 \left( \frac{\sigma_1^2 - 3 \sigma_2}{\sigma_3} Z - \frac{\sigma_1^2}{\sigma_3} \bar{Z} \right)
\]

This is the \( K = 0 \) cubic of the Kieper RH construct. See the next section for more details.
Proposition 20.4.31. Resulting pencils. We have the following pencils:

1. The pencil generated by \(pK_{\text{Neuberg}}\) and \(pK_{\text{shortest}}\) is the set of all the circular \(pK\) cubics. Their pivots are at infinity.
2. The pencil generated by \(pK_{\text{McKay}}\) et \(pK_{\text{Neuberg}}\) is the set of the \(pK\) cubics that goes through the \(X(3), X(4)\) pair. Their pivots are on the Euler line.
3. The pencil generated by \(pK_{\text{McKay}}\) et \(pK_{\text{shortest}}\) is the set of \(pK\)-cubics whose pivots are on the line \(X(3), X(523)\), i.e. the line through \(X(3)\) and perpendicular to the Euler line. The common points of these cubics are on \(\Delta\) (horrible formula, with a 24th degree radicand).

Exercise 20.4.32. Does it exist a cubic XXX such that:

1. XXX, K001, K003 provide a basis of the \(ZU\) cubics space
2. XXX contains "many" ETC points
3. Pencils (XXX,K001) and (XXX,K003) contain "many" known cubics
4. Lubin equation of XXX remains practicable

20.4.5 Darboux and Lucas cubics

In this section, \(P \in \text{Darboux}\) and \(U \in \text{Lucas}\), while pole and pivot are noted otherwise.

20.4.5.1 Presentation of K004 and K007

Definition 20.4.33. The Darboux cubic K004 is the locus of point \(P\) such that the pedal triangle of \(P\) is the Cevian triangle of some other point \(U\), while the Lucas cubic K007 is the locus of point \(U\) such that the cevian triangle of \(U\) is the pedal triangle of some other point \(P\).

Proposition 20.4.34. Darboux cubic is a \(pK\) cubic, with \(X(6)\) as pole and \(X(20)\) as pivot. \(X(20)\) is the de Longchamps point. Lucas cubic is a \(pK\) cubic, with \(X(2)\) as pole and \(X(69)\) as pivot. \(X(69)\) is the anticomplement of \(X(6)\). Their equations are:

\[
\begin{align*}
\det(X_{20}, P, \text{isog}(P)) &= 0 \quad (20.7) \\
\det(X_{69}, U, \text{isot}(U)) &= 0 \quad (20.8)
\end{align*}
\]

Moreover, K004 has a reflection center at \(X(3)\), the circumcenter. Using Morley affixes and expanding, we have:

\[
pK_{\text{Darboux}} = -Z^3 + \frac{\sigma_2}{\sigma_3} Z^2 - \sigma_1 Z^2 + \frac{\sigma_1}{\sigma_3} Z^2 + \frac{3\sigma_1 \sigma_3 - \sigma_1^2}{\sigma_3^2} T^2 Z
\]

\[
pK_{\text{Lucas}} = \begin{pmatrix}
\frac{3}{s_3} Z^2 - \frac{s_3}{s_1} Z^2 - \frac{s_3}{s_2} Z^2 - 3 s_3 Z^2 - \left(\frac{s_1}{s_3} + \frac{s_2}{s_3}\right) Z T^2 + \left(\frac{s_1}{s_3} + \frac{s_2}{s_3}\right) Z T^2
- \left(\frac{s_1}{s_3} + \frac{s_2}{s_3} - \frac{s_3}{s_2}\right) Z T^2 + \left(\frac{s_1}{s_3} + \frac{s_2}{s_3} - \frac{s_3}{s_2}\right) T^3
\end{pmatrix}
\]

Their common points are \(A, B, C, X(4), X(20)\) and four other points.

Proof. Straightforward from (9.1) and (3.6).

Fact 20.4.35. The barycentric equations of these cubics can be rewritten as:

\[
\text{Darboux} = \sum \left(2 S^2 - S_6 S_5\right) x \left(b^2 z^2 - c^2 y^2\right) ; \text{Lucas} = \sum S_a x \left(y^2 - z^2\right)
\]

while, as of 2019, the following points are known:

<table>
<thead>
<tr>
<th>(P)</th>
<th>1</th>
<th>3</th>
<th>4</th>
<th>20</th>
<th>40</th>
<th>64</th>
<th>84</th>
<th>1490</th>
<th>1498</th>
<th>2130</th>
<th>2131</th>
<th>3182</th>
<th>3183</th>
</tr>
</thead>
<tbody>
<tr>
<td>(U)</td>
<td>7</td>
<td>2</td>
<td>4</td>
<td>69</td>
<td>8</td>
<td>253</td>
<td>189</td>
<td>329</td>
<td>20</td>
<td>14362</td>
<td>?</td>
<td>5932</td>
<td>14361</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>(P)</th>
<th>3345</th>
<th>3346</th>
<th>3347</th>
<th>3348</th>
<th>3353</th>
<th>3354</th>
<th>3355</th>
<th>3472</th>
<th>3473</th>
<th>3637</th>
</tr>
</thead>
</table>

Asymptotes of the Darboux cubic are the perpendicular bisectors of the sidelines.
Definition 20.4.36. The \( \psi \) transform sends a point \( P \) onto the intersection of lines \( AA_P \) and \( BB_P \) where \( A_P B_P C_P \) is the pedal triangle of \( P \), while the \( \psi^{-1} \) transform sends a point \( U \) onto the intersection of the perpendicular to \( BC \) through \( U_A \) with the perpendicular to \( AC \) through \( U_B \) where \( U_A U_B U_C \) is the cevian triangle of \( U \).

Proposition 20.4.37. When defined that way, i.e. using the same vertex of \( ABC \) to play the non-symmetric role, then \( \psi \) and \( \psi^{-1} \) are reciprocal Cremona transforms of the whole plane and satisfy:

\[
\psi \begin{pmatrix} p \\ q \\ r \end{pmatrix} \simeq \begin{pmatrix} (pb^2 + qS_c)(ra^2 + pS_b) \\ (qa^2 + pS_c)(rb^2 + qS_a) \\ (ra^2 + pS_b)(rb^2 + qS_a) \end{pmatrix} \tag{20.9}
\]

\[
\psi^{-1} \begin{pmatrix} u \\ v \\ w \end{pmatrix} \simeq \begin{pmatrix} a^2(uw b^2 + u v S_a - v w S_c) \\ b^2(v w a^2 + u v S_b - u w S_c) \\ v w S_b S_c + u w S_a S_c - u v S_a S_b + 4S^2 w^2 \end{pmatrix} \tag{20.10}
\]

Points of indeterminacy of \( \psi \) are \( a^2 : -S_c : -S_b, -S_c : b^2 : -S_a \) (the directions of \( AH \) and \( BH \)) and \( a^2S_a : b^2S_b : -S_a S_b \) (the antipode of \( C \), i.e. the point \( 2O - C \)) while points of indeterminacy of \( \psi^{-1} \) are \( A, B \) and \( G_c = A + B - C \).

Proof. Computation proves that we have an exact reciprocity on the whole plane when taking éclatements into account.

Proposition 20.4.38. When angle in \( C \) is not a straight one, then \( P \in \text{Darboux} \) is equivalent to the alignment of \( P, \psi(P) \) with \( X(20) \), while \( U \in \text{Lucas} \) is equivalent to the alignment of \( U, \psi^{-1}(U) \) with \( X(20) \).

Proof. Both determinants are the product of \( S_c \) by the equation of the corresponding cubic. When \( S_c = 0 \), then \( X(20) \) comes at \( G_c \) and degeneracy is not a surprise.
Lemma 20.4.43. When restricted to both cubics, ψ and ψ⁻¹ define two reciprocal central transforms and doesn’t depend on the choice of the special vertex.

Proposition 20.4.40. Darboux, i.e. pK(6, 20), is invariant by isog and by sym3, the reflection about X(3), while Lucas, i.e. pK(2, 69) is invariant by isot and cycl, the cyclopedal transform (as defined at Section 13.23). Moreover, ψ⁻¹ ◦ isot ◦ ψ = sym3 holds over the whole plane, while ψ⁻¹ ◦ cycl ◦ ψ = isog holds when restricted to the cubics.

Proof. When P is on Darboux, then isog (P) is also on Darboux. By Section 9.3, they share the same pedal circle. Therefore, the corresponding U ∈ Lucas are cyclocevian. The other formula is easily computed.

Corollary 20.4.41. The known points on Lucas cubic can be used to build the following chains. They were emphasized in Kimberling (2002a). When P = X₃, then gP = X₄, and the center of the cyclopedal circle is X₅; when P = X₁₀, this center is X₅⁹⁴; when P = X₁, this center is X₁₁₅₈, etc.

\[
\begin{array}{cccccccc}
X₁ & \xrightarrow{\text{sym}³} & X₂₀ & \xrightarrow{\text{isog}} & X₈ & \xrightarrow{\text{isog}} & X₁₄₉₀ & \xrightarrow{\text{isog}} & X₃₁₈₂ & \xrightarrow{\text{isog}} & X₃₄₄₇ \\
1 \downarrow \psi & & 1₁₅₈ & \downarrow \psi & \downarrow \psi & \downarrow \psi & \downarrow \psi \\
X₇ & \xrightarrow{\text{isot}} & X₈ & \xrightarrow{\text{cycl}} & X₁₈₉ & \xrightarrow{\text{isot}} & X₃₂₉ & \xrightarrow{\text{cycl}} & X₁₀₃₄ & \xrightarrow{\text{cycl}} & X₅₉₃₂ \\
\downarrow \psi & & 5₈₉₄ & \downarrow \psi & \downarrow \psi & \downarrow \psi & \downarrow \psi \\
X₃ & \xrightarrow{\text{isot}} & X₄ & \xrightarrow{\text{sym}³} & X₂₀ & \xrightarrow{\text{isog}} & X₆₄ & \xrightarrow{\text{isog}} & X₁₄₉₈ & \xrightarrow{\text{isog}} & X₃₄₄₆ & \xrightarrow{\text{sym}³} & X₃₁₈₃ \\
\downarrow \psi & & 5₈₉₄ & \downarrow \psi & \downarrow \psi & \downarrow \psi & \downarrow \psi \\
X₂ & \xrightarrow{\text{cycl}} & X₄ & \xrightarrow{\text{isot}} & X₆₉ & \xrightarrow{\text{cycl}} & X₂₅₃ & \xrightarrow{\text{isot}} & X₂₀ & \xrightarrow{\text{cycl}} & X₁₀₃₂ & \xrightarrow{\text{isot}} & X₁₄₃₆₁ \\
\end{array}
\]

Since X₁ is fixed by isog and X₃ by sym³, these chains are unidirectional.

Claim 20.4.42. Let Q = (sym³ ◦ isog ◦ sym³ ◦ isog ◦ sym³) P. Then P is on darboux if and only if cevamul (P, Q) = X(20). When P is on the branch of X₃, so is Q (obvious).

When P is not on Darboux, ??? In any case, a simple division by polynomial (20.7) isn’t sufficient.

20.4.5.2 The Orion bundle

Lemma 20.4.43. The nine intersections of K₀₀₄ and K₀₀₇ are: A, B, C, X(4) = H, X(20) = 2O – H and four other points called the Orion points. Their affixes are the solutions of:

\[
6₄Z^₄ - 1₆ \left( s₁ + \frac{s₃²}{s₃} \right) TZ^³ + \left( -₄ s₁ + \frac{s₁² s₃²}{s₃} + 1₈ s₁s₂ - ₄ \frac{s₁² s₃²}{s₃} - ₇₅ s₁s₃ + ₁₆ s₂² \right) T^⁴ \\
+ \left( ₁₆ \frac{s₁² s₃²}{s₃} + ₆₄ s₂ - ₄₈ s₁² \right) T²Z² + \left( ₂₈ s₁ - ₇ \frac{s₁² s₃²}{s₃} - ₇₅ s₁s₂ + ₁₂ \frac{s₃²}{s₃} + ₁₂₅ s₃ \right) T³Z
\]

Proof. Direct elimination from the equations.

Definition 20.4.44. Reflect point P ≼ p : q : r through the sidelines of its cevian triangle ApBpCp. The obtained triangle is perspective with triangle ABC, and the perspector is called the Orion transform of P (Ehrmann, 2003). Notation and barycentrics are:

\[
\mathcal{D} (M) \simeq \left[ \begin{array}{c}
(b²r² + c²q² + ₂ S₃ r) p³ - a²q²r²p \\
(a²r² + c²p² + ₂ S₃ p) q³ - b²p²q²r \\
(a²q² + b²p² + ₂ S₃ p) r³ - c²p²q²r
\end{array} \right]
\]

Remark 20.4.45. When M is on the sidelines, \( \mathcal{D} (M) = M \).
Example 20.4.46. One can identify the following pairs:

<table>
<thead>
<tr>
<th></th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
<th>6</th>
<th>7</th>
<th>8</th>
<th>9</th>
<th>10</th>
<th>11</th>
<th>12</th>
<th>13</th>
<th>14</th>
<th>15</th>
<th>16</th>
<th>17</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>35</td>
<td>69</td>
<td>2055</td>
<td>24</td>
<td>2056</td>
<td>57</td>
<td>2057</td>
<td>11581</td>
<td>11582</td>
<td>2058</td>
<td>2059</td>
<td>2060</td>
<td>2061</td>
<td>2062</td>
<td>2063</td>
<td>10419</td>
<td>2064</td>
</tr>
</tbody>
</table>

Definition 20.4.47. The A-Orion cubic $K_A$ is the locus of points $M$ such that $M M_a \perp M_b M_c$ where $M_a M_b M_c$ is the cevian triangle of $M$.

Proposition 20.4.48. For $M$ not on the sidelines, $M \in K_A$ is equivalent to the alignment of $A, M, \Omega(M)$. Equation of $K_A$ is

$$K_A \simeq c^2 xy^2 - b^2 xz^2 + S_b y^2 z - S_c yz^2$$

so that $K_A$ goes once through $B, C$ and twice through vertex $A$. Tangent at $B, C$ are going through $T_a = 2O - A$ (on circumcircle and Darboux as well), while tangents at $A$ are the bissectors of angle $A$. The 6th point on $\Gamma$ is $a^2 : c^2 - b^2 : b^2 - c^2$, the 3rd one on $BC$ is $0 : S_c : S_b$ (the foot of the $A$-altitude).

Proof. Direct computation, using the gradient (at $B, C$) and the hessian (at $A$).

Proposition 20.4.49. The four Orion points are the only ones which are the orthocenter of their cevian triangle. Moreover, the three Orion cubics $K_A, K_B, K_C$ generate the bundle of all the cubics through the 3 vertices and the 4 Orion points.
Proof. Since $B$ counts twice on $K_B$, it counts also twice in $K_B \cap K_C$. The same with $C$, while $A$ counts for one. Thus it remains four other common points. For each of them, $M$ is on two altitudes of $M_AM_BM_C$ and therefore belongs to the third one, so that $M \in K_A$. Therefore the cevian and the pedal triangle of $M$ are equal, and $M$ is on both K004 and K007. Moreover, neither X(4) nor X(20) are the orthocenter of their cevian triangle.

**Proposition 20.4.50.** Let $P \simeq p : q : r$ be a fixed point. When $M$ is not on the sidelines, the alignment of $P, M, \Delta(M)$ is equivalent to $M \in K_P$ where $K_P$ is defined by $K_P \simeq pK_A + qK_B + rK_C$.

**Proof.** Determinant is linear wrt any column. Moreover, one can check that

\[ \text{lucas} = K006 = K_a + K_b + K_c \]
\[ \text{darboux} = K004 = a^2S_aK_a + b^2S_bK_b + c^2S_cK_c \]

**Proposition 20.4.51.** When $P$ is on the Thomson cubic, then $K_p$ is a $pK(F, U)$ cubic and we have

\[ F^2 \simeq \text{cevadiv}(G, P) \simeq \begin{pmatrix} (q + r - p) p \\ (r + p - q) q \\ (p + q - r) r \end{pmatrix} ; U \simeq \text{anticompl} P^* \simeq \begin{pmatrix} b^2rp + c^2pq - a^2qr \\ c^2pq + a^2qr - b^2rp \\ a^2qr + b^2rp - c^2pq \end{pmatrix} \]

**Proof.** $P \in K002$ comes from elimination. Then symmetric formula for $F^2$ and $U$ can be checked modulo $K002$.

**Example 20.4.52.** Here are some of the $K_p$ cubics.

<table>
<thead>
<tr>
<th>$P$</th>
<th>$F^2$</th>
<th>$U$</th>
<th>Kxxx</th>
<th>name</th>
</tr>
</thead>
<tbody>
<tr>
<td>X(1)</td>
<td>X(9)</td>
<td>X(8)</td>
<td>K199</td>
<td>Lucas</td>
</tr>
<tr>
<td>X(2)</td>
<td>X(2)</td>
<td>X(69)</td>
<td>K007</td>
<td>Lucas</td>
</tr>
<tr>
<td>X(3)</td>
<td>X(6)</td>
<td>X(20)</td>
<td>K004</td>
<td>Darboux</td>
</tr>
<tr>
<td>X(6)</td>
<td>X(3)</td>
<td>X(2)</td>
<td>K168</td>
<td></td>
</tr>
</tbody>
</table>

**Proposition 20.4.53.** Consider the six conics

\[ f_a(x, y, z) = b^2y^2 - c^2z^2 + x(S_cy - S_bz) \text{, etc} \]
\[ g_a(x, y, z) = a^2(c^2y^2 - b^2z^2) + x(c^2S_cy - b^2S_bz) \text{, etc} \]

Then the $f_j$ are going through the isotomic conjugates of the Orion points while the $g_j$ are going through their isotogonal conjugates.

**Proof.** Since $A$ is double in $K_A$, then $x^2yz$ comes in factor at both isomot $K_A$ and isogonal $K_A$.

### 20.4.6 Equal areas (second) cevian cubic aka K155

**Definition 20.4.54.** Cubic shadow. Triangle centers on a cubic $K$ yield non-central points on the cubic; e.g., if $Q_1$ and $Q_2$ are on $K$, then the line $Q_1Q_2$ meets $K$ in a "third" point, $L(Q_1, Q_2)$, possibly $Q_1$ or $Q_2$. If $A'B'C'$ is a central triangle (cf Section 2.2), $R$ a triangle center, $A'' = L(R, A')$ and cyclically, then triangle $A''B''C''$ is a central triangle on $K$.

**Definition 20.4.55.** Cubic EAC2, the equal areas (second) cevian cubic is K155 in Gibert (2004-2024). This cubic is $pK(X31, X238)$, i.e self-isogonic wrt $P = X_{31} = a^3: b^3: c^3$ and pivotal wrt $U = X_{238} = a^3 - abc : b^3 - abc : c^3 - abc$.

**Proposition 20.4.56.** It happens that $P \in EAC2$. When a point $Q$ is on EAC2, its isoconjugate $Q_p$ aka $X31 \div Q$ is on EAC2 too. In the following table, for each $(I, J)$, the centers $X(I)$ and $X(J)$ are on EAC2 and are an isoconjugate pair. Each pair is collinear with the pivot $X(238)$.
Proposition 20.4.57. Let $A'$, $B'$, $C'$ be the reflections of a point $M$ into the sidelines $BC$, $CA$, $AB$. When triangle $A'B'C'$ is perspective with $ABC$, point $M$ lies on the Neuberg cubic $K001$, while the resulting perspector $N$ lies on another cubic ($K060$).

**Proof.** The matrix of the reflection into the sideline $BC$ is:

$$\sigma_A \simeq \begin{pmatrix} -a^2 & 0 & 0 \\ a^2 + b^2 - c^2 & a^2 & 0 \\ a^2 - b^2 + c^2 & 0 & a^2 \end{pmatrix}$$

Start from $M = p : q : r$. Compute $A' = \sigma_A(M)$, etc and obtain:

$$A'B'C' \simeq \begin{pmatrix} -pa^2 & (a^2 + b^2 - c^2)q + pb^2 & (a^2 - b^2 + c^2)r + pc^2 \\ (a^2 + b^2 - c^2)p + a^2q & -qb^2 & (b^2 + c^2 - a^2)r + qc^2 \\ (a^2 - b^2 + c^2)p + a^2r & (b^2 + c^2 - a^2)q + b^2r & -rc^2 \end{pmatrix}$$

Table 20.2 gives some cubic shadows on EAC2. Column 1 gives the perspector $R \in K$. Column 2 gives $A' \in K$, the A vertex of the original triangle. Columns 3 and 4 give $A'' \in K$, the $A$ vertex of the shadow triangle. When expressions are growing, these coordinates are given in two rows. For example, in row 1, the perspector is the centroid, the original triangle is $ABC$ itself and $A'' = -a^2 : bc : bc$, the third coordinate being obtained by swapping $b$ and $c$. Points $X_2$, $A' = A$ and $A''$ are collinear. Obviously, $(A'')_p$, $(B'')_p$, $(C'')_p$, is another central triangle inscribed in $K$.

## 20.4.7 The cubic K060

**Proposition 20.4.57.** Let $A'$, $B'$, $C'$ be the reflections of a point $M$ into the sidelines $BC$, $CA$, $AB$. When triangle $A'B'C'$ is perspective with $ABC$, point $M$ lies on the Neuberg cubic $K001$, while the resulting perspector $N$ lies on another cubic ($K060$).

**Proof.** The matrix of the reflection into the sideline $BC$ is:

$$\sigma_A \simeq \begin{pmatrix} -a^2 & 0 & 0 \\ a^2 + b^2 - c^2 & a^2 & 0 \\ a^2 - b^2 + c^2 & 0 & a^2 \end{pmatrix}$$

Start from $M = p : q : r$. Compute $A' = \sigma_A(M)$, etc and obtain:

$$A'B'C' \simeq \begin{pmatrix} -pa^2 & (a^2 + b^2 - c^2)q + pb^2 & (a^2 - b^2 + c^2)r + pc^2 \\ (a^2 + b^2 - c^2)p + a^2q & -qb^2 & (b^2 + c^2 - a^2)r + qc^2 \\ (a^2 - b^2 + c^2)p + a^2r & (b^2 + c^2 - a^2)q + b^2r & -rc^2 \end{pmatrix}$$

— pldx : Translation of the Kimberling’s Glossary into barycentrics ——
Then \( \det(AA', BB', CC') \) is computed and identified with \( K001 \). Now, start from \( N = u : v : w \).

Compute \( \delta_A = \sigma_A (AN) = (A \wedge N) \cdot \frac{1}{[\sigma_A]} \), etc and obtain:

\[
\begin{pmatrix}
\delta_A \\
\delta_B \\
\delta_C \\
\end{pmatrix} \simeq \begin{pmatrix}
2vS_b - 2wS_c & -wa^2 & va^2 \\
wS_b^2 - 2wS_a & 2wS_c - 2uS_a & -wb^2 \\
uS_c^2 - wS_a & uc^2 & 2uS_a - 2vS_b \\
\end{pmatrix}
\]

Then \( \det(\delta_A, \delta_B, \delta_C) \) is computed, and we obtain yet another \( pK \) cubic, defined as \( K060 \).

**Proposition 20.4.58.** *K060*, the \( pK \) (1989, 265) cubic. The pole \( P \), the pivot \( U \) and the Morley equations of this cubic are respectively:

\[
P = \frac{1}{b^2c^2 - 4S_a^2} \text{ etc} ; \quad z_P = \frac{\sigma_1^2 \sigma_2^2 - \sigma_1 \sigma_2 - (4 \sigma_1^4 - 9 \sigma_1^2 \sigma_2 + 9 \sigma_2^2)}{3 \sigma_1^2 \sigma_2^2 - 6 \sigma_3^2 + (9 \sigma_1 \sigma_2 - 6 \sigma_1^2)} \sigma_3
\]

\[
U = \frac{S_a}{b^2c^2 - 4S_a^2} : \frac{S_b}{a^2c^2 - 4S_b^2} : \frac{S_c}{a^2b^2 - 4S_c^2} ; \quad Z_U = \frac{\sigma_1^2 - \sigma_2}{\sigma_1}
\]

\[
\begin{pmatrix}
s_3^2 Z^2 - s_1 Z^2 + \left( \frac{2s_1}{s_3} \right) T Z^2 + \left( s_1^2 - 2s_2 \right) T^2 Z + & 0 \\
\left( \frac{s_2^2 s_1}{s_3^2} + \frac{s_2^2 s_1}{s_3^2} \right) T^2 Z + & \left( \frac{s_2^2 s_1}{s_3^2} - s_1 \right) T^2 Z + \left( \frac{s_2^2 s_1}{s_3^2} - s_1 \right) T^3
\end{pmatrix}
\]

Both umbilics belong to the curve. The corresponding asymptotes intersect at \( X(3448) \). The real asymptote:

\[ [\sigma_1 \sigma_2^2, 2 \sigma_3 \sigma_1^2 - 2 \sigma_2^2, -\sigma_1^2 \sigma_2 \sigma_3] \]

is parallel to the Euler line. The sixth intersection with the circumcircle is \( X(1141) \).

**Proof.** Direct inspection.

D on \( K001 \), F=isoD on \( K001 \)

\[
Nd = \text{antig}(D) = (\text{isg} \circ \text{inv} \circ \text{isg})(D) \\
NF = \text{antig}(F) = (\text{isg} \circ \text{inv})(D)
\]

### 20.4.8 Eigentransform

**Definition 20.4.59.** The mapping \( U \mapsto \text{cevadiv} \left( U, U_P^B \right) \) is called eigentransform of \( U \) wrt pole \( P = F^2 \).

In ETC, \( F = X(1) \), i.e. \( P = X(6) \), is assumed, and notation \( ET(U) \) is used. Here, the same notation is used, but isogonal conjugation isn’t assumed.

**Example 20.4.60.** Assuming \( P = X(6) \), pairs (I,J) such that \( X(J) = ET(X(I)) \) include:

| 1 | 1 | 13 | 62 | 81 | 3293 | 174 | 266 | 664 | 2082 | 1156 | 1 |
| 2 | 3 | 14 | 61 | 86 | 3294 | 190 | 1 | 673 | 1 | 1492 | 1 |
| 3 | 1075 | 19 | 2128 | 88 | 1 | 512 | 2142 | 694 | 384 | 1821 | 1 |
| 4 | 155 | 20 | 2130 | 92 | 47 | 648 | 185 | 771 | 1 | 1942 | 1941 |
| 5 | 2120 | 30 | 2132 | 94 | 49 | 651 | 1 | 799 | 1 |
| 6 | 194 | 37 | 2134 | 99 | 39 | 653 | 1 | 811 | 2083 |
| 7 | 218 | 57 | 2136 | 100 | 1 | 655 | 1 | 823 | 1 |
| 8 | 2122 | 63 | 1712 | 101 | 2140 | 658 | 1 | 897 | 1 |
| 9 | 2124 | 69 | 2138 | 110 | 5 | 660 | 1 | 1113 | 3 |
| 10 | 2126 | 75 | 2172 | 162 | 1 | 662 | 1 | 1114 | 3 |

**Proposition 20.4.61.** For any point \( U \) not on a sideline of triangle \( ABC \), the following properties of eignetransform are easy to verify:
1. The barycentrics of ET(U) are (cyclically):
\[ vwf^2 (u^2v^2h^2 + u^2w^2g^2 - v^2w^2f^2) \]

2. ET(U) is the eigencenter of the cevian triangle of U as well as the eigencenter of the anticevian triangle of \( U^*_P \).

3. ET(U) = \( F = f : g : h \) (fixed point of the isocubics) if and only if \( U = F \) or \( U \) lies on the CC(F) circumellipse. When \( P = X(6) \), then \( F = X(1) \) and this locus is the Steiner circumellipse: \( yz + zx + xy = 0 \).

4. Points U, ET(U) and \( (ET(U))^\#_P \) are collinear points of the cubic \( pK(P, U) \).

5. Points F, ET(U) and \( (cevadiv(U, F))^\#_P \) are collinear. The last point is also on the cubic.

6. ET(U) is the tangential of \( U^*_P \).

### 20.5 Non pivotal isocubics \( nK(P, U, k) \) and \( nK_0(P, U) \)

**Definition 20.5.1.** The non pivotal isocubic with pole \( P \), root \( U \) and parameter \( k \) is defined by the equation:
\[ nK(P, U, k) = wx (y^2r + qz^2) + vy (z^2p + rx^2) + wz (qx^2 + py^2) + kxyz \quad (20.11) \]

When \( k = 0 \), the cubic is noted \( nK_0(P, U) \).

**Remark 20.5.2.** The more efficient method for specifying \( k \) is to indicate a point that belongs to the cubic. This is noted \( nK(F, U, X) \).

**Proposition 20.5.3.** The "third intersections" of a \( nK(P, U, k) \) with the sidelines are the cocevians of the root \( U \). Therefore, they are aligned. This is to be compared with the fact that, for a \( pK \) cubic, these points are the cevians of the pivot.

**Proof.** Direct inspection. \( \Box \)

**Remark 20.5.4.** In the general case, a \( nK_0(P, U) \) contains neither \( P \) nor \( U \) nor any of the four fixed points \( F \) of the conjugacy.

**Definition 20.5.5.** We define the \( F \)-crosssum of two points \( U = u : v : w \) and \( X = x : y : z \) that aren’t lying on a sideline of \( ABC \) as:
\[ \text{crosssum}_F(U, X) = f^2 (wy + vz) : g^2 (uz + wx) : h^2 (vx + uy) \]

**Remark 20.5.6.** In ETC, \( F = X(1) \) is assumed. Defined as above, the operation \( (F, U, X) \mapsto \text{crosssum}_F(U, X) \) is globally type-keeping and provides a point when the entries are points \( (F) \) is any of the four fixed point of the conjugacy \( X \mapsto X^*_P \).

**Definition 20.5.7.** The \( NK_P(X) \) point is the pole of the line \( XX^*_P \) with respect to the circumconic that passes through \( X \) and \( X^*_P \) (Bernard Gibert, 2003/10/1). Using barycentrics and \( P = p : q : r = f^2 : g^2 : h^2 \), we have:
\[ NK_P(X) = px (ry^2 + qz^2) : qy (pz^2 + rx^2) : rz (qx^2 + py^2) \]
\[ = \text{crosssum}_F(X, X^*_P) = \text{crossmul}(X, X^*_P) = (cevamul(X, X^*_P))^\ast_P \]

**Proof.** Direct computation. \( \Box \)

**Remark 20.5.8.** Here, \( X^*_P \) is the perspector of the cevian triangle of \( NK_P(X) \) and the anticevian triangle of \( X \).

**Example 20.5.9.** Using \( F = X(1) \), i.e. \( P = X(6) \), we have \( NK(X(1)) = X(J) \) for these \( (I, J) \):
\[
\begin{array}{cccccccccccc}
I & 1 & 2 & 3 & 4 & 6 & 9 & 19 & 31 & 57 & 63 \\
J & 1 & 39 & 185 & 185 & 39 & 2082 & 2083 & 2085 & 2082 & 2083 \\
\end{array}
\]

— pldx : Translation of the Kimberling’s Glossary into barycentrics ——
Proposition 20.5.10. When \( NK_P (X) \) belongs to the tripolar line of \( U_P^\prime \), then point \( X \) belongs to the cubic \( nK0 (P, U) \).

Proof. Direct computation. In Kimberling (1998, p. 240), notation \( Z + (XY) \) is used to denote the \( nK0 (\#1, U) \) cubic where the pole \( U \) is the isogonal of the tripole of line \( XY \). Therefore,

\[
\begin{align*}
Z^+(X_1X_6) &= nK0 (\#1, 513) \\
Z^+(X_3X_6) &= nK0 (\#1, 523) \\
Z^+(X_1X_2) &= nK0 (\#1, 649) \\
Z^+(X_1X_3) &= nK0 (\#1, 650)
\end{align*}
\]

\( \square \)

20.5.1 Conicopivotal isocubics \( cK(\#F, U) \)

Definition 20.5.11. A conico-pivotal isocubic \( cK(\#F, U) \) (Ehrmann and Gibert, 2005) is a non pivotal isocubic \( nK (P, U, k) \) that contains one of the fixed points of the isoconjugacy \( (F \neq U \text{ is assumed}) \). Using \( F = f : g : h \) instead of \( P = p : q : r = f^2 : g^2 : h^2 \), \( k = -2 (ghu + fhv + fgw) \) and equation becomes:

\[
x(gz - hy)^2u + y(fz - hx)^2v + z(fy - gx)^2w = 0 \quad (20.12)
\]

Proposition 20.5.12. The pivotal conic is defined as the conic \( C \) tangent to the six lines \( F_B F_C \), \( AA_U^\prime \) and cyclically where \( F_AF_BF_C^+ \) is the anticentric of \( F \) and \( AA_U^\prime B_B^\prime C_C^\prime \) the cocevian of \( U \). Then the dual conic of \( C \) is conicev \( \{1/F, 1/U\} \) and \( C \) itself has equation:

\[
\sum_{\text{cyclic}} (gw - hv)^2 x^2 - 2 (gw^2h + 3f (gw + vh) u + f^2vw) zy = 0
\]

Proof. We have the equations:

\[
\begin{align*}
(F_BF_C) &= F_B \land F_C = \left( \begin{array}{c} f \\ -g \\ h \end{array} \right) \land \left( \begin{array}{c} f \\ g \\ -h \end{array} \right) = \left[ 0, 2fh, 2fg \right] = \left[ 0, \frac{1}{g}, \frac{1}{h} \right] \\
AA_U &= U_B U_C = \left[ 0, \frac{1}{v}, \frac{1}{w} \right]
\end{align*}
\]

The equation of \( C \) follows by duality. Barycentrics of the center are \( 2fu - (v + w) f - (g + h) u, \) etc.

\( \square \)

Proposition 20.5.13. The contact conic \( (K) \) is defined as the circumconic whose perspector is

\[
K \simeq \left( 2 \frac{f}{u} + \frac{g}{v} + \frac{h}{w} \right) f, \text{ etc}
\]

Assuming \( F \neq U \), three of the intersections of the pivotal and contact conics are the three contacts of \( cK \) with \( C \), the fourth point being:

\[
T_4 \simeq \left( 2 \frac{f}{u} + \frac{g}{v} + \frac{h}{w} \right) \div (gw - vh), \text{ etc}
\]

Proof. Eliminate \( z \) between \( (K) \) and \( C \). Obtain \( P_1 (x, y) P_3 (x, y) \), where degrees are respectively 1 and 3. Solving for \( P_1 \) gives directly \( T_4 \). Eliminate \( z \) between \( cK \) and \( C \). This leads again to \( P_3 \), proving that each common point is a contact and belongs also to \( (K) \).

\( \square \)

Example 20.5.14. A special case is obtained when \( U = F \), i.e. when the root is a fixed point of the isoconjugacy. Then \( C = (K) \) is a circumconic. The Tucker cubic \( K015 \) is obtained with \( F = X(2) \), while \( K228 \) is obtained with \( F = X(1) \) and \( K229 \) is obtained with \( F = X(6) \).
Centroid $G = X_2$ is isolated, but belongs nevertheless to the cubic.

Figure 20.6: The Simson cubic (as depicted in Gibert-CTP)

20.5.2 Simson cubic, aka K010

Notation 20.5.15. In this section, the involved pole is the centroid, so that $X^* = \text{isot}(X)$. Due to the nature of the cubic, the key point is $F = X_2$ (involved as fixed point) rather than $P = X_2$ (involved as pole). Therefore, letter $P$ has been used not to describe the pole, but the independent moving point of various parametrization.

Definition 20.5.16. The Simson cubic is the locus of the tripoles of the Simson lines. Depicted as K010 (cf. Figure 20.6) in Gibert (2004-2024). Founding paper is Ehrmann and Gibert (2001).

Proposition 20.5.17. The Simson cubic (K010) is $c_{K}(\#X_2, X_{69})$. Centroid $G = X_2$ belongs to K010 (Simson line of $Q$ is $L_\infty$ when $Q \in L_\infty$). Apart from this isolated point, a parametrization of K010 is given in (26.7), starting from $L_\infty$. Another parametrization, using the barycentrics of the involved point on $\Gamma$ is as follows:

\[
\begin{pmatrix}
u \\
w
\end{pmatrix} \in \Gamma \mapsto \begin{pmatrix}
\left( b^2 wv^2 - c^2 vu^2 + (b^2 - c^2) wvu \\
c^2 uv^2 - a^2 v^2 w + (c^2 - a^2) wuv \\
a^2 v^2 w - w^2 b^2 u + (a^2 - b^2) wuv
\end{pmatrix} \in K010
\]

Definition 20.5.18. Cubic K162 is the isogonal transform of the Simson cubic (that can also be obtained by $Q \mapsto Q \div bX_6$). Therefore, K162 is $c_{K}(\#X_6, X_3)$.

Definition 20.5.19. The Gibert-Simson transform is another parametrization of the Simson cubic that also uses an $U \in \Gamma$:

\[
\text{GS}(U) = \text{cyclic} \left[ \left( \frac{b^2 (c^2 + a^2 - b^2)}{va^2} - \frac{c^2 (a^2 + b^2 - c^2)}{a^2 w} \right) u \right]
\]

The lack of uniqueness is due to the binding relation $\sum a^2 vw = 0$. Definition introduced in ETC on 2003/10/19, leading to points $X(2394)$ - $X(2419)$ on the Simson cubic and points $X(2420)$-$X(2445)$ -their isoconjugates- on K162.

Remark 20.5.20. Regarding triangle centers that do not lie on the circumcircle, GS(X(I)) = X(J) for these (I,J): (32,669), (48,1459), (187,1649), (248,879), (485,850), (486,850). Of course, other realizations of $U \mapsto K$ give other results. Here again, only parametrization (26.7) ensures uniqueness.

Example 20.5.21. Use $^tP = ^tX_{525}$ as entry point Figure 20.7 (arrow at the left of the bottom diagram). Obtain $X_{30} = Q_1 \in L_\infty$ by (7.15), then $X_{74} = U_1 \in \Gamma$ by isogonal conjugacy. The
Steiner line $S_{1}$ hasn’t received any name, while the Simson line $S_{1}$ of $U_{1}$ is $^{t}X_{247}$. The trilinear pole of this line, i.e. $X_{2394} = K_{1} \in K_{010}$, can be obtained by $isot \circ (\cdot )$ from $S_{1}$, by $gs$ from $U_{1}$ and also directly from $P$ (the dotted line) using parametrization (26.7).

![Diagram](image)

**Figure 20.7: The Simson diagram**

**Proposition 20.5.22** (Fools’ Day Theorem). Direct arrows $U_{1} \mapsto K_{2}$ and $U_{2} \mapsto K_{1}$ have a geometrical meaning : $(gs(U))^*$ is the eigencenter of the pedal triangle of $U$.

**Proof.** Point $K_{2}$ has no other choice : he *is* the unary cofactor of the pedal triangle of $U$, and therefore the perspector of this triangle with anything else.

**Proposition 20.5.23.** The barycentric equation of the Simson cubic is

$$
\sum_{cyclic} x(z^2 + y^2)(b^2 + c^2 - a^2) - 2(a^2 + b^2 + c^2)x y z
$$

(20.13)

In other words, the Simson cubic is $nK(X_{2}, X_{69}, \#X_{2394})$. Moreover, one of the fixed points (namely $G = X_{2}$) belongs to the cubic and the Simson cubic is in fact $cK(\#X_{2}, X_{69})$.

**Proof.** Obtained from the parametric representation. The converse property is more easily obtained from next coming proposition.

**Definition 20.5.24. Special points wrt $K_{010}$.** Points on sidelines of triangle $ABC$ or of antimedial triangle, together with the centroid are said to be special wrt $K_{010}$ (because quite every “general” formula turns wrong when dealing with these points).
Proposition 20.5.25. Among the special points, the following are the sole and only elements of $K010$:

(i) the centroid itself (fixed point under isoconjugacy)
(ii) the vertices of triangle $ABC$, the cocevians of $X_{69}$ (the root) and points $b + c : b - c : c - b$ or $b - c : b + c : -b - c$ and cyclically.

Proof. Direct inspection.

Proposition 20.5.26. When point $X$ is on the Simson cubic but not $G, A, B, C$ then:

(i) the trilinear polars of $X$ and $X^*$ are perpendicular
(ii) the trilinear polars of $X$ and $X^*$ are concurrent on the nine-point circle

Conversely, when $X$ is not special and either property holds, then $X$ is on the cubic.

Proof. When $X$ is on the Simson cubic, tripolar $(X)$ is a Simson line and conclusion follows from tripolar $= t \circ \text{isot}$. Conversely, if (i) then points $tX^* \wedge L_\infty$ and $^tX \wedge L_\infty$ are the infinity points of both tripolars. They have to be the orthopoint of each other, and (20.13) is re-obtained by elimination. If (ii) then dividing $\text{nineq}(X \wedge X^*)$ by (20.13), leads to $\prod (y + z) / x^2$. When $X$ is on a sideline of $ABC$, conjugacy is no more defined, and when $y + z = 0$ (implying $X$ on the sidelines of the antimedial triangle) then $X \wedge X^*$ is ever $0 : 1 : 1$. Outside of these six lines, both conditions are equivalent.

20.5.3 Brocard second cubic aka K018

Definition 20.5.27. The Brocard second cubic is inventoried as K018 in (Gibert, 2004-2024). This cubic is $nK0(X_6, X_{523})$. It is a circular isogonal focal nK cubic with root $X(523)$ and singular focus $X(111)$. The real asymptote is parallel to GK. It is also the orthopivotal cubic $O(X_6)$ and $Z+(L)$ with $L = X(3)X(6)$ in TCCT p.241. See also $Z+(O) = CL025$ and CL034.

Proposition 20.5.28. The barycentric equation of $K018$ is:

$$\sum_{\text{cyclic}} x(b^2z^2 + y^2c^2)(b^2 - c^2) = 0$$ (20.14)
20.5. Non pivotal isocubics $nK(P,U,k)$ and $nK_0(P,U)$
Chapter 21

Tripolar curves

Definition 21.0.1. Given three fixed distinct points $E, F, G$, and three real numbers $u, v, w$, "this branch of a tripolar curve" $W(u, v, w)$ is the locus of points $M$ such that:

$$u |EM| + v |FM| + w |GM| = 0$$

On the contrary, the corresponding algebraic tripolar curve is the locus of points $M$ such that:

$$\pm u |EM| \pm v |FM| \pm w |GM| = 0 \quad (21.1)$$

Remark 21.0.2. When one of the $u, v, w$ vanishes, a tripolar curve degenerates into an ordinary conic (this is excluded in what follows).

Notation 21.0.3. In this chapter, quantities $\alpha, \beta, \gamma, \delta$ are related to points $E, F, G, H$, while $u, v, w$ are some multipliers. Quantities $S_Q, S_u, S_v, S_w$ are related to these multipliers $u, v, w$. They mimic the usual area and Conway symbols. In other words,

$$S_Q^2 = \frac{1}{16} (u + v + w) (-u + v + w) (u - v + w) (u + v - w), \quad S_u = \frac{1}{2} (v^2 + w^2 - u^2)$$

On the contrary, symmetric functions are related to $\alpha, \beta, \gamma, \delta$. In other words,

$$q_1 = \sum_4 \alpha ; \quad q_2 = \sum_6 \alpha \beta ; \quad q_3 = \sum_4 \alpha \beta \gamma ; \quad q_4 = \alpha \beta \gamma \delta \quad (21.2)$$

Remark 21.0.4. Section Section 21.1 provides some background by studying the whole space of the bicircular quartics. Then a section describes how to use Maple and GeoGebra to draw the tripolar curves. A later section gives some further properties of these curves.

21.1 The bicircular space

Definition 21.1.1. A general quartic $q$ requires $5+4+3+2+1 = 15$ coefficients. When $q$ is singular at each umbilic, its gradient has to vanish there, so that

$$q_{40} = q_{31} = q_{13} = q_{04} = q_{03} = q_{30} = 0$$

The set of all bicircular quartics is therefore a copy of $\mathbb{P}_C(C^3)$. In this chapter, "quartic" is assumed everywhere, and this space will simply be depicted as "the bicircular space $\mathbb{P}(Q)$".

Proposition 21.1.2. The equation of any curve $q \in \mathbb{P}(Q)$ can be written in the following matrix form:

$$q(M) = \begin{pmatrix} Z^2 \\ ZT \\ T^2 \end{pmatrix} \cdot \begin{pmatrix} q_{22} & q_{21} & q_{20} \\ q_{12} & q_{11} & q_{10} \\ q_{02} & q_{01} & q_{00} \end{pmatrix} \cdot \begin{pmatrix} Z^2 \\ ZT \\ T^2 \end{pmatrix} = 0$$

Therefore, any homography $H$ acting over the points of $\mathbb{P}_C(C^3)$ according to

$$(Z : T : \bar{Z}) \mapsto \left( \frac{aZ + bT}{cZ + dT} : 1 : \frac{a'\bar{Z} + b'\bar{T}}{c'\bar{Z} + d'\bar{T}} \right)$$

induces an action which is linear over the bicircular space $\mathbb{P}(Q)$ according to
The bicircular quartic admit four singular foci, namely

\[ -q_{21} \pm \sqrt{q_{21}^2 - 4q_{22}q_{20}} : 1 : -q_{12} \pm \sqrt{q_{12}^2 - 4q_{22}q_{20}} \]

Proof. Direct computation: cut by \( M \Omega x \), factor, substitute \( T = 0 \), \( Z = 1 \) and equate to 0. \( \square \)

Proposition 21.1.4. Among the elements of \( \mathbb{P}(Q) \), we have (1) the union (=product) of two cycles; (2) the image of any conic by an homography.

Proof. Direct computation. \( \square \)

Proposition 21.1.5. Whatever could be meaning of \( \sqrt{ux} \), etc, then the three terms relation (TTR)

\[ \sqrt{ux} + \sqrt{vy} + \sqrt{wz} = 0 \]  

(21.3)

can be rewritten as:

\[ \text{TTR} \iff u^2x^2 + v^2y^2 + w^2z^2 - 2(uvx + vwy + wuz) = 0 \]

\[ \iff [x, y, z] \cdot \begin{bmatrix} u^2 & -uv & -uw \\ -uv & v^2 & -uw \\ -uw & -vw & w^2 \end{bmatrix} \cdot [x, y, z] = 0 \]

1. When \( x, y, z \) are the coordinates of a line wrt the reference trigone, the TTR describes a tangential conic \( \mathcal{C}^* \). The associated punctual conic is the circumscribed conic

\[ \mathcal{C} \simeq \begin{bmatrix} 0 & w & v \\ w & 0 & u \\ v & u & 0 \end{bmatrix} \]

2. When \( x, y, z \) are circles’ equations, one obtains the equation of a quartic \( \mathcal{Q} \). And this quartic is bitangent to each of these circles.

Proof. Direct computation. \( \square \)

Proposition 21.1.6. Assume that \( u, v, w \) be three coefficients and \( C_j, j = 1 \ldots 3 \) be three circles, with centers \( z_j \) and radiuses \( r_j \). This defines a common orthogonal cycle \( C_0 \), and a \( z_j \)-circumscribed conic \( \mathcal{C} \). A moving point \( P \in \mathcal{C} \) defines a circle \( \mathcal{C}_P \) orthogonal to \( C_0 \). Then, the envelope of the circles \( \mathcal{C}_P \) is the bicircular quartic \( \mathcal{Q} \) of the just above proposition.

Proof. The usual property of the enveloppes! (Casey, 1871, §5, p.459) \( \square \)

Example 21.1.7. All the three sub-figures of Figure 21.1 have been drawn with \( z_1 = -2 - 3i \), \( z_2 = 2 + 3i \), \( z_3 = 3 - i \) and \( u = 5 \), \( v = 6 \), \( w = 11 \). Thus, they share the same focal conic. But using 1, 2, 1, resp 3, 2, 1 and 3, 2, 3 as radiuses lead to different orthogonal circles. As a result, there are resp. four, none and two visible intersections of \( \mathcal{C} \) with \( C_0 \), are visible on the top. A not visible point is alluded with its \( Z : T \) part, noted \( F_j \). Then a *true* intersection is \( F_{34} \simeq F'_3 : 1 : F'_4 \), etc (on the \( C_0 \) circle), while \( F'_3 \) and \( F'_4 \) are inverse in the focal circle.

Proposition 21.1.8. The four intersections of the focal conic and the orthogonal cycle, visible or not, are four of the 16 foci of the quartic. (Casey, 1871, §16, p.464) On the other hand, the foci of the focal conic are the singular foci of the quartic.

Proof. (1) Each of these intersections define a null circle which is bitangent to the curve. But a null circle is the product of the isotropic lines through its center. (2) Direct computation. \( \square \)
Figure 21.1: Does the focal conic intersect the orthogonal circle?
21.2 Define and draw

**Maple 21.2.1.** The Maple library `tcurv.m` deals with the following objects:

- `zptE1` to `zptH1`: the four focuses
- `zptDe_1` to `zptDg_1`: the diagonal triangle
- `VALdelta1`: value of $\delta$ from $a, b, c, \alpha, \beta, \gamma$
- `VALeq11`: raw $Z, T, \bar{Z}$ equation from $a, \alpha, b, \beta, c, \gamma$
- `VALmeth2`: new values of $a, b, c$ when changing the poles
- `ledeb`: vector $a:b:c$
- `valabc_1` to `valabc_2`: values of $a, b, c$ from $\alpha, \beta, \gamma, \delta, \nu$
- `zptK0_2` to `zptX6g_1`: the four points on the focal circle.
- `rotEFG` to `rotEFG3`: rotate the variables

**Maple 21.2.2.** contains the procedures `geotcurv` and `buildmeth`.

The procedure receives $u, v, w \in \mathbb{R}$ together with $E, F, G \in \mathbb{C}$ and produces $\alpha, \beta, \gamma, \delta \in \Gamma$ (the reduced coordinates of the four focuses). The seventh argument $\nu \in \Gamma$ is used to select a connected piece of the curve. When given, the procedure checks that $\nu$ in one of the four acceptable values. Otherwise, one of the four possibilities is returned.

The result is two Maple sequences (to be used in Lubin1 or Lubin2 context), together with a set of commands to be transmitted to geogebra.

**Maple 21.2.3.** The curve, already depicted using a set of three focuses, is depicted using the other triples of focuses. And then, one can check that all the four descriptions are leading to the same cartesian equation $eqq1$.

21.3 The generic case

**Proposition 21.3.1.** The tripolar curve is globally invariant under the reflection $\mu$ into the EFG cycle.

**Proof.** This is obvious if $EFG$ are aligned. Otherwise, let $U$ be the center of the reflection circle. Then, for generic $P, Q$, the triangles $(U, P, Q)$ and $(U, \mu(P), \mu(Q))$ are similar, leading to

$$|\mu(P)\mu(Q)| = |PQ| \times \frac{R^2}{|UP||UQ|}$$

therefore $|EP(P)| = |EP| \times \frac{R}{|UP|}$

$\square$

**Notation 21.3.2.** For this reason, we will use turns $\alpha, \beta, \gamma, \delta$ to represent the points $E, F, G$ (and the next coming point $H$). Part of the time, Lubin-1 will be used, and results are written as $z_E = \alpha$, etc. Part of the time Lubin-2 will be used and results are written as $z_E = \frac{\alpha^2}{2}$. Increasing the degree allows to split some algebraic equations at the price of lengthening the expressions. Coexistence of both systems requires to tag any equal sign. This third option is the one we have chosen.
Listing 21.1: The geotcurv procedure
Proposition 21.3.3. When $S \neq 0$, the equation of the tripolar curve can be written as

$$W(M) = \frac{1}{2} \left( Z^2\overline{Z}^2 + T^4 \right) + \left( Z\overline{Z} + T^2 \right) (W_1 Z + W_{12} \overline{Z}) T + T^2 \left( W_{20} Z^2 + W_{11} Z\overline{Z} + W_{02} \overline{Z}^2 \right)$$

$$= \begin{pmatrix} \overline{Z}^2 \\ \overline{Z}T \\ T^2 \end{pmatrix} \begin{pmatrix} 1 & W_{12} & W_{02} \\ W_{21} & W_{11} & W_{12} \\ W_{20} & W_{21} & 1 \end{pmatrix} \begin{pmatrix} Z^2 \\ ZT \\ T^2 \end{pmatrix}$$

where

$$W_{21} = -2 \sum_3 \frac{u^2}{8} \frac{1}{Q} \alpha^2 ; \quad W_{12} = -2 \sum_3 \frac{u^2}{8} \frac{1}{Q} \alpha^2 ; \quad W_{20} = \frac{1}{16S^2} \sum_4 \left( \frac{u \alpha + v \beta + w \gamma}{u \alpha + v \beta + w \gamma} \right)$$

and the curve is a bicircular quartic. Coefficients in $H_{21}, H_{12}$ are the usual normalized barycentrics of the circumcenter (as of now, no interpretation has been given). When $u \pm v \pm w = 0$, the curve degenerates into a simply-circular cubic. Otherwise, the visible part of $W$ is bounded.

Proof. This equation can be rationalized into:

$$\sum_3 \left( u^4 |EM|^4 \right) - 2 \sum_3 \left( v^2 w^2 |FM|^2 |GM|^2 \right) = 0$$

leading to a not so huge expression and, when making $T = 0$, it only remains:

$$(u + v + w)(-u + v + w)(+u - v + w)(+u + v - w) \times \alpha^2 \beta^2 \gamma^2 Z^2 \overline{Z}^2 \Box$$

Proposition 21.3.4. There are four intersections with the circumcircle. One has:

$$K_0 = \begin{pmatrix} \nu \\ 1 \end{pmatrix} \approx \begin{bmatrix} u \alpha + v \beta + w \gamma \\ u / \alpha + v / \beta + w / \gamma \\ 1 \\ u \alpha + v \beta + w \gamma \end{bmatrix}$$

January 3, 2024 21:08 published under the GNU Free Documentation License
together with $K_\alpha K_\beta K_\gamma$ obtained by replacing the corresponding $\alpha$ by its opposite.

Proof. The main occasion to use Lubin-2 here. It must be noted that the $\alpha, \beta, \gamma$ used in this
Lubin-2 formula are only defined up to a change of sign, so that none of the four is the "true
$u \alpha + v \beta + w \gamma$"

Proposition 21.3.5. The tripolar curve has four focuses, the already involved $E, F, G = \alpha, \beta, \gamma$
and a fourth point, $H$, also on the unit circle and given by:

$$\delta = \frac{1}{\alpha \beta \gamma} \mathrm{id}_{\text{conj}} \left( \sum_3 u^2 (\alpha^2 - \alpha \beta - \alpha \gamma + \beta \gamma) \right)$$

(21.6)

Proof. We cut by a line $P \Omega_g$. Since the curve is bi-circular, the degree falls and we only have to nullify the discriminant of a second degree equation. One can notice that:

$$\frac{\delta^2}{2} = \frac{\mathrm{id}_{\text{conj}}}{2} \left( \frac{1}{\alpha \beta \gamma} [\alpha^2, \beta^2, \gamma^2] \cdot \begin{bmatrix} \mathcal{M} \end{bmatrix}^t [\alpha^2, \beta^2, \gamma^2] \right)$$

(21.7)

where $\mathcal{M}$ is the usual matrix (7.18) build on $(u, v, w)$.

Definition 21.3.6. There are three bi-transpositions of the set $\{E, F, G, H\}$. We note them $\sigma_j$
by using who is paired with $H$. Thus $\sigma_0$ notes $\alpha \leftrightarrow \gamma, \beta \leftrightarrow \delta$. This convention will also be
used to describe the Cremona homography of the whole plane (see Definition 17.1.5) specified by
$\alpha \leftrightarrow \gamma, \beta \leftrightarrow \delta$. We have

$$\sigma_0 : \begin{pmatrix} Z \\ T \\ Z \end{pmatrix} \mapsto \begin{pmatrix} (\delta \beta - \alpha \gamma) Z + ((\beta + \delta) \alpha \gamma - (\alpha + \gamma) \beta \delta) T \\ (\beta + \delta - \alpha - \gamma) Z - (\delta \beta - \alpha \gamma) T \\ ((\beta + \delta) \alpha \gamma - (\alpha + \gamma) \beta \delta) Z - (\delta \beta - \alpha \gamma) T \end{pmatrix}$$

Theorem 21.3.7. The tripolar curve remains globally unchanged by the $2 \times 2 \times 2$ group $\Omega$ generated
by $\mu$ and the $\sigma_j$.

Proof. Invariance by $\mu$ has been proven at Proposition 21.3.1. Invariance by $\sigma_0$ is direct computation. The type $2 \times 2 \times 2$ of the group $\Omega$ comes from the underlying action on $\{E, F, G, H\}$.

Definition 21.3.8. Transformation $\pi_0 = \mu \circ \sigma_0$ is involutory and is necessarily a reflection into a
circle $C_\beta$.

$$\pi_0 : \begin{pmatrix} Z \\ T \\ Z \end{pmatrix} \mapsto \begin{pmatrix} ((\beta + \delta) \alpha \gamma - (\alpha + \gamma) \beta \delta) Z - (\delta \beta - \alpha \gamma) T \\ (\delta \beta - \alpha \gamma) Z + (\alpha + \gamma - \beta - \delta) T \\ ((\beta + \delta) \alpha \gamma - (\alpha + \gamma) \beta \delta) Z - (\delta \beta - \alpha \gamma) T \end{pmatrix}$$

Let $D_\beta$ and $\rho_\beta$ be the center and radius of $C_\beta$.

Proposition 21.3.9. Triangle $D_\alpha D_\beta D_\gamma$ is the common diagonal triangle to both the quadrangle
$D, E, F, G$ and the quadrangle of the $K_j$, while the four circles $\Gamma$ and $C_j$ are orthogonal to each other.

Proof. By construction, the set $\{E, F, G, H\}$ is invariant by $\sigma_0$. Since $\Gamma$ and $W$ are globally
invariant, so is their intersection, the set of the four $K_j$. And the first conclusion follows, since
object and image of a reflection in a circle are aligned with the center. Moreover, orthogonality is
required for a given circle be invariant by reflection into another circle.

We can also take the representatives of all these circles

$$\begin{pmatrix} \beta + \gamma - \alpha - \delta \\ 2(\delta \alpha - \beta \gamma) \\ (\alpha + \delta) \beta \gamma - (\beta + \gamma) \alpha \delta \\ \delta \alpha - \beta \gamma \end{pmatrix} \quad \begin{pmatrix} \gamma + \alpha - \beta - \delta \\ 2(\delta \beta - \gamma \alpha) \\ (\beta + \delta)(\gamma \alpha - (\gamma + \alpha) \beta \delta \\ \delta \beta - \gamma \alpha \end{pmatrix} \quad \begin{pmatrix} \alpha + \beta - \delta - \gamma \\ 2(\delta \gamma - \alpha \beta) \\ (\gamma + \delta)(\alpha \beta - (\alpha + \gamma) \gamma \delta \\ \delta \gamma - \alpha \beta \end{pmatrix}$$
then use the generic formulas to obtain center and power

\[
D_\beta \simeq \frac{1}{1 \left( \beta \delta (\alpha + \gamma) - \alpha \gamma (\beta + \delta) \right)} \beta \delta - \alpha \gamma
\]

and finally conclude by taking the Gramm matrix of the four circles.

\[\square\]

Exercise 21.3.10. Prove that product \( \rho_\alpha \rho_\beta \rho_\gamma \) is imaginary, so that one of the circles is imaginary.

Exercise 21.3.11. The 12 fixed points of the three homographies \( \sigma_j \) are the intersections of the four circles \( \Gamma \) and \( C_j \). To obtain the visible points among them, we have to discard the imaginary circle (see previous exercise).

Exercise 21.3.12. The action of \( \pi_A \) on the four intersections with the circumcircle amounts to change the sign of \( a \) in the formulas (21.5). Hint: use Lubin2, and substitute \( \delta^2 \) from (21.7).

21.4 Cross-ratios

Lemma 21.4.1. Equation (21.1) can be rewritten using cross-ratio, transforming \( M \in W \) into:

\[
W(M) = \frac{1}{(\gamma - \beta)\sqrt{\alpha}} + \frac{v}{(\alpha - \gamma)\sqrt{\beta}} \frac{(M - F) (G - E)}{(M - E) (G - F)} + \frac{w}{(\beta - \alpha)\sqrt{\gamma}} \frac{(M - G) (F - E)}{(M - E) (F - G)}
\]

(21.9)

Proof. One has \(|GF| = |G - F| = \pm i (\beta - \gamma) \div \sqrt{\beta \gamma} \), etc.

\[\square\]

Proposition 21.4.2. The defining equation (21.1) can be rewritten for each triple of focuses. And we have

\[
\begin{pmatrix} u \\ v \\ w \end{pmatrix}_{E,F,G} \mapsto \begin{pmatrix} d \\ e \\ f \end{pmatrix}_{E,H,G} \simeq \frac{1}{\beta \gamma} \begin{pmatrix} +w \sqrt{\frac{\alpha}{\gamma}} (\gamma - \beta) (\gamma - \delta) \\ -v \sqrt{\frac{\alpha}{\beta}} (\beta - \alpha) (\beta - \gamma) \\ +u \sqrt{\frac{\alpha}{\gamma}} (\alpha - \delta) (\alpha - \beta) \end{pmatrix}
\]

Iterating the process, we get the final table:

<p>| | | | | |</p>
<table>
<thead>
<tr>
<th></th>
<th></th>
<th></th>
<th></th>
<th></th>
</tr>
</thead>
<tbody>
<tr>
<td>E</td>
<td>F</td>
<td>G</td>
<td>u</td>
<td>v</td>
</tr>
<tr>
<td>F</td>
<td>G</td>
<td>H</td>
<td>w</td>
<td>v</td>
</tr>
<tr>
<td>E</td>
<td>G</td>
<td>H</td>
<td>w</td>
<td>v</td>
</tr>
<tr>
<td>E</td>
<td>F</td>
<td>H</td>
<td>v</td>
<td>u</td>
</tr>
</tbody>
</table>

Proof. On the one hand, substitute \([E = H, F = G, G = F]\) (and nothing else) into formula (21.9). On the other hand, substitute not only \([E = H, F = G, G = F]\) but also \([\alpha = \delta, \beta = \gamma, \gamma = \beta]\) and \([u = k_h, v = k_g, w = k_f]\). And equate both results. The second move is only a flat application of the formula, the first one is using the invariance of the curve under the cross-ratio preserving homography \( \sigma_\alpha \).

\[\square\]

21.5 Introducing the cut parameter

Definition 21.5.1. Choosing one of the four intersections of \( W \) with \( \Gamma \) as \( K_0 \simeq \nu : 1 : 1/\nu \) breaks the symmetry of the problem. But this provides what is required to split the action of \( \mu, \sigma, \pi \).
Proposition 21.5.2. Using the cut parameter, we obtain: $t(u,v,w) = \frac{1}{\alpha\beta\gamma\sqrt{\delta}} \left( \sqrt{\alpha}(\beta - \gamma) \left( (\alpha + \delta - \beta - \gamma)\nu - 2(\delta \alpha - \beta \gamma) + \frac{\delta \alpha (\beta + \gamma) - \beta \gamma (\delta + \alpha)}{\nu} \right) \right.$

\[
\left. \sqrt{\beta}(\gamma - \alpha) \left( (\beta + \delta - \gamma - \alpha)\nu - 2(\delta \beta - \gamma \alpha) + \frac{\delta \beta (\gamma + \alpha) - \alpha \beta (\delta + \beta)}{\nu} \right) \right.
\]

\[
\left. \sqrt{\gamma}(\alpha - \beta) \left( (\gamma + \delta - \alpha - \beta)\nu - 2(\delta \gamma - \alpha \beta) + \frac{\delta \gamma (\alpha + \beta) - \alpha \beta (\delta + \gamma)}{\nu} \right) \right. \in \mathbb{R}^3
\]

Proof. Compute the $D_j$ from the $K_0, K_j$ and compare with those obtained at (21.8). Then the simultaneous reality of the multipliers is proven by their invariance under conjugacy.

Corollary 21.5.3. Equation of $W$ can be rewritten using the symmetric functions (21.2) The $\mathbb{Z}^2\mathbb{Z}T$ coefficients are a sum of terms like

\[
\frac{-2q_1^4 - 32q_1q_3 + 32q_2^2 - 128q_4}{q_4^4}\nu^4
\]

i.e. a power $\nu^k$ (where $-4 \leq k \leq 4$) times a $\mathbb{Z}[q_1, q_2, q_3, q_4]$ polynomial divided by a power of $q_4$, leading to a zero global degree in $\alpha, \beta, \gamma, \delta, \nu$. It remains to say that the resulting expressions have a rather huge length:

\[
\begin{array}{cccc}
H_{22} & H_{12} & H_{02} & H_{11} \\
705 & 800 & 773 & 876 \\
\end{array}
\]

Fact 21.5.4. We can use a varying $\nu$ as in Figure 21.2 to see that focuses cut the circle into arcs that contains either 2 or 0 intersections with the curve. Starting from violet arcs $EG$ and $FH$, we get something like two bananas surrounding both pairs of focuses. Then they cross each other at circle $cinvG$ and reduce back to the original arcs. Then we get arcs $EF$ and $HG$, etc.
Example 21.5.5. Figure 21.2 has been constructed by choosing
\[ u = 11\sqrt{5}, \nu = 5\sqrt{5}, w = 26, \alpha = -4/5 + 3/5 i, \beta = 4/5 + 3/5 i, \gamma = -i \]
Then the fourth focus is \( \delta = 1 \). The multipliers are given by columns 4,5,6 to obtain the branch containing \( \nu = (-28 - 45i)/53 \) and are given by columns 7,8,9 to obtain the branch containing \( \nu' = (80 + 39i)/89 \).

<table>
<thead>
<tr>
<th></th>
<th></th>
<th>( \nu )</th>
<th>( \nu' )</th>
</tr>
</thead>
<tbody>
<tr>
<td>( E )</td>
<td>( F )</td>
<td>( G )</td>
<td>( 11\sqrt{5} )</td>
</tr>
<tr>
<td>( F )</td>
<td>( G )</td>
<td>( H )</td>
<td>( 13\sqrt{5} )</td>
</tr>
<tr>
<td>( E )</td>
<td>( G )</td>
<td>( H )</td>
<td>( 13\sqrt{5} )</td>
</tr>
<tr>
<td>( E )</td>
<td>( F )</td>
<td>( H )</td>
<td>( 5\sqrt{5} )</td>
</tr>
</tbody>
</table>

21.6 Barycentrics wrt the diagonal triangle

Proposition 21.6.1. When using the diagonal triangle \( D_\alpha D_\beta D_\gamma \) as barycentric basis, we obtain:
\[
H_0 \simeq \langle f : g : h \rangle |_{b,1} \simeq \left( \frac{(\alpha \delta - \gamma \beta)(\alpha - \delta)}{26(\alpha - \gamma)(\alpha - \beta)}, \frac{(\beta \delta - \alpha \gamma)(\beta - \delta)}{28(\beta - \gamma)(\beta - \alpha)}, \frac{(\gamma \delta - \alpha \beta)(\gamma - \delta)}{28(\gamma - \beta)(\gamma - \alpha)} \right)
\]
where the \( f, g, h \) are real with \( f + g + h = 1 \), while the other three focuses \( H_e H_f H_g \) are given by \( \pm f : \pm g : \pm h \), leading to an anticevian configuration.

Proof. Direct computation.

Proposition 21.6.2. The point \( O \), center of the focal circle, is the orthocenter of the diagonal triangle. Its matrix is
\[
\text{diag} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}
\]
with \( \text{diag} \) defined in (21.3).

Proof. From the \( \pm f : \pm g : \pm h \) property, the matrix of \( \Gamma \) has to be diagonal. But the orthocentroidal circle is the only one to share this property.

Definition 21.6.3. For each triple of focuses, we introduce the Lemoine center \( L_h \) and the mass center \( M_h \) by:
\[
L_h \simeq \frac{(FG)^2 \cdot E + (GE)^2 \cdot F + (EF)^2 \cdot G}{(FG)^2 + (GE)^2 + (EF)^2}, \quad M_h \simeq \frac{(u^2E + v^2F + w^2G)}{(u^2 + v^2 + w^2)}
\]
The first time, we ponder by the squared lengths of the sidelines, the second time, we ponder by the square of the coefficients involved in (21.1).

Proposition 21.6.4. Wrt the diagonal triangle, the barycentric coordinates of \( L_0 \) and \( K_0 \) are
\[
L_0 \simeq \begin{pmatrix} \delta(\alpha - \gamma \beta)(\beta - \gamma)(\beta - \alpha) \\ \delta(\beta - \alpha \gamma)(\gamma - \alpha)(\beta - \gamma) \\ \delta(\gamma - \beta \alpha)(\alpha - \beta)(\gamma - \beta) \end{pmatrix}, \quad K_0 \simeq \begin{pmatrix} \delta(\alpha - \gamma \beta)(\beta - \gamma)(\beta - \alpha) \\ \delta(\beta - \alpha \gamma)(\gamma - \alpha)(\beta - \gamma) \\ \delta(\gamma - \beta \alpha)(\alpha - \beta)(\gamma - \beta) \end{pmatrix}
\]
and we have \( M_j = K_j * K_j * L_j \; z \; E_j \; z \; E_j \).

Proof. Everything remains in the Lubinis domain, since we are using \( K_0 \simeq \nu : 1 : \nu^{-1} \). A direct computation is easy. One can also use the fact that:
\[
K_0 \simeq H_0 * \begin{pmatrix} u/FG \\ v/GE \\ w/EF \end{pmatrix}, \quad L_0 \simeq H_0 * \begin{pmatrix} HE/FG \\ HF/GE \\ HG/EF \end{pmatrix}, \quad M_0 \simeq H_0 * \begin{pmatrix} w^2HE/FG \\ v^2HF/GE \\ w^2HG/EF \end{pmatrix}
\]
21.7 When $E$ is on the curve

$u \cdot |EE| + v \cdot |EF| + w \cdot |EG| = 0$ implies $v : w \simeq |EG| : |EF|$.

**Proposition 21.7.1.** ***pompous*** Point $D_e$ is on the curve when either $S = 0$ (this case will be studied in detail at Section 21.8) or one of the $F, G$ is on the curve (focuses are supposed to be different).

**Proof.** When substituting (21.8) into (21.4), we get:

$$S^2 \left( \frac{\alpha^4 \cdot \beta^4 \cdot |EF|^4 \cdot |GE|^4 \left( u^2 \cdot |FG|^2 - u^2 \cdot |GE|^2 \right) \left( v^2 \cdot |EF|^2 - w^2 \cdot |FG|^2 \right) \left( u^2 \cdot |EF| - w^2 \cdot |FG| \right)^2}{(\sqrt{\beta} \cdot \sqrt{\gamma} \cdot |GE| \cdot |EF| \cdot (v^2 + \sqrt{\alpha} \cdot |EF| \cdot |FG| \cdot v^2 + \sqrt{\beta} \cdot |FG| \cdot |GE| \cdot w^2)} \right) $$

$$\square$$

21.8 The tripolar circular cubics

**Definition 21.8.1.** In this section $\pm u \pm v \pm w = 0$ is assumed. Parametrization $u : v : w \simeq 1 : s : -1 - s$ will be overused. And notation $K$ will be used when asserting properties that doesn’t apply to the whole family of tripolar curves $W$.

**Example 21.8.2.** Figure 21.5 has been drawn using the following data:

$$u = 1, v = 2, w = 3, E_0 = 12i, F_0 = 5, G_0 = 0$$

This results into

$$\alpha = \frac{-5 + 12i}{13}, \beta = \frac{5 - 12i}{13}, \gamma = \frac{-5 - 12i}{13}, \nu = \frac{12 - 35i}{37}$$

$$\delta = \frac{-1555 - 48348i}{48373}, \delta = \frac{153 - 158i}{793} \sqrt{13}$$

while the coefficients describing the connected piece containing $\nu$ are

$$\begin{array}{ccc}
E & F & G \\
F & G & H \\
E & G & H \\
E & F & H \\
\end{array} \begin{array}{ccc}
1 & 2 & -3 \\
33 & 28 & -61 \\
33 & -155 & 122 \\
28 & 155 & -183 \\
\end{array}$$
Exercice 21.8.3. Values  \( a = 1, \ b = 2, \ c = -3, \ \alpha = \frac{-12 + 5i}{13}, \ \beta = \frac{12 - 5i}{13}, \ \gamma = \frac{5 - 12i}{13} \) are given. Apply everything.

Proposition 21.8.4. Coefficients from focuses. When \( \pm u \pm v \pm w = 0, \) but \( uvw \neq 0, \) the resulting curve is a circular cubic. It admits four concyclic focuses as any other \( W. \) When parametrizing \( u : v : w \) by \( 1 : s : -1 - s, \) formula (21.6) can be reverted into:

\[
\begin{align*}
s^\parallel &= -\frac{\beta (\alpha^2 - \gamma^2)(\alpha \gamma - \beta \delta)}{\alpha (\beta^2 - \gamma^2)(\beta \gamma - \alpha \delta)}; \\
 s^\perp &= \frac{\beta (\alpha^2 - \gamma^2)(\alpha \gamma + \beta \delta)}{\alpha (\beta^2 - \gamma^2)(\beta \gamma + \alpha \delta)}
\end{align*}
\]

And then the coefficients can be re-obtained as:

\[
\begin{pmatrix} a \\ b \\ c \end{pmatrix} \approx \begin{pmatrix} 1 \\ s \\ -1 - s \end{pmatrix} \approx \begin{pmatrix} \alpha (\beta^2 - \gamma^2)(\alpha \delta - \beta \gamma) \\ \beta (\gamma^2 - \alpha^2)(\beta \delta - \alpha \gamma) \\ \gamma (\alpha^2 - \beta^2)(\delta \gamma - \beta \alpha) \end{pmatrix} = \begin{pmatrix} FG (\alpha \delta - \beta \gamma) \\ GE (\beta \delta - \gamma \alpha) \\ EF (\gamma \delta - \alpha \beta) \end{pmatrix}
\]

Proof. Obvious computation. Naming both values of \( s \) as "parallel" and "perpendicular" will be explained later. When \( s \) is known, the sign of \( \delta \) is supposed to be chosen so that \( s = s^\parallel (\delta) \), rather than the contrary.

Proposition 21.8.5. Asymptotes. Using \( q_1 = \alpha + \beta + \gamma + \delta, \) etc (the so called symmetric functions), the equation of the tripolar cubic becomes:

\[
\kappa (\alpha, \beta, \gamma, \delta) = 4 \left( T^2 + Z \mathbb{Z} \right) (q_1^3 \mathbb{Z} - \mathbb{Z}) (q_1^2 q_4 - q_3^2) + T \left( 8 q_1 q_3^2 q_4^2 - 4 q_2 q_3^2 q_4 + q_3^4 \right) \mathbb{Z}^2 + \left( q_1^4 - 4 q_1^2 q_2 + 8 q_1 q_3 \right) \mathbb{Z}
\]

The third point at infinity is \( U(\delta) = \alpha \beta \gamma \delta : 0 : 1. \) Thus the asymptotes \( \Delta^\parallel \) and \( \Delta^\perp \) are orthogonal. Their intersection is the gravity center of \( EFGH. \) When \( s \) is known, sign of \( \delta \) is supposed to be chosen so that \( \kappa = \kappa^\parallel (\delta), \) rather than the contrary.

Proof. When substituting, one can see the symmetry of the expression, allowing to use the \( q_j. \) Then straightforward computations. Equations found for both asymptotes are:

\[
\Delta^\parallel \approx \left[ 4, -q_1^2 + q_3^2/q_4, -4 q_4 \right]; \quad \Delta^\perp \approx \left[ 4, -q_1^2 + q_2 - q_3^2/q_4 + 4 q_4 \right]
\]
Corollary 21.8.6. The leading factor \((q_1^2q_4 - q_3^2)\) forbids a possible degeneracy into a conic and the line at infinity.

Proposition 21.8.7. Tangentials. Each of the \(K^\|\) and \(K^\perp\) curves are invariant under the \(\Omega\) group. Their 9 = 3 \times 3 common points are \(\Omega_x, \Omega_y, O\), the \(D_e\) and their inverses \(D'_e\), etc. in the unit circle. They form an orbit under the \(\Omega\) group... when taking into account the indeterminacy at \(O\) for the inversion into \(\Gamma\). Moreover:

1. \(U^\|\) is the tangential of \(O, D_e, D_f, D_g\) wrt \(K^\|\).
2. The tangentials of \(U^\|\) and \(U^\perp\) wrt their own curve are:

\[
U^\|_t \approx \begin{pmatrix} 4q_2q_3q_4 - 8q_1q_3q_1^2 - q_3^4 + 16q_1^4 \\ 4q_4(q_1^2q_4 - 4q_2q + q_3^2) \\ q_4(q_2^2q_2 - 8q_1q_3 - q_1^4 + 16q_4) \end{pmatrix} ; \quad U^\perp_t \approx \begin{pmatrix} -q_3(q_1q_2^3 - 4q_2q_3q_4 + q_3^3) \\ 4q_4(q_1^2q_4 - q_3^2) \\ q_1q_4(q_3^3 - 4q_2q_3 + 8q_3) \end{pmatrix}
\]

Point \(U_t\) is also the tangential of \(D'_e, D'_f, D'_g\). And it happens that \(U_t(-\delta)\) is the singular focus of \(K(\delta)\).

3. Apart from \(\Omega^\perp\), the 8 intersections of both \(K\) and the unit circle are obtained from \(-\sigma_3 + \sigma_1\) by changing the signs of \(\alpha, \beta, \gamma, \tau\) (odd changes move to \(K^\perp\), even ones remain on \(K^\|\)). Each group of 4 forms an orbit under \(\Omega\). Point \(O\) is the common tangential of all these points.

4. Both curves are orthogonal at each of their intersections.

Proof. Due to the choice \(z_A = \alpha^2\), everything factors and computations are easy. The six \(D_j, D'_j\) depend only on the squares \(\alpha^2\), etc: that is the reason why they can belong to both curves. 

Exercise 21.8.8. Consider the straight lines which are the bisectors of the angles \((\vec{OE}, \vec{OH})\) and \((\vec{OF}, \vec{OG})\). And then consider the gudule which bisects these two lines. How to help some constructions with this object?

21.8.1 When the fourth focus is moving

Exercise 21.8.9. The locus of \(U_t, F_s\) is a circular quintic, and the directions of sidelines are the other points at infinity. The corresponding asymptotes are going through \((2A + B + C)/4\), etc while it exists a singular focus at \(X(143)\).
Exercise 21.8.10. The envelope of $\Delta$ is the deltoid:

$$M(\tau) = \left(\frac{1}{4} s_1^2 - \frac{1}{2} s_2\right) - \frac{1}{4} \tau^2 - \frac{s_3}{2\tau}$$

whose center is $X(140) = (A + B + C + O)/4$, its internal and external radiiuses being $1/4$ and $3/4$. Cups are given by $\tau^3 = s_3$, i.e. the Morley’s directions. Moreover $\Delta \parallel \cap \Delta \perp -$ describes the inner circle. Hint: use $d\tau = ik\tau$ where $k$ is evanescent.

Exercise 21.8.11. The reflection of $\mathcal{K}$ wrt a circle centered at a focus leads to Cartesian ovals whose focuses are the images of the remaining three original focuses (see Figure 21.7). Choosing $\rho = 1$ is a great choice, which leads to $O \mapsto O$.

Figure 21.6: Angels
Figure 21.7: Cartesian ovals by inversion
Chapter 22

Special Triangles

Central triangles have been defined in Section 2.2.

22.1 Changing coordinates, functions and equations

**Proposition 22.1.1.** Any triangle $\mathcal{T}$ can be used as a barycentric basis instead of triangle $ABC$. When columns of triangle $\mathcal{T}$ are synchronized, the old barycentrics $x : y : z$ (relative to $ABC$) can be obtained from the new ones $\xi : \eta : \zeta$ (relative to $\mathcal{T}$) by:

$$
\begin{pmatrix}
  x \\
  y \\
  z
\end{pmatrix} = \mathcal{T} \cdot 
\begin{pmatrix}
  \xi \\
  \eta \\
  \zeta
\end{pmatrix}
$$

while the converse transformation can be done using the adjoint matrix.

**Proof.** A column of synchronized barycentrics acts on the matrix of rows containing the projective coordinates of the vertices of the reference triangle by the usual matrix multiplication. \( \square \)

**Remark 22.1.2.** By definition of synchronized barycentrics, $\mathcal{L}_\infty \cdot \mathcal{T} \simeq \mathcal{L}_\infty$ and the line at infinity is (globally) invariant.

**Proposition 22.1.3.** Let $\alpha, \beta, \gamma$ the side lengths of triangle $\mathcal{T}$ (computed using Theorem 7.2.4). Consider a central punctual transformation $\Phi$ that can be written as:

$$p : q : r \mapsto u : v : w = \phi(a, b, c, p, q, r)$$

with all the required properties of symmetry and homogeneity. Consider now the corresponding punctual transformation $\Phi'$ with respect to triangle $\mathcal{T}$ (written in its normalized form) and define $\phi_\mathcal{T}$ as the action of $\Phi'$ on the old barycentrics (the ones related to $ABC$). Then:

$$\phi_\mathcal{T} \left( a, b, c, \begin{pmatrix} p \\ q \\ r \end{pmatrix} \right) = \mathcal{T} \cdot \phi \left( \alpha, \beta, \gamma, \mathcal{T}^{-1} : \begin{pmatrix} p \\ q \\ r \end{pmatrix} \right)$$

**Example 22.1.4.** Applied to the isogonal transform and some usual triangle, this leads to formulas given in Figure 22.1. The term "complementary conjugate" is a synonym for "medial isogonal conjugate", as is "anticomplementary conjugate" for "anticomplementary isogonal conjugate". Also, "excentral isogonal conjugate" is "X(188)-aleph conjugate" and "orthic isogonal conjugate" is "X(4)-Ceva conjugate".

**Proposition 22.1.5.** Let $\alpha, \beta, \gamma$ the side lengths of triangle $\mathcal{T}$. Consider a conic $\Phi$ whose matrix can be written as $M(a, b, c)$ with the required properties of symmetry and homogeneity. Consider now the corresponding conic $\Phi'$ with respect to triangle $\mathcal{T}$ (written in its normalized form) and define $M_\mathcal{T}$ as the matrix defining $\Phi'$ wrt the old barycentrics. Then:

$$M_\mathcal{T}(a, b, c) = \mathcal{T}^{-1} \cdot M(a, b, c) \cdot \mathcal{T}^{-1}$$

**Example 22.1.6.** Applied to the circumcircle and some usual triangles, this leads to formulas given in Figure 22.1.
22.2 Residual triangles

Definition 22.2.1. The residual triangles of a triangle \(A'B'C'\) inscribed in a bigger one \(ABC\) are the triangles \(AB'C', A'BC', A'B'C'.\) Mind the fact that the residuals are oriented counterclockwise wrt triangles \(ABC\) and \(A'B'C'.\)

Proposition 22.2.2. Suppose that \(A'B'C'\) is inside the convex hull \(ABC\) and note \(A, A_0, A_a, A_b, A_c\) the absolute values of the areas of the five triangles. Then \(A = A_a + A_b + A_c + A_0.\) Moreover, \(A_0 \geq \min (A_a, A_b, A_c).\) More precisely (Bottema et al., 1969), \(A_0 \geq \min (\sqrt{A_a A_b}, \sqrt{A_b A_c}, \sqrt{A_a A_c}).\) And equality occurs only when \(A'B'C'\) are the mid-points.

Proof. Write \(A' \simeq 0 : x : x', B' \simeq y' : 0 : y, C' \simeq z' : 0 : z\) where \(x' = 1 - x,\) etc. Then

\[ A_0 = xyz + x'y'z' \geq 2\sqrt{xyzy'z'} = 2\sqrt{A_a A_b A_c} \]

Suppose \(A_a \geq A_b \geq A_c.\) If \(A_a \leq 1/4,\) then \(A_0 \geq 1/4 \geq \sqrt{A_a A_b A_c}.\) If \(A_a \geq 1/4\) then \(2\sqrt{A_a} \geq 1.\)

Equality occurs only if \(xyz = x'y'z'\) and \(A_a = 1/4.\)

Definition 22.2.3. The residual cevian triangles associated with point \(P\) are the triangles \(AP_a P_c, P_b BP_c, P_a P_b P_c\) where \(P_a P_b P_c\) is the cevian triangle of \(P\) (mind the order...).

Proposition 22.2.4. The \(A\)-residual of the orthic (i.e. wrt \(X(4)\)) and the intouch (i.e. wrt \(X(7)\)) triangles have the following sidelengths:

\[
[a, \beta, \gamma]_{orthic} = \frac{S_a}{bc} [a, b, c]
\]

\[
[a^2, \beta^2, \gamma^2]_{intouch} = \frac{(b+c-a)^2}{4bc} [(a+b-c)(a-b+c), bc, bc]
\]

The incenters of the orthic residuals are the orthocenters of the intouch residuals.

22.3 Incentral triangle

Definition. Cevian triangle of the incenter \(X(1)\)

Pythagoras (strong values)

\[
\alpha^2 = \frac{abc}{(a+b)(a+c)} \left( a^3 + a^2b + a^2c - ab^2 - ac^2 + 3abc - b^3 + b^2c + bc^2 - c^3 \right)
\]

Barycentrics (normalized)

\[
\mathbf{c}_1 = \begin{pmatrix} 0 & a & a \\ b & a + c & a + b \\ c & b + c & 0 \\ b + c & a + b & 0 \end{pmatrix}
\]

\((f, g)\) \((0; b)\)

22.4 Excentral triangle

Definition. Triangle of the excenters. And thus, anticevian triangle of the incenter \(X(1)\)

Pythagoras (strong values)

\[
[a^2, \beta^2, \gamma^2] = \frac{2R}{\rho} [a(b+c-a), b(c+a-b), c(a+b-c)]
\]

thus similar with the intouch triangle (center \(X(57)\), ratio \(2R/\rho\)
barycentrics (normalized)

\[
A_1 = \begin{pmatrix}
\frac{-a}{b+c-a} & \frac{a}{c+a-b} & \frac{a}{a+b-c} \\
\frac{b+c-a}{b} & \frac{c+a-b}{c} & \frac{a+b-c}{a} \\
\frac{b+c-a}{c} & \frac{c+a-b}{b} & \frac{a+b-c}{c}
\end{pmatrix}
\]

\((f, g) \quad (-a; b)\)

### 22.5 Medial triangle

definition cevian triangle of the centroid \(X(2)\)

side_length (strong values)

\([\alpha, \beta, \gamma] = \frac{1}{2} [a, b, c]\)

barycentrics (normalized)

\[
C_2 = \frac{1}{2} \begin{pmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{pmatrix}
\]

### 22.6 Antimedial triangle

definition anticevian triangle of the centroid \(X(2)\)

side_length (strong values)

\([\alpha, \beta, \gamma] = 2 [a, b, c]\)

barycentrics (normalized)

\[
A_2 = \begin{pmatrix} -1 & 1 & 1 \\ 1 & -1 & 1 \\ 1 & 1 & -1 \end{pmatrix}
\]

### 22.7 Orthic triangle

definition cevian triangle of the orthocenter \(X(4)\)

side_length (strong values)

\([\alpha, \beta, \gamma] = \frac{1}{abc} [a^2S_a, b^2S_b, c^2S_c]\)

circumcircle center=\(X(5)\), \(R_{orthic} = \frac{1}{2} R_{ABC}\)

incircle center=\(X(4)\), \(r_{orthic} = \frac{S_aS_bS_c}{2abcS}\)

barycentrics (synchronized)

\[
C_4 = \begin{pmatrix} 0 & S_c/b^2 & S_b/c^2 \\ S_c/a^2 & 0 & S_a/c^2 \\ S_b/a^2 & S_a/b^2 & 0 \end{pmatrix}
\]

angles \(\pi - 2A, \pi - 2B, \pi - 2C\)
22.8 Tangential triangle

**definition**  The sidelines are the tangents to the $ABC$-circumcircle at the vertices.

**key_property**  Anticevian triangle of $X(6)$.

**side_length (strong values)**

\[
[\alpha, \beta, \gamma] = \frac{abc}{2S_aS_bS_c} \left[ a^2S_a, b^2S_b, c^2S_c \right]
\]

Therefore, this triangle is similar to the orthic triangle.

**barycentrics (synchronized)**

\[
[A_6] = \begin{pmatrix}
\frac{a^2}{S_a} & \frac{a^2}{S_b} & \frac{a^2}{S_c} \\
\frac{b^2}{S_a} & \frac{b^2}{S_b} & \frac{b^2}{S_c} \\
\frac{c^2}{S_a} & \frac{c^2}{S_b} & \frac{c^2}{S_c}
\end{pmatrix}
\]

22.9 Brocard triangle (first)

Remember that Brocard points are defined by $\omega^+ = a^2b^2 : b^2c^2 : c^2a^2$ and $\omega^- = c^2a^2 : a^2b^2 : b^2c^2$ (see Proposition 7.7.1).

**definition**  $A_1 \doteq B\omega^- \cap C\omega^+ \simeq a^2 : c^2 : b^2$ etc. This triangle is inscribed in the Brocard 3-6 circle, see , diameter $X(3)X(6)$.

**side_length (strong values)**

\[
[\alpha, \beta, \gamma] = \frac{W_2}{2S_w} [a, b, c]
\]

and therefore $Broc_1$ is (anti-)homothetic to $ABC$. Perspector $X(76)$, the third Brocard point. Moreover, anti-similar to $ABC$ with center $X(2)$.

**barycentrics (synchronized)**

\[
[Broc_1] = \begin{pmatrix}
a^2 & c^2 & b^2 \\
c^2 & b^2 & a^2 \\
b^2 & a^2 & c^2
\end{pmatrix}
\]

22.10 Brocard triangle (second)

**definition**  $U_a \simeq b^2 + c^2 - a^2 : b^2 : c^2$ is the projection of $X(3)$ on the $A$-symmedian, etc. (and therefore belongs to the 3-6 Brocard circle).

**key_property**  Triangle $ABC$ and triangle $[Broc_2]$ share the same isodynamic centers $X(15)$, $X(16)$.

**Exercise 22.10.1.** The center $E_a$ of circle $A, X(15), X(16)$ is the inverse in circumcircle of $U_a$, etc.

22.11 Brocard triangle (third)

**definition**  $A_3 \doteq C\omega^- \cap A\omega^+ \simeq b^2c^2 : b^4 : c^4 = isog(A_1)$.

perspector with $ABC$: $X(32)$
22.12 Intouch triangle (contact triangle)

definition Cevian triangle of the Gergonne point X(7).

key_property contacts of the incircle and the sidelines.

pythagoras (strong equality)

\[ [\alpha^2, \beta^2, \gamma^2] = \frac{\rho}{2R} [a(b + c - a), b(c + a - b), c(a + b - c)] \]

thus similar with the excentral triangle (ratio \(\rho/2R\))

barycentrics (normalized)

\[ C_7 = \begin{pmatrix} 0 & a + b - c & c + a - b \\ a + b - c & b & c \\ c + a - b & b - a + c & c \\ a & b - a + c & b \end{pmatrix} \]

22.13 Extouch triangle

definition cevian triangle of the Nagel point X(8).

side_length (strong values)

\[ [\alpha, \beta, \gamma] = \]

barycentrics (normalized)

\[ C_8 = \begin{pmatrix} 0 & b - a + c & b - a + c \\ b - a + c & b & c + a - b \\ a + b - c & a + b - c & c \\ a + b - c & a + b - c & b \end{pmatrix} \]

22.14 Hexyl triangle

definition symmetric of the excentral triangle \(J_aJ_bJ_c\) wrt the circumcircle. Thus \(H_aJ_bJ_cJ_aH_bJ_c\) is an hexagon whose opposite sides are parallel.

key_property Vertex \(H_a\) is the point in which the perpendicular to \(AB\) through the excenter \(J_b\) meets the perpendicular to \(AC\) through the excenter \(J_c\).

pythagoras (strong values)

\[ [\alpha^2, \beta^2, \gamma^2] = \frac{2R}{\rho} [a(b + c - a), b(c + a - b), c(a + b - c)] \]

thus similar with the intouch triangle (center \(I\), ratio \(2R/\rho\)).

barycentrics (normalized) \(H_a \simeq a(aS_a + bS_b + cS_c + abc) : b(aS_a + bS_b - cS_c - abc) : c(aS_a - bS_b + cS_c - abc)\)

circumcircle centered at X(1), radius \(2R\).

22.15 Fuhrmann triangle

**Definition 22.15.1. Fuhrmann triangle** is \(A''B''C''\) where \(A'B'C'\) is the circumcevian triangle of \(X_1\) and \(A''\) is the reflection of \(A'\) in sideline \(BC\) (and cyclically for \(B''\) and \(C''\)).
Proposition 22.15.2. Side length of Fuhrmann triangle are:

\[ W_4 \times \left( \frac{a}{(a+b-c)(a-b+c)} \right) \left( \frac{b}{(b+c-a)(b-c+a)} \right) \left( \frac{c}{(c+a-b)(c-a+b)} \right) \]

where \( W_4 = \sqrt{a^3+b^3+c^3-(a^2b+a^2c+ab^2+ac^2+b^2c+bc^2)+3abc} \)

Quantity \( W_4 \) is the Fuhrmann square root (13.14). Circumcircle of the Fuhrmann triangle is the Fuhrmann circle of \( ABC \) (whose diameter is \([X_4, X_8]\)).

Remark 22.15.3. As noticed in (Dekov, 2007), \( OIFuNa \) and \( OIFu \) are parallelograms, whose respective centroids are the Spieker center and nine-point center, respectively, where \( IFu = 2NI \) or \( IFu = R - 2r \).

### 22.16 Star triangle

**Definition 22.16.1. Star triangle.** Consider the midpoints \( A'B'C' \) of the sidelines of triangle \( ABC \). Draw from each midpoint the perpendicular line to the corresponding bisector. These three lines determine a triangle \( A'B'C' \). This is our star.

**Proposition 22.16.2.** The synchronized barycentrics and the sidelengths of the star triangle are:

\[
T_* \simeq \begin{pmatrix} b+c & c-b & b-c \\ c-a & c+a & a-c \\ b-a & a-b & b+a \end{pmatrix} \cdot \begin{pmatrix} b+c-a & 0 & 0 \\ 0 & c+a-b & 0 \\ 0 & 0 & a+b-c \end{pmatrix}^{-1}
\]

\[ \left[ a^2, \beta^2, \gamma^2 \right] = \left( \frac{R}{2\rho} \right)^2 \times [a(b+c-a), b(c+a-b), c(a+b-c)] \]

Similar to the intouch triangle \( (k^2 = R^3/2\rho^3) \) and to the excentral triangle \( (k^2 = R/8\rho) \). We have the following central correspondences:

<table>
<thead>
<tr>
<th>( T_* )</th>
<th>( T )</th>
<th>( T_* )</th>
<th>( T )</th>
<th>( T_* )</th>
<th>( T )</th>
<th>( T_* )</th>
<th>( T )</th>
</tr>
</thead>
<tbody>
<tr>
<td>2</td>
<td>3817</td>
<td>133</td>
<td>121</td>
<td>542</td>
<td>2801</td>
<td>2393</td>
<td>527</td>
</tr>
<tr>
<td>3</td>
<td>946</td>
<td>134</td>
<td>122</td>
<td>647</td>
<td>3835</td>
<td>2501</td>
<td>4885</td>
</tr>
<tr>
<td>4</td>
<td>10</td>
<td>135</td>
<td>123</td>
<td>690</td>
<td>3887</td>
<td>2574</td>
<td>3308</td>
</tr>
<tr>
<td>5</td>
<td>5</td>
<td>136</td>
<td>124</td>
<td>804</td>
<td>926</td>
<td>2575</td>
<td>3307</td>
</tr>
<tr>
<td>6</td>
<td>142</td>
<td>137</td>
<td>125</td>
<td>924</td>
<td>522</td>
<td>2679</td>
<td>1566</td>
</tr>
<tr>
<td>20</td>
<td>4301</td>
<td>138</td>
<td>126</td>
<td>974</td>
<td>1387</td>
<td>2777</td>
<td>2802</td>
</tr>
<tr>
<td>25</td>
<td>3452</td>
<td>139</td>
<td>127</td>
<td>1112</td>
<td>3035</td>
<td>2781</td>
<td>528</td>
</tr>
<tr>
<td>30</td>
<td>517</td>
<td>143</td>
<td>140</td>
<td>1154</td>
<td>30</td>
<td>2782</td>
<td>2808</td>
</tr>
<tr>
<td>39</td>
<td>2140</td>
<td>184</td>
<td>226</td>
<td>1205</td>
<td>3254</td>
<td>2790</td>
<td>2810</td>
</tr>
<tr>
<td>51</td>
<td>2</td>
<td>185</td>
<td>1</td>
<td>1495</td>
<td>908</td>
<td>2794</td>
<td>2809</td>
</tr>
<tr>
<td>52</td>
<td>3</td>
<td>235</td>
<td>1329</td>
<td>1503</td>
<td>518</td>
<td>2797</td>
<td>2821</td>
</tr>
<tr>
<td>53</td>
<td>141</td>
<td>389</td>
<td>1125</td>
<td>1510</td>
<td>523</td>
<td>2799</td>
<td>2820</td>
</tr>
<tr>
<td>65</td>
<td>178</td>
<td>403</td>
<td>3814</td>
<td>1531</td>
<td>1512</td>
<td>2848</td>
<td>2832</td>
</tr>
<tr>
<td>113</td>
<td>119</td>
<td>418</td>
<td>2051</td>
<td>1562</td>
<td>4904</td>
<td>3258</td>
<td>3259</td>
</tr>
<tr>
<td>114</td>
<td>118</td>
<td>427</td>
<td>2886</td>
<td>1568</td>
<td>1532</td>
<td>3564</td>
<td>971</td>
</tr>
<tr>
<td>115</td>
<td>116</td>
<td>428</td>
<td>3740</td>
<td>1596</td>
<td>3820</td>
<td>3566</td>
<td>3900</td>
</tr>
<tr>
<td>125</td>
<td>11</td>
<td>511</td>
<td>516</td>
<td>1637</td>
<td>4928</td>
<td>3574</td>
<td>442</td>
</tr>
<tr>
<td>128</td>
<td>113</td>
<td>512</td>
<td>514</td>
<td>1824</td>
<td>2090</td>
<td>3575</td>
<td>960</td>
</tr>
<tr>
<td>129</td>
<td>114</td>
<td>520</td>
<td>3667</td>
<td>1843</td>
<td>9</td>
<td>3917</td>
<td>1699</td>
</tr>
<tr>
<td>130</td>
<td>115</td>
<td>523</td>
<td>513</td>
<td>1986</td>
<td>214</td>
<td></td>
<td></td>
</tr>
<tr>
<td>131</td>
<td>117</td>
<td>525</td>
<td>3309</td>
<td>1990</td>
<td>3834</td>
<td></td>
<td></td>
</tr>
<tr>
<td>132</td>
<td>120</td>
<td>526</td>
<td>900</td>
<td>2052</td>
<td>3840</td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

For example, orthocenter \( X(4, T_*) \) is Spieker center \( X(10, T) \).
Proof. Straightforward computations. In fact, $A'A^*$ and $B^*C^*$ are orthogonal and the orthic triangle of $T_*$ is the medial triangle of $T$. 

\[ \square \]
\[ \phi_T \text{ when } \phi \text{ is the isogonal conjugacy} \]

<table>
<thead>
<tr>
<th>Type</th>
<th>Formula</th>
</tr>
</thead>
<tbody>
<tr>
<td>medial</td>
<td>( C_2 ) ( (v + w - u) ((u + v - w) b^2 + (u - v + w) c^2) )</td>
</tr>
<tr>
<td>antimedial</td>
<td>( A_2 ) [-a^2 ] ( \frac{v + w}{u + w} + \frac{u + v}{u + v} ) ( u (-S_a u + S_b v + S_c w) )</td>
</tr>
<tr>
<td>orthic</td>
<td>( C_4 ) (-a^2 ) ( \frac{b^2}{b} + \frac{c^2}{c} + \frac{c^2}{a} v ) ( c^2 )</td>
</tr>
<tr>
<td>tangential</td>
<td>( A_6 ) ( a \left( \frac{c^2 v + wb^2}{c^2} + \frac{uc^2 + wa^2}{uc^2} + \frac{b^2 + a^2 v}{b^2 + a^2 v} \right) )</td>
</tr>
<tr>
<td>excentral</td>
<td>( A_1 ) ( a \left( \frac{1}{b + c - a (bv + bw)} + \frac{1}{a - b + c (bu + aw)} + \frac{1}{b + a - c (bv + bw)} \right) )</td>
</tr>
</tbody>
</table>

Figure 22.1: Special Triangles

Figure 22.2: The star triangle
Chapter 23

Formal operations

Let us examine again some already defined operations, and consider in more details their formal properties.

23.1 Unary operators

1. DP1, see 3.13
   \[ DP_1(U) = u^2(v + w) : v^2(w + u) : w^2(u + v) \]

2. DP2, see 3.14
   \[ DP_2(U) = u(v - w)^2 : v(w - u)^2 : w(u - v)^2 \]

23.2 cevamul, cevadiv, crossmul, crossdiv

These operations were considered in Kimberling to formalize various properties related to cevian nests.

Remark 23.2.1. cevadiv has been defined in Section 3.11 by:

\[
\text{cevadiv}(P; U) \triangleq \text{persp}(\text{Cevian}(\text{dividend}, ABC), \text{Anticevian}(\text{divisor}, ABC)) \\
\approx u(-qru + rpv + pqw) : v(qru - rpv + pqw) : w(qru + rpv - pqw)
\]

Property \( X = \text{cevadiv}(P, U) \) is sometimes stated as "\( X \) is the \( P \)-cevaconjugate of \( U \)". This operation is clearly a type-keeping \((P, U)\)-map and an involutory \( U \)-map (when \( P \) is fixed).

Remark 23.2.2. The cevamul operation is the converse of the previous one, and is sometimes called cevapoint.

\[
\text{cevamul}(u: v: w, x: y: z) = (uz+wx)(uy+vx) : (vz+w) : (wy+vx) : (vz+wy)(uz+w)
\]

This operation is clearly commutative and type-keeping.

Remark 23.2.3. The crossdiv has been defined in Section 3.10 by:

\[
\text{crossdiv}(P; U) \triangleq \text{persp}(ABC, \text{Cevian}(\text{dividend}, \text{Cevian}(\text{divisor}, ABC))) \\
\approx \frac{u}{quw + ruv - pew} : \frac{v}{pew + ruv - quw} : \frac{w}{pew + quw - ruw}
\]

Property \( X = \text{crossdiv}(P, U) \) is sometimes stated as "\( X \) is the \( P \)-crossconjugate of \( U \)". This operation is clearly a type-keeping \((P, U)\)-map and an involutory \( U \)-map (when \( P \) is fixed).

Remark 23.2.4. The crossmul operation is the converse of the previous one, and is sometimes called crosspoint.

\[
\text{crossmul}(u: v: w, x: y: z) = (vz+wy)u : (uz+w) : (uy+v)w
\]

This operation is clearly commutative and type-keeping.
Proposition 23.2.5. 'div' formula. The isoconjugacy that exchanges $P, U$ also exchanges $P \leftrightarrow U$; crossdiv $(P, U) \leftrightarrow$ cevadiv $(U, P)$; crossdiv $(U, P) \leftrightarrow$ cevadiv $(P, U)$

Proof. We have crossdiv $(P, U) \ast$ cevadiv $(U, P) = P \ast U$.

Proposition 23.2.6. 'mul' formula. For any isoconjugacy, we have:

cевамул $(U, X) = \text{crossmul} \left( U^{\#}, X^{\#} \right)^{\#}_F$

crossmul $(U, X) = \text{cevamul} \left( U^{\#}, X^{\#} \right)^{\#}_F$

Proof. Direct examination is shorter, using the "div" formula just above is more stratospheric.

23.3 Formal operators and conics

Since they are all Cremona transforms of second degree, operators wrt a triangle can be transformed into operators wrt a well chosen conic.

Proposition 23.3.1. barymul. Let $\psi$ be the $ABC$-isoconjugacy that swaps points $P, U$ (not on the sidelines). Then $\psi$ swaps $C_{PU} \cong \text{conic} (A, B, C, P, U)$ and $L_{PU} \cong \text{line} (P, U)$. The pole $F^2$ of the conjugacy is the barymul of $P, U$ and therefore the "barysquare" of the four fixed points of $\psi$:

$F^2 \cong P \ast U$; $f^2 = kp \, u$, $g^2 = kq \, v$, $h^2 = kr \, w$, $k \neq 0$

Proposition 23.3.2. We have the various 'cross' formulas

<table>
<thead>
<tr>
<th>formula</th>
<th>name</th>
<th>usefulness</th>
<th>inverse</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\frac{1}{qw} - \frac{1}{rv}$</td>
<td>crossmul</td>
<td>intersections of the tripolars, perspector of $C_{PU}$</td>
<td>$X \in C_{PU}$</td>
</tr>
<tr>
<td>$\frac{1}{rv} + \frac{1}{qw}$</td>
<td></td>
<td></td>
<td>crossdiv</td>
</tr>
<tr>
<td>$\frac{1}{qw - rv}$</td>
<td>cevadiv</td>
<td></td>
<td>$X \in L_{PU}$</td>
</tr>
<tr>
<td>$\frac{1}{qw + rv}$</td>
<td></td>
<td></td>
<td>cevadiv</td>
</tr>
</tbody>
</table>

Thus the 'odd ones'\(^1\) aren't Cremona transforms.

23.4 crossdiff, crosssum, polarmul, polardiv

Definition 23.4.1. The $F$-crossdiff of two points $U = u : v : w$ and $X = x : y : z$ that aren't lying on a sideline of $ABC$ is defined by:

crossdiff $(U, X) = f^2 (wy - vz) : g^2 (uz - wx) : h^2 (vx - uy)$

Remark 23.4.2. In ETC, $F \cong \chi(1)$ is assumed. Defined as above, the operation $(F, U, X) \mapsto$ crossdiff $(U, X)$ is globally type-keeping and provides a point when the entries are points ($F$ is any of the four fixed point of the conjugacy $X \mapsto X^*$).

Proposition 23.4.3. The $F$-crosssum of $U, X$ that was defined at Definition 20.5.5 is constructible using crossmul, cevadiv and isoconjugacies. Therefore a better definition is the globally type-keeping function:

\[
\text{crosssum}_F (U, X) = f^2 (wy + vz) : g^2 (uz + wx) : h^2 (vx + uy) = (\text{cevamul} (U, X))^{\#}_F = \text{crossmul} \left( U^{\#}_F, X^{\#}_F \right)
\]

where $F = f : g : h$ is any of the fixed points of the conjugacy.

\(^1\)English joke. Can not be translated.
Remark 23.4.4. Defined that way, crosssum\(_F(U,X)\) is really different from the polar line of \(X\) wrt the circumconic \(CC(U) : uyz + vxz + wxy = 0\) since this line is the next coming polarmul.

**Proposition 23.4.5.** Given \(U\) and \(P \doteq \text{crosssum}_{U}(U,X)\) one can find \(X\) using
\[
X \simeq \text{cevadiv} \left( P^\#_F, U \right)
\]

**Proof.** Direct inspection (here \(P\) is generic, not the "square" of the fixed point \(F\)). \(\square\)

**Proposition 23.4.6.** The polarmul of two points \(U, X\) is a line \(\Delta\). When \(U = u : v : w\) and \(X = x : y : z\) are not on the sidelines, then line \(\Delta\) is defined as the conipolar of the point \(X\) wrt the circumconic \(CC(U)\). We have polarmul \((U,U) = \text{tripolar}(U)\) and
\[
\text{polarmul}(U,X) = \text{crossmul}(\text{tripolar}(U), \text{tripolar}(X)) = \text{tripolar} \left( \text{cevamul} \left( U, X \right) \right)
\]

Operation polarmul is commutative and type-crossing (i.e output is a line when entries are points).

**Proposition 23.4.7.** The polarmul of two lines \(D, \Delta\) is the point \(P\) obtained by applying "the same rules" as above, i.e.
\[
\text{polarmul} \left( [u,v,w], [x,y,z] \right) \doteq wy + vz : uz + wx : vx + uy
\]

Then polarmul \((\Delta, \Delta)\) is tripolar \((\Delta)\) and polarmul \((D, \Delta)\) is the conipolar of line \(\Delta\) wrt the inconic whose perspector is tripolar \((D)\).

**Proof.** Direct inspection. Remember that the dual of an inconic "goes through" the sidelines. \(\square\)

**Definition 23.4.8.** The polardiv of a line \(\Delta \simeq [\rho; \sigma; \tau]\) and a point \(U \simeq u : v : w\) is the reverse operation of the previous one. One has the formula:
\[
\text{polardiv}(\Delta, U) = \text{cevadiv}(\text{tripolar}(\Delta), U)
\]
\[
= (\sigma v + \tau w - \rho u) u : (\tau w + \rho u - \sigma v) v : (\rho u + \sigma v - \tau w) w
\]

that correctly defines a point when \(P\) is a line and \(U\) is a point.

### 23.5 Complementary and anticomplementary conjugates

![Figure 23.1: The complementary and anticomplementary conjugates](image)

**Figure 23.1:** The complementary and anticomplementary conjugates

**Definition 23.5.1.** comcon, anticomcon. For points \(P = f^2 : g^2 : rh^2\) and \(U = u : v : w\), neither lying on a sideline of \(ABC\), the \(P\)-complementary and \(P\)-anticomplementary conjugates of \(U\) are defined as in Figure 23.1, where \(k\) is the \(P\)-isoconjugacy. Most of the time, \(P = X_1\) and the conjugacy reduces to isogonal conjugacy.

— pdlx : Translation of the Kimberling’s Glossary into barycentrics —
23.6 Hirstpoint aka Hirst inverse

**Definition 23.6.1. Hirstpoint.** Suppose \( P = p : q : r \) and \( U = u : v : w \) are distinct points, neither lying on a sideline of \( ABC \). The hirstpoint \( X \) is the point of intersection of the line \( PU \) and the polar of \( U \) with respect to the circumconic \( CC(P) \) conic:

\[
p y z + q z x + r x y = 0.
\]

**Proposition 23.6.2.** We have the following properties:

(i) \( H \) is a type-keeping operation as a "ramified product" of type-crossing transforms and:

\[
\text{hirstpoint} (P,U) = (P \wedge U) \wedge \text{polarmul} (P,U) = u^2 q r - p^2 v w : p v^2 r - u q^2 w = p q w^2 - u v r^2
\] (23.1)

(ii) \( H \) is commutative from the duality properties of polarization.

(iii) \( H (P,U) = 0 : 0 : 0 \) occurs only when \( U = P \).

(iv) \( H (P,U) = \) if and only if \( U \) lies on the polar line of \( P \).

(v) \( H (P, H (P, U)) \) is either \( 0 : 0 : 0 \) or \( U \). Indeterminate form is obtained (a) on the polar line of \( P \) and (b) on a conic containing \( P \) and having \( P \) has perspector ... i.e. a conic whose only real point is \( P \).

*Proof.* Direct inspection for all properties. To be precise, these properties are valid only on the real part of the world. For example, (i) gives \( U = p : j q : j^2 r \) where \( j^2 = 1 \), i.e. \( U = P \) and two other "imagined" solutions.

All these properties show that "Hirst inverse" is a poorly chosen term, since we aren’t dividing, but multiplying. Concerning the designation "Hirst inverse," see the contribution Gunter Weiss: http://mathforum.org/kb/message.jspa?messageID=1178474.

23.7 Line conjugate

Suppose \( P = p : q : r \) and \( U = u : v : w \) are distinct points, neither equal to \( A, B, \) or \( C \). The \( P \)-line conjugate of \( U \) is the point whose trilinears are given by:

\[
p(v^2 + w^2) - u(qv + rw) : q(w^2 + u^2) - v(rw + pu) : r(u^2 + v^2) - w(pu + qv)
\]

This is the point of intersection of line \( PU \) and the tripolar of the isogonal conjugate of \( U \).

Using the same formula with barycentrics, another point is obtained, that is the intersection of \( PU \) and the dual of \( U \). So what?

23.8 Collings transform

**Lemma 23.8.1.** Let \( M_i, 1 \leq i \leq 5 \), be five (different) points, not four of them on the same line, such that midpoint \((M_1, M_2) = \text{midpoint} (M_3, M_4) = P \). They determine uniquely a conic whose center is \( P \) and contains reflection \((P, M_5)\).

*Proof.* Take \( P \) as origin of the euclidian coordinate system and consider determinant \( \gamma \) whose lines are \([x_1^2, x_1 y_1, y_1^2, x_1, y_1, 1]\), the last line \((i = 6)\) referring to the generic point of the plane. Since \( x_2 = -x_1, \ldots \gamma \) can be factored into \((x_1 y_3 - x_3 y_1)\) times an expression without terms of first degree in \( x_6, y_6 \).

**Lemma 23.8.2.** Let \( M_i, 1 \leq i \leq 4 \), be four different points, not on the same line. The locus:

\[
\Theta = \{ \text{center} (\gamma) \mid \gamma \text{ is a conic and } M_1, M_2, M_3, M_4 \in \gamma \}
\]

is a conic. It contains the six midpoint \((M_i, M_j)\) and its center is \( K = \sum M_i/4 \).

*Proof.* \( \Theta \) is a conic since degree is two. When \( P = \text{midpoint} (M_1, M_2) \), the preceding lemma can be applied to \( M_1, M_2, M_3, \text{reflection} (P, M_3), M_4 \), defining a conic whose center is \( P \), so that \( P \in \Theta \).

Now, the lemma can be applied to \( \Theta \) itself, since \( K = \text{midpoint} (\text{midpoint} (M_1, M_2), \text{midpoint} (M_3, M_4)) \) –and cyclically.
**Proposition 23.8.3.** Let $P$ be a point not on a sideline of $ABC$, and $A', B', C'$ the reflections of $A, B, C$ in $P$.

(i) It exists a conic $\gamma$ through the six points $A, B, C, A', B', C'$, its center is $P$ and its perspector is $U = P \ast \text{anticompl}_{b}(P)$, so that $\gamma = CC'(U)$. This conic intersects the circumcircle at point $Q = \text{isotom}(X_{6} \setminus U)$, i.e. the tripole of line $UX_{6}$. Moreover, the circumcircles of triangles $AB'C'$, $A'BC'$ and $A'B'C$ are also passing through point $Q$.

(ii) Conversely, when $Q$ is given on the circumcircle, the locus of $P$ is $\text{conicev}(X_{2}, Q)$. This conic goes through the three $AB \cap CQ$ points, the three midpoint $(A, Q)$ and through four fixed points : the vertices of medial triangle and through the circumcenter $X_{3}$. Moreover, this conic is a rectangular hyperbola, its center is $K = (A + B + C + Q) / 4$ and belongs to the nine points circle of the medial triangle.

(iii) The anticomplement of this RH is the rectangular $ABC$-circumhyperbola whose center is the complement of $Q$.

**Proof.** For (i), only $Q$ belongs to circumcircle of $AB'C'$ has to be proved. Barycentric computation. For (ii), point $AB \cap CQ$ lead to the degenerate conic $AB \cup CQ$ (and cyclically) while the six midpoints come from the lemma. When $P = X_{3}$, conic $\gamma$ is the circumcircle... and passes through $A, B, C, Q$ and $X_{3} \in \Theta$. But $X_{3}$ of $ABC$ is $X_{4}$ of the medial triangle, and $\Theta$ contains an orthic configuration, characteristic property of a rectangular hyperbola.

For the sake of exhaustivity, if barycentrics of $P$ are $p : q : r$ then barycentrics of $Q$ are

$$Q = \frac{1}{r(p + q - r) b^2 - q(p - q + r) c^2}$$

(and cyclically in $a, b, c$ and $p, q, r$ too). The transformation $P \mapsto Q$ was described by Collings (1974) and was further discussed by (Grinberg, 2003b).

**Example 23.8.4.** Examples are as follows:

— pldx : Translation of the Kimberling’s Glossary into barycentrics —
23.8. Collings transform

Figure 23.3: Collings locus is a ten points rectangular hyperbola

<table>
<thead>
<tr>
<th>Q</th>
<th>points on the conic</th>
<th>wrt medial triangle</th>
</tr>
</thead>
<tbody>
<tr>
<td>$X_{74}$</td>
<td>125</td>
<td></td>
</tr>
<tr>
<td>$X_{98}$</td>
<td>115, 868</td>
<td></td>
</tr>
<tr>
<td>$X_{99}$</td>
<td>2, 39, 114, 618, 619, 629, 630, 641, 642, 1125</td>
<td>Kiepert hyperbola</td>
</tr>
<tr>
<td>$X_{100}$</td>
<td>1, 9, 10, 119, 142, 214, 442, 1145</td>
<td>Feuerbach hyperbola</td>
</tr>
<tr>
<td>$X_{107}$</td>
<td>4, 133, 800, 1249</td>
<td></td>
</tr>
<tr>
<td>$X_{110}$</td>
<td>5, 6, 113, 141, 206, 942, 960, 1147, 1209</td>
<td>Jerabek hyperbola</td>
</tr>
<tr>
<td>$X_{476}$</td>
<td>30</td>
<td></td>
</tr>
</tbody>
</table>
Chapter 24

Sondat theorems

24.1 Perspective and directly similar

Remark 24.1.1. When triangle $ABC$ is translated into $A'B'C'$, these triangles are in perspective and the perspector is the direction of the translation. The converse situation is not clear, so that translations will be excluded from what follows, implying the existence of a center. When using composition, we have to examine if nevertheless translations are reappearing.

Lemma 24.1.2. When $\sigma$ is a direct similarity (but not a translation) with center $S = z : 0 : \zeta$ and ratio $k\kappa$ ($k$ is real while $\kappa$ is unimodular, and $k\kappa \neq \pm 1$) then its matrix in the Morley space is:

$$
\sigma = \begin{pmatrix}
    k\kappa & z & (1 - k\kappa) & 0 \\
    0 & 1 & 0 & 0 \\
    0 & \zeta & (1 - \frac{k}{\kappa}) & \frac{k}{\kappa}
\end{pmatrix}
$$

Proof. One can check that umbilics are fixed points of this transform. The characteristic polynomial of matrix $\sigma$ is $\chi(\mu) = (\mu - 1)(\mu - k\kappa)(\mu - k/\kappa)$. Excluding $k\kappa = \pm 1$ ensures the existence of a center.

Proposition 24.1.3. When a direct central similarity $\sigma(S, k\kappa)$ and a perspector $P \neq S$ are given, the locus $C$ of points $M$ such that $P, M, M' = \sigma(M)$ are collinear is the circle through $S, P, \sigma^{-1}P$.

Proof. The locus contains certainly the five points such that $M = P$ or $M' = P$ or $M = M'$ i.e. $S$ and both umbilics. The general case results from the fact that $\det [P, M, M']$ is a second degree polynomial in $Z, \overline{Z}, T$ so that $C$ is a conic.

Proposition 24.1.4. Suppose that triangles $T_1$ and $T_2$ are together in perspective (center $P$) and strictly similar (center $S$, ratio $k\kappa$, $\kappa \neq \pm 1$). Then $P$ and $S$ are the two intersections of their circumcircles ($P = S$ cannot occur).

Proof. Use Lubin coordinates relative to $T_1 = ABC$, and note $S \simeq z : t : \zeta$. Then $A'B'C'$ is obtained as:

$$
A'B'C' \simeq \sigma \begin{pmatrix} A \ B \ C \end{pmatrix}
$$

The determinant of lines $AA', BB', CC'$ factors as:

$$
\frac{\sigma_4}{\sigma_3} \frac{k}{\kappa} (1 - \kappa^2) \left(1 - \frac{k}{\kappa}\right) (1 - k\kappa) \times \left(\frac{z\zeta - t^2}{t^2}\right)
$$

proving $S \in \Gamma$. Then perspector is also on this circle (from preceding proposition). We even have the more precise result:

$$
P = \begin{pmatrix}
    \omega & 0 & 0 \\
    0 & 1 & 0 \\
    0 & 0 & \omega^{-1}
\end{pmatrix} \cdot S \quad \text{where} \quad \omega = \frac{1 - k\kappa}{1 - \frac{k}{\kappa}}
$$

349
Proposition 24.1.5. Consider a fixed triangle $ABC$, and describe the plane using the Lubin frame. Consider points $P = \Phi : 1 : 1/\Theta$ and $S = \Theta : 1 : 1/\Theta$ on the unit circle ($P$ as Phi, and $S$ as Sigma. But Sigma is sum, use the next Greek letter). Assume $P \neq S$. Then all triangles $A'B'C'$ that are $P$-perspective and $S$-similar to triangle $ABC$ are obtained as follows. Let point $O'$ on the perpendicular bisector of $(P, S)$ be defined by property $(SO, SO') = \kappa$ where $\kappa^2$ is a given turn. Draw circle $\gamma$ centered at $O'$ and going through $P$ and $S$. Then $A' = \gamma \cap SA$, etc.

Proof. Point $O'$ can be written as $P + S + x (\Theta \Phi : 0 : 1)$. This point is the $\sigma(S, \kappa \kappa)$ image of $O$ if and only if:

$$k = \kappa \frac{\Phi - \Theta}{\Phi - \kappa^2 \Theta} \sigma \simeq \left( \begin{array}{ccc} \kappa^2 (\Phi - \Theta) & (1 - \kappa^2) \Theta \Phi & 0 \\ 0 & \Phi - \kappa^2 \Theta & 0 \\ 0 & 1 - \kappa^2 & \Phi - \Theta \end{array} \right)$$

As it should be, $\sigma$ depends only of $\kappa^2$, while the sign of $k$ depends on the choice of $\kappa$ among the square roots of $\kappa^2$. The matrix of circle $\gamma$ is:

$$\gamma = \left( \begin{array}{c} \sigma \end{array} \right) \left( \begin{array}{c} 0 \\ 1 - \kappa^2 \\ \kappa^2 \Theta - \Phi \end{array} \right) \kappa^2 \Theta - \Phi$$

$$= \left( \begin{array}{ccc} 0 & 1 - \kappa^2 & \kappa^2 \Theta - \Phi \\ 1 - \kappa^2 & 2 (k^2 \Phi - \Theta) & (1 - \kappa^2) \Theta \Phi \\ \kappa^2 \Theta - \Phi & (1 - \kappa^2) \Theta \Phi & 0 \end{array} \right)$$

And it can be checked that $P, A, A' = \sigma(A)$ are collinear. \hfill \square

Proposition 24.1.6. With same hypotheses, the perspectrix of triangles $ABC$ and $A'B'C'$ is the line:

$$XYZ = [(\Phi - \kappa^2 \Theta) \Phi, -\Phi \Theta (\Phi - \kappa^2 \sigma_1) + \kappa^2 (\kappa^2 \sigma_3 - \sigma_2 \Phi), \kappa^2 (\Phi - \kappa^2 \Theta) \sigma_3]$$

Points $S, C, C', X, Y$ are concyclic (and circularly). When $\kappa^2$ reaches $\Phi/\alpha$, then $A'$ moves to $P$ and $XYZ$ becomes the sideline $BC$. When $S$ is not a vertex $A, B, C$, the envelope of line $XYZ$ is the Steiner parabola of point $S$ (focus at $S$, directrix the Steiner line of $S$). The tangential equation of this parabola is given by matrix:

$$\left( \begin{array}{c} P' \end{array} \right) \simeq \left( \begin{array}{ccc} 2 \Theta \sigma_3 & \sigma_3 & \sigma_2 - \sigma_1 \Theta \\ \sigma_3 & 0 & -\Theta \\ \sigma_2 - \sigma_1 \Theta & -\Theta & -2 \end{array} \right)$$

Proof. Since $XYZ$ is given by second degree polynomials, the envelope is a conic. It can be obtained by diff and wedge, then eliminate. Parabola comes from the central 0. Focus is obtained in the usual way, and one recognizes $S$. Directrix $\Delta$ is the locus of the reflections of the focus in the tangents. From the special cases, this is the Steiner line of $S$.

Last point, compute circle $(S, X, Y)$. Special cases $\Theta = \alpha$, etc, and $\Phi = \kappa^2 \Theta (O'$ at infinity) are appearing in factor. Otherwise, the equation is:

$$\alpha (\Phi - \kappa^2 \Theta) Z \bar{Z} + (\kappa^2 \alpha - \Phi) (Z + \alpha \Theta \bar{Z}) T + (\Phi \Theta - \alpha^2 \kappa^2) T^2$$

and this circle goes through $A$ and $A'$. \hfill \square

Proposition 24.1.7. When $A, B, C, S, P$ are according the former hypotheses, let $H, H'$ be the respective orthocenters of $ABC$ and $A'B'C'$. Then midpoint of $H, H'$ belongs to $XYZ$ if and only if $\kappa^2 = -1$ (so that $\kappa$ is a quarter turn) or:

$$\kappa^2 = -\frac{\Phi^2 (\Theta - \sigma_1)}{\sigma_2 \Theta - \sigma_3}, \quad \text{i.e.} \kappa = (BC, OP) + (SA, SH)$$

In the second case, $XYZ$ is the perpendicular bisector of $(H, H')$.

Proof. We have $H' = \sigma(H)$ and equation in $\kappa^2$ is straightforward. Then we have:

$$\omega^2 (BC) = -\beta \gamma, \omega^2 (OP) = \Phi^2, \omega^2 (SA) = -\Theta \alpha, \omega^2 (SH) = -\sigma_3 \Theta \frac{\Theta - \sigma_1}{\sigma_2 \Theta - \sigma_3}$$

\hfill \square
Corollary 24.1.8. Start from triangle ABC, and assume that XYZ = [ρ, σ, τ] while κ is a quarter turn. Then X, Y, Z are X = 0 : τ : −σ, etc. Lines XδA = [S, σ + Sbτ, a2σ, a2τ], etc are the perpendicular at X to BC, etc. Finally, point A′ is YδB ∩ ZδC, etc. In other words:

\[ A' \simeq \begin{pmatrix} \rho (\rho SbS_c + \sigma S_s + \tau S_aS_b) - 4 S^2 \sigma \tau \\ \rho^2 \rho (\tau c^2 - \sigma S_a - \rho S_b) \\ c^2 \rho (\sigma b^2 - \tau S_a - \rho S_c) \end{pmatrix}, \]

The perspector and the similicenter are:

\[ P = \text{isogon} (\text{M}^{-1} \Delta); S = \text{isogon} ((\sigma - \tau) \rho : \sigma (\tau - \rho) : \tau (\rho - \sigma)) \]

while the ratio of the similarity is:

\[ k = \frac{(a^2 + b^2 + c^2) \rho \sigma \tau - S_a \rho (\sigma^2 + \tau^2) - S_b \sigma (\rho^2 + \tau^2) - S_c \tau (\rho^2 + \sigma^2)}{2 S (\sigma - \tau) (\rho - \tau) (\rho - \sigma)} \]

Proof. Straightforward computation.

24.2 Perspective and inversely similar

Lemma 24.2.1. A circumscribed rectangular hyperbola goes through A, B, C, H, Gu where H is the orthocenter and Gu is the gudulic point, the intersection of the RH and the circumcircle. Directions of axes are given by the bisectors of, for example, AGu and BC. Then directions of asymptotes are obtained by a 45° rotation (or taking again the bisectors).

Lemma 24.2.2. When a rectangular hyperbola H is known by its implicit equation

\[ \kappa^2 \overline{\mathbf{Z}}^2 - \frac{1}{\kappa^2} \mathbf{Z}^2 + (W \overline{\mathbf{Z}} + V \mathbf{Z}) T + Q T^2 \]

then points M ∈ H can be parametrized as:

\[ M = \frac{1}{2} \begin{pmatrix} +V \kappa^2 \\ 1 \\ -W \kappa^2 \end{pmatrix} + X \begin{pmatrix} \kappa \\ 0 \\ 1 \kappa \end{pmatrix} + Y \begin{pmatrix} +i \kappa \\ 0 \\ -i \kappa \end{pmatrix} \]

(24.1)

where X, Y are real quantities linked by:

\[ YX = \frac{-i}{16} \left( \kappa^2 V^2 + 4 Q - \frac{W^2}{\kappa^2} \right) \]

Proof. This way of writing may look weird but, most of the time, hyperbola equations are appearing that way.

Lemma 24.2.3. Consider four points M1 on a rectangular hyperbola, parametrized by (24.1). These points form an orthocentric quadrangle if and only if:

\[ 256 x_1 x_2 x_3 x_4 = \left( \kappa^2 V^2 + 4 Q - \frac{W^2}{\kappa^2} \right)^2 \]

Proof. We write that \((M_1 \cap M_2) : M_3 \cap M_4 = 0\), and obtain this condition. The conclusion follows from the symmetry of the result.

Lemma 24.2.4. When ψ is a central inverse similarity (reflections in a line are allowed, but not the other isometries), then its matrix in the Morley space can be written as:

\[ [\psi] = \begin{pmatrix} 0 & z & \zeta & k \kappa^2 \\ 0 & 1 & 0 \\ k & \zeta & k & z \\ \kappa^2 & \zeta & \kappa^2 & 0 \end{pmatrix} \]

Point \( S = z : 0 : \zeta \) is the center, axes are directed by ±k2 and ratio is k (k is unimodular while k is real and k = ±1 is a dubious case). One can check that umbilics are exchanged by this transform.
Proof. Let \( z_2 : t_2 : \zeta_2 \) be the image of the origin \( 0 : 1 : 0 \). The characteristic polynomial of matrix:

\[
\begin{pmatrix}
0 & z_2/t_2 & k \zeta_2^2 \\
0 & 1 & 0 \\
k & \zeta_2/z_2 & 0
\end{pmatrix}
\]

is \( \chi (\mu) = (\mu - 1)(\mu - k)(\mu + k) \). Excluding \( k = \pm 1 \) ensures the existence of a center. Consider the reflection \( \delta \) about line through \( S \) and \( \zeta^2 : 0 : 1 \). We have:

\[
\delta = \text{subs} \left( k = \pm 1, \psi \right) ; \quad \psi \cdot \delta = \delta \cdot \psi
\]

Remark 24.2.5. The unimodular \( k \) was an intrinsic quantity when we were dealing with direct similarities. Now, \( k^2 \) measure the angle between the real axis and one of the axes of the skew similarity.

Proposition 24.2.6. When a central skew similarity \( \psi (S, kS) \) and a perspector \( P \neq S \) are given, the locus \( H \) of points \( M \) such that \( P, M, M' = \psi (M) \) are collinear is the conic through \( S, P, \psi^{-1}P \) and directions of the \( \psi \)-axes (and this conic is a rectangular hyperbola).

Proof. The locus contains certainly the five points such that \( M = P \) or \( M' = P \) or \( M = M' \) i.e. \( S \) and both directions \( \pm \zeta^2 : 0 : 1 \). The general case results from the fact that \( \det [P, M, M'] \) is a second degree polynomial in \( Z, \zeta, T \) so that \( C \) is a conic.

Proposition 24.2.7. Suppose that triangles \( T_1 \) and \( T_2 \) are together in perspective (center \( P \)) and strictly antisimilar (center \( S \), ratio \( k \neq \pm 1 \), direction of axes \( \pm \zeta^2 : 0 : 1 \)). Let \( X \) be one of the points \( S, P, \pm \zeta^2 : 0 : 1 \). Then the other three are obtained as the remaining intersections between conic \( H \) through \( A, B, C, H, X \) and conic \( H' \) through \( A', B', C', H', X \).

Proof. Use Lubin coordinates relative to \( T_1 = ABC \), and note \( S \simeq z : t : \zeta \). Then \( A'B'C' \) is obtained as:

\[
[A'B'C'] \simeq \psi^{-1} \cdot ABC
\]

The determinant of lines \( A\bar{A}', B\bar{B}', C\bar{C}' \) factors as:

\[
\frac{\sigma_3 - k (k^2 - 1)}{t^2 \sigma_3} \times \text{conic}
\]

proving that \( S \) belongs to the rectangular hyperbola whose implicit and parametric equations are:

\[
-1 \frac{1}{k^2} Z + k^2 Z^2 + \left( \frac{\sigma_1}{\sigma_3} - \frac{\sigma_3}{\sigma_2} \right) T Z + \left( \frac{\sigma_2}{\sigma_3} - \frac{\sigma_3}{\sigma_2} \right) T Z + \left( \frac{\sigma_1}{\sigma_3} - \frac{\sigma_2}{\sigma_3} \right) T^2
\]

Then perspector is also on this hyperbola (from preceding proposition). We even have the more precise result:

\[
k = \frac{x_S - x_P}{x_S + x_P}
\]
Proposition 24.2.8. Consider a triangle $A, B, C$, its orthocenter $H$ and two points $S, P$ such that the six points $A, B, C, H, P, S$ are on the same conic $\mathcal{H}$. Then it exists exactly one triangle $A'B'C'$ that is together $S$-antisimilar and $P$ perspective with $ABC$. Moreover $A', B', C', H', P, S$ are on the same conic $\mathcal{H}'$, both conics are rectangular hyperbolas and share the same asymptotic directions. Finally, the gudulic point $G_u$ of conic $\mathcal{H}$ sees triangle $ABC$ at right angles with trigone $A'B'C'$, and the fourth intersection of conic $\mathcal{H}'$ with circle $\Gamma'$ sees triangle $A'B'C'$ at right angles with trigone $ABC$.

Proof. A conic through $A, B, C, D$ is a rectangular hyperbola. Consider one of its asymptotes and draw a parallel $\Delta$ to this line through point $S$. Let $A'', B'', C'', H''$ be the reflections of $A, B, C, H$ into $\Delta$. Then we have $A' = S A'' A'$, $P A'' A'$, $S A''$ etc. Final result comes from $\square$

Proposition 24.2.9. When $A, B, C, H, P$ are fixed, the direction of the perspectrix $XYZ$ is also fixed.

Proposition 24.2.10. When $A, B, C, H, S$ are fixed and $P$ moves onto the $A, B, C, H, S$ hyperbola, the envelope of the perspectrix $XYZ$ is the parabola inscribed in triangle $ABC$ whose directrix is line $HS$.

### 24.3 Parallelogy

Definition 24.3.1. Two triangles $T_1$ and $T_2$ are parallelogic when a point $M_1$ exists that sees triangle $T_1$ with rays parallel to the sidelines of $T_2$. In other words, the lines drawn from the vertices of $T_1$ and parallel to the corresponding sides of $T_2$ are concurrent at a point $M_1$ (the ray source).

Proposition 24.3.2. Parallelogy is a symmetric relation between triangles (defining a ray source $M_1$ that sees $T_1$ and a ray source $M_2$ that sees $T_2$).

Proof. Using computer, symmetry is straightforward : condition of concurrence is the product of $\det T_2$ by a polynomial that is invariant by exchange of the two triangles. $\square$

Proposition 24.3.3. When $T_1, T_2$ are parallelogic, the formula $\phi = T_2 \phi T_1$ (columns are supposed to be synchronized !) defines a collineation $\phi$ such that $L_{\infty} \mapsto L_{\infty}$ together with $A_1 \mapsto A_2$, etc. Then $\phi(M_1) = M_2$. Moreover, any other triangle is parallelogic with its image by $\phi$ (so that $\phi$ itself can be called a parallelogy).

Conversely, a collineation $\phi$ is a parallelogy when (1) $L_{\infty} \mapsto L_{\infty}$ and (2) $L_{\infty} \phi = \trace(\phi) L_{\infty}$.

Proof. Direct computation, assuming that $T_1, T_2$ are parallelogic. $\square$

Exercise 24.3.4. (spoiler). Morley spaces: conditions are $\phi_{21} = \phi_{23} = 0$ to enforce $L_z \mapsto L_z$ and $\phi_{11} + \phi_{33} = 0$ to have the right trace.

### 24.4 Orthology

Definition 24.4.1. We say that point $P$ sees triangle $A'B'C'$ at right angles to trigone $ABC$ when $P$ is different from $A', B', C'$ and verifies $PA' \perp BC$ etc.

Remark 24.4.2. Should point $P$ be at infinity, all sidelines of $ABC$ would have the same direction, and triangle $ABC$ would be degenerate (flat). Some problems are to be expected...

Lemma 24.4.3. The orthodir of the $BC$ sideline is $\delta_A = a^2 : -S_c : -S_b$. This is also the direction of line $HA$, where $H$ is the orthocenter of the triangle.

Proposition 24.4.4. Consider two non degenerate finite triangles $ABC$, $A'B'C'$ and suppose that it exists a finite point $P$, different from $A', B', C'$, that sees triangle $A'B'C'$ at right angles to trigone $ABC$. Then it exist a point $U$ that sees triangle $ABC$ at right angles to trigone $A'B'C'$.
Proof. From hypothesis, $A'$ belongs to line $P\delta_A$. Then it exists a real number $k_A \neq \infty$ such that $A' = k_A P + \delta_A$. And the same for $B'$, $C'$. Now compute:

$$A'' = [M] \cdot (B' \cap C') = -2S(p + q + r) \begin{pmatrix} -k_B - k_C \\ k_B \\ k_C \end{pmatrix}$$

This comes from $P \cap P = 0$, together with $[M]^{-1}(\delta_B \leq \delta_C) = [M]^{-1}L_\infty = 0$. Since $p + q + r \neq 0$ and $k_j \neq 0$ is assumed, column $A''$ really defines a direction. We can therefore simplify and obtain:

$$A''B''C'' = \begin{pmatrix} -k_B - k_C \\ k_B \\ k_C \end{pmatrix} \begin{pmatrix} k_A \\ -k_A - k_C \\ -k_A - k_B \end{pmatrix}$$

This triangle is perspective to $ABC$, with perspector $U = k_A : k_B : k_C$. Therefore point $U$ sees triangle $ABC$ at right angles to trigone $A'B'C''$.

\[\square\]

**Definition 24.4.5.** We say that two triangles are orthologic to each other when it exists a point $M_1$ that sees triangle $T_1$ with rays orthogonal to the sidelines of trigone $T_2$ and a point $M_2$ that sees triangle $T_2$ at right angles to the sidelines of trigone $T_1$.

In other words, the lines drawn from the vertices of $T_1$ and orthogonal to the corresponding sidelines of $T_2$ are concurrent at a point $M_1$ (the ray source), etc. In this definition, flat triangles and centers at infinity are allowed.

**Remark 24.4.6.** This property cannot be reworded in a shorter form (the so-called symmetry), since $P \notin L_\infty$ is required to be sure of the existence of $U$, but this is not sufficient to be sure of $U \notin L_\infty$.

**Example 24.4.7.** Cevian triangles of X(2) and X(7) are orthologic. Orthology centers are X(10) and X(1).

**Proposition 24.4.8.** Let $ABC$ be the reference triangle, $P$ a point not on the sidelines, $Q$ its isogonal conjugate and $P_AP_BP_C$, $Q_AQ_BQ_C$ their respective pedal triangles. Then $ABC$ and $P_AP_BP_C$ are orthologic. The center which looks at $P_AP_BP_C$ is obviously $P$, while the center which. looks at $A,B,C$ is $Q$. Let $EF$ be the line through $A$ and parallel to $Q_BQ_C$, etc. Then $ABC$ is the pedal triangle of $P$ wrt $DEF$, and $ABC$ is orthologic with $DEF$. The center which looks at $D,E,F$ is $K = 20 - P$, whose barycentrics wrt $DEF$ are $p : q : r$. The triangle $DEF$ is called the anti-pedal triangle of $P$ and we have: $DEF \approx$

\[
\begin{pmatrix}
+qr(a^2q + S_c)p \quad a^2r + S_bp \\
-qr(b^2p + S_cq) \quad a^2r + S_bq \\
-qr(c^2p + S_br) \quad a^2r + S_bq \\
pr(a^2q + S_c)p \quad (b^2r + S_ap)q \\
-pr(b^2p + S_cq) \quad (b^2r + S_ap)q \\
-pr(c^2p + S_br) \quad (b^2r + S_ap)q \\
-pq(a^2r + S_b)p \quad (c^2q + S_ar) \\
pq(b^2p + S_cq) \quad (c^2q + S_ar) \\
pq(c^2p + S_br) \quad (c^2q + S_ar)
\end{pmatrix}
\]

Proof. Using the mkortho routine, one obtains the coordinates of $Q$, and $Q_AQ_BQ_C$. Then one computes the lines $EF$, etc and obtain the points $D,E,F$. Using again the mkortho routine, one obtains $K$. See Figure 24.1. The cyan circles are illustrating the cyclopedal property (Section 9.3).

\[\square\]

**Proposition 24.4.9.** When $T_1$, $T_2$ are orthologic, the formula $[\phi] = [T_2]^{-1} [T_1]$ defines a collineation $\phi$ such that $L_\infty \Rightarrow L_\infty$ together with $A_1 \mapsto A_2$, etc. Then $\phi(M_1) = M_2$. Moreover, any other triangle is orthologic with its image by $\phi$ (so that $\phi$ itself can be called an orthology).

Conversely, a collineation $\phi$ is an orthology when (1) $L_\infty \Rightarrow L_\infty$ and (2) trace $\{\phi, \text{OrtO}\} = 0$.

**Proof.** Direct computation, assuming that $T_1$, $T_2$ are orthologic.

\[\square\]

**Exercise 24.4.10.** (spoiler). Morley spaces: conditions are $\phi_{21} = \phi_{23} = 0$ to enforce $L_z \Rightarrow L_z$ and $\phi_{11} = \phi_{33}$ to have the right trace.
Proposition 24.4.11. Assume that triangle $ABC$ is not degenerate and let the finite points $P, U$ be described by their barycentrics $P = p : q : r$ and $U = u : v : w$ (here $P = U$ is allowed). Moreover, assume that $U$ is not on the sidelines of $ABC$. Then all triangles $A'B'C'$ such that point $P$ sees triangle $A'B'C'$ at right angles to trigone $ABC$ and point $U$ sees triangle $ABC$ at right angles to trigone $A'B'C'$ are given by formula:

$$A'B'C' \simeq \begin{pmatrix} pu + a^2 \vartheta & pv - S_c \vartheta & pw - S_b \vartheta \\ qu - S_b \vartheta & qv + b^2 \vartheta & qw - S_a \vartheta \\ ru - S_a \vartheta & rv - S_a \vartheta & rw + c^2 \vartheta \end{pmatrix} = (P \cdot t U) + 2S \vartheta M$$ (24.2)

where $\vartheta$ describes an homothety centered at $P$.

Proof. Since $u \neq 0$, relation $A' \in P \delta A$ can be written as $uP + \vartheta A \delta A$, and the same holds for the other points. Now, compute:

$$\det \begin{bmatrix} uP + \vartheta A \delta A, vP + \vartheta B \delta B, [M(C \land U)] \end{bmatrix} = 2uvS(p + q + r)(\vartheta_A - \vartheta_B)$$

Due to the hypotheses, all the $\vartheta$ must have the same value, leading to the formula. Converse is obvious, when a triangle is as described by the formulas, both orthologies are verified.

Remark 24.4.12. If $P$ were at infinity, $PA', PB'$, $PC'$ would have the same direction, and also the sidelines of $ABC$. In the formula, this would lead to $A'B'C'$ at infinity. If coordinate $u$ was 0 then $UB \parallel UC$ so that $A'B'C'$ would lead to $A'$ at infinity.

24.5 Simply orthologic and perspective triangles

Definition 24.5.1. When triangles are orthologic with $P \neq U$, we say they are simply orthologic. When $P = U$, we say they are bilologic.

Notation 24.5.2. In this chapter, when triangles $ABC$ and $A'B'C'$ are in perspective, their perspector will be noted $\Omega$ (and never $P$, nor $U$), while the perspectrix will be noted $XYZ$ with $X = BC \cap B'C'$, etc.

Theorem 24.5.3 (First Sondat Theorem). Assume that triangle $ABC$ is finite and non degenerate; $P$ is at finite distance; $U$ is different from $P$ and is not on the sidelines. Then it exists exactly one triangle $A'B'C'$ such that (1) $P$ sees $A'B'C'$ at right angles to trigone $ABC$ (2) $U$ sees $ABC$ at right angle to trigone $A'B'C'$ (3) $A'B'C'$ is perspective to $ABC$. When $A'B'C'$ is chosen that way, the corresponding perspector $\Omega$ is collinear with points $P, U$ and the perspectrix $XYZ$ is orthogonal to $PU$. 

—– pldx : Translation of the Kimberling’s Glossary into barycentrics —–
Proof. We have assumed that $P = p : q : w$ is different from $U = u : v : w$. Then $\vartheta$ is fixed by the condition of being perspective and we have:

$$\vartheta = -u(vS_b - wS_c) qr + v(wS_c - uS_a) pr + w(uS_a - vS_b) pq$$

$$\Omega = \frac{S_c w - S_b v}{qw - rv} ; \frac{S_c w - S_a u}{pw - qu} ; \frac{S_c w - S_a u}{pw - qu}$$

First assertion is proved by $\Omega \simeq \alpha U - \beta P$ where:

$$\begin{bmatrix} \alpha \\ \beta \end{bmatrix} \simeq \begin{bmatrix} (qw - rv) puS_a + (ru - pw) vqS_b + rw (pv - qu) r w S_c \\ (qw - rw) v^2 S_a + (ru - pw) v^2 S_b + (pv - qu) w^2 S_c \end{bmatrix}$$

Second assertion is proved by checking that

$$\Delta \cdot \text{Ort} \cdot (\text{normalized}(P) - \text{normalized}(U)) = 0$$

24.6 Bilogic triangles

Proposition 24.6.1. Bilogic triangles are ever in perspective. When orthology center is $U = u : v : w$ is finite and not on the sidelines, formula (24.2) becomes

$$A'B'C' \simeq \begin{pmatrix} u^2 + a^2 \vartheta & uv - S_c \vartheta & uw - S_b \vartheta \\ uv - S_c \vartheta & u^2 + b^2 \vartheta & vw - S_a \vartheta \\ uw - S_b \vartheta & vw - S_a \vartheta & r^2 + c^2 \vartheta \end{pmatrix} = (U \cdot \Omega)^{-1} + 2\Theta \cdot M$$

(24.3)

while the perspector and the tripole of the perspectrix are respectively:

$$\Omega \simeq (\vartheta vw - S_a)^{-1} ; (\vartheta vu - S_a)^{-1} ; (\vartheta w - S_a)^{-1}$$

$$\text{tripolar}(XYZ) \simeq \begin{pmatrix} (vwa^2 + u(vS_b + wS_c - uS_a)) \vartheta - 4S^2 \\ (uwb^2 + v(wS_c + uS_a - vS_b)) \vartheta - 4S^2 \\ (uc^2 + w(uS_a + vS_b - wS_c)) \vartheta - 4S^2 \end{pmatrix}$$

Proof. Straightforward computation.

Proposition 24.6.2. When triangle $ABC$ is fixed and the bilogic center $U$ is given, the locus of the perspector $\Omega$ of the bilogic triangles $A'B'C'$ is the rectangular hyperbola that goes through $A, B, C, U, H = X(4)$. The perspector of this circumconic is:

$$u(vS_b - wS_c) : v(wS_c - uS_a) : w(uS_a - vS_b)$$

while the envelope of the perspectrix is the inconic whose perspector is:

$$(vS_b - wS_c)^{-1} ; (wS_c - uS_a)^{-1} ; (uS_a - vS_b)^{-1}$$

Proof. Eliminate $\vartheta$ from $\Omega$. Then eliminate $\vartheta$ from $XYZ$ and take the adjoint matrix.

Proposition 24.6.3. Let $ABC$, $A'B'C'$ be two bilogic triangles, with orthology center $U$, perspector $\Omega$ and perspectrix $(XYZ)$ where $X = BC \cap B'C'$, etc. Then lines $U\Omega$ and $XYZ$ are orthogonal. Moreover we have $(UX) \bot (AA')\Omega$, etc.
Figure 24.2: Orthology and perspective.
Proof. We compute the orthodir $\delta$ of line $XYZ$ and find that:

$$\delta = \begin{bmatrix}
(u^2 (w + v) S_a - S_c w w^2 - S_b v v^2) \theta + S_b S_c (v + w) - a^2 S_a u \\
(v^2 (w + u) S_b - S_a v u^2 - S_c v v^2) \theta + S_c S_a (w + u) - b^2 S_b v \\
(w^2 (v + u) S_c - S_a w u^2 - S_b w v v^2) \theta + S_a S_b (u + v) - c^2 S_c w
\end{bmatrix}$$

It remains to check that $\text{normalized}(\Omega) - \text{normalized}(U)$ is proportional to $\delta$. \qed
Chapter 25

Linear Families of Inscribed Triangles

This chapter is an extended version of Douillet (2014c) and summarises discussions with pappus, Poulbot and others at http://www.les-mathematiques.net

Notation 25.0.1. In all parts of this chapter, the relations

\[ f + g + h = u + v + w = \rho + \sigma + \tau \]

will ever be assumed. These rules are related to the asymmetric parametrization described at Definition 25.2.18.

25.1 General linear families of triangles

Definition 25.1.1. Suppose that \( a_0, b_0, c_0 \) and \( a_1, b_1, c_1 \) are six points at finite distance, together with \( a_0 \neq a_1, b_0 \neq b_1, c_0 \neq c_1 \). Then we say that

\[ T_t = t T_1 + (1 - t) T_0 \]

defines a (general) linear family of triangles.

Remark 25.1.2. Property \( a_0 / \in \mathcal{L}_\infty \), etc is required to allow the projective definition:

\[ a_t = t \frac{a_1}{\mathcal{L}_\infty \cdot a_1} + (1 - t) \frac{a_0}{\mathcal{L}_\infty \cdot a_0} \]

while property \( a_0 \neq a_1 \), etc is required to ensure the existence of \( \mathcal{L}_a = a_0 \land a_1 \), etc.

Remark 25.1.3. Such a family can also be defined as

\[
\begin{pmatrix}
1 - t (v_{12} + v_{13}) - p_{12} - p_{13} & tv_{21} + p_{21} & tv_{31} + p_{31} \\
v_{12} + p_{12} & 1 - t (v_{21} + v_{23}) - p_{21} - p_{23} & tv_{32} + p_{32} \\
v_{13} + p_{13} & tv_{23} + p_{23} & 1 - t (v_{31} + v_{32}) - p_{31} - p_{32}
\end{pmatrix}
\]

where triangles \( T_t \) are in a normalized form, while \( \left( T_t - T_s \right) / (t - s) \) is a set of three non-zero constant vectors.

Definition 25.1.4. An equicenter \( E \) is a fixed point which has the same barycentrics wrt all the triangles \( T_t \) of the family. The column \( F = f_a : f_b : f_c \) of these barycentrics is called the Neuberg column of the family (and doesn’t depends on the barycentric frame used to compute them).

Definition 25.1.5. An areal center \( S \) is a fixed point which verifies:

\[ \forall s, t : \text{area}(S, a_t, a_s) = \text{area}(S, b_t, b_s) = \text{area}(S, c_t, c_s) \]

For reasons given later, this point is also called the slowness center of the family.
Construction 25.1.6. The Neuberg's (1921) construction of $E, T$ is as follows (see Figure 25.1).
From an auxiliary point $O$, draw points $O_a \doteq O + a_1 - a_0$, etc and compute $f_a = \text{area}(O, O_b, O_c)$, etc.
Then $O$ is the barycenter of the $O_j$ with coefficients $f_j$, and this implies that $T_f$. $F$ is ever equal
to $f_a b_0 + f_b b_0 + f_c c_0$,
And now, draw the lines $L_A \doteq a_0 \wedge a_1$, etc, obtain the points $T = L_b \wedge L_c$, etc. and take their
barycenter, i.e. compute $S \doteq \text{ABC}$. $F$.

Discussion. If $O$ is chosen as $1 - y - z : y : z$ then

$$f_a = \det \begin{bmatrix} 1 - y - z & 1 - y - z - v_{21} & -y - z - v_{31} \\ y & y + v_{21} + v_{23} & y - v_{32} \\ z & z - v_{23} & z + v_{31} + v_{32} \end{bmatrix}$$

and, therefore:

$$F = \begin{pmatrix} v_{21} v_{31} + v_{21} v_{32} + v_{23} v_{31} \\ v_{31} v_{12} + v_{12} v_{32} + v_{32} v_{13} \\ v_{12} v_{23} + v_{21} v_{13} + v_{23} v_{13} \end{pmatrix}$$

When $f_a + f_b + f_c = 0$, the magenta triangle of Figure 25.1 if flat and point $E$ is at infinity. When $f_a = f_b = f_c = 0$, we have the special-special case, discussed at Proposition 25.13.3

The trigone $(L_j)$ degenerates when quantity $\det (L_a, L_b, L_c) =

(p_{12} v_{13} - p_{13} v_{12}) f_a + (-p_{21} v_{23} + p_{23} v_{21}) f_b + (p_{31} v_{32} - p_{32} v_{31}) f_c + v_{12} v_{23} v_{31} - v_{13} v_{21} v_{32}$

vanishes. Otherwise... everything becomes simpler when seen from $ABC$, the fixed circumscribed triangle. See next coming section for a discussion of what happens when $S$ is at infinity. □

25.2    Slowness- and equi-center of a LFIT

25.2.1    Slowness center

Definition 25.2.1. We say that triangle $abc$ is inscribed in triangle $ABC$ when $a \in BC$, etc. Conversely, we say that $ABC$ is circumscribed to triangle $abc$.

Definition 25.2.2. LFIT. When the triangles $T_f$ of the former section are all inscribed into a fixed (non degenerate) triangle $ABC$, this situation is described as a Linear Family of Inscribed Triangles. Using the values relative to $t = 0$ and $t = 1$, this can be written as:

$$T_f = \frac{a_j b_i c_i}{\begin{pmatrix} a_0 b_0 c_0 \doteq (1-t)(a_0 b_0 c_0) + (t)(a_1 b_1 c_1) \\ \begin{pmatrix} 0 & 1 & 0 \end{pmatrix} & \begin{pmatrix} r_1 & 1 - r_1 & 1 - r_1 \\ r_0 & 0 & 0 \end{pmatrix} \end{pmatrix} \begin{pmatrix} 0 & 1 & 0 \end{pmatrix} \begin{pmatrix} p_1 & 0 & 1 - r_1 \\ p_0 & 0 & 1 - r_1 \end{pmatrix} \begin{pmatrix} 0 & 1 & 0 \end{pmatrix} \begin{pmatrix} q_1 & 0 & 0 \end{pmatrix} \begin{pmatrix} 0 & 1 & 0 \end{pmatrix} \begin{pmatrix} q_0 & 0 & 0 \end{pmatrix} \begin{pmatrix} 0 & 1 & 0 \end{pmatrix} \begin{pmatrix} 0 & 1 & 0 \end{pmatrix} + (t)}$$

(25.1)

Construction 25.2.3. Geoebra: given $a \in BC$, the temporal parameter $t$ is obtained as

$$t = \text{real} \left( \frac{(a_t - a_0)/(a_1 - a_0)} \right) ; \text{real is required}$$

$$b_t = (t)*b_1+(1-t)*b_0 + 0*I ; \text{not a vector}$$

Definition 25.2.4. As in Figure 25.2, the velocities $v_a^T$, etc are defined by $v_a^T \doteq a_1 - a_0 = [0, p_1 - p_0, p_0 - p_1]$, while the so called "speed vector" is defined by $\overrightarrow{v} \doteq [p_1 - p_0, q_1 - q_0, r_1 - r_0]$.

Remark 25.2.5. Many things are to be computed from the speed vector, with the following strange property: everything becomes simpler when introducing the reciprocal of these quantities (remember that $t \mapsto a$, etc are non constant). This leads to the following:
Spoiler: $\omega = (S + E)/2$ is the center of the pilar conic $\mathcal{C}$.

Figure 25.1: The Neuberg construction
Definition 25.2.6. The **slowness center** of a linear family of triangles is:

\[ S \doteq f : g : h \simeq \frac{1}{p_1 - p_0} : \frac{1}{q_1 - q_0} : \frac{1}{r_1 - r_0} \]

leading to the following description (where indices 0 have been omitted):

\[
\begin{pmatrix}
0 & 1 - q & t & \frac{r + t}{h} \\
p + \frac{t}{f} & 0 & 1 - r - \frac{t}{h} \\
1 - p - \frac{t}{f} & q + \frac{t}{g} & 0
\end{pmatrix}
\]  

(25.2)

Remark 25.2.7. Considering \( S \) as a projective quantity is only recognizing that choosing a time unit or another is unessential. Caveat: when used together with the later introduced \( E \simeq u : v : w \), the projective object is \( f : g : h : u : v : w \), so that neither \( f : g : h \) nor \( u : v : w \) can be changed "at will", i.e. independently of the other.

Theorem 25.2.8 (Main result). The slowness center is the **areal center** of the family. In other words, the areas swept out from \( S \) by the \( T \)-vertices are equivalent for any pair \( t_1, t_2 \).

Proof. Suppose that \( S \) is at finite distance, compute the area, obtain:

\[
\text{area } (S, a(t_1), a(t_2)) = (t_2 - t_1) S \div (f + g + h)
\]

and conclude from symmetry. Suppose now that \( S \) is at infinity (i.e. \( h = -f - g \)), compute the width of the strip created by the parallel lines \( \Delta_1 \doteq S \wedge a(t_1) \) and \( \Delta_2 \doteq S \wedge a(t_2) \), obtain:

\[
\Delta_2 - \Delta_1 = [t_1 - t_2 : t_1 - t_2 : t_1 - t_2]
\]

and here again conclude from symmetry.

25.2.2 Equicenter

Theorem 25.2.9 (Neuberg). If we apply to \( abc \) the barycentrics of \( S \) wrt \( ABC \), we obtain a fixed point, the so called equicenter \( E \) (except in the special-special case where \( E \) degenerates into \( 0 : 0 : 0 \), see Proposition 25.13.3).

Proof. From equation (25.2), it is obvious that \( E \doteq \underline{abc} \cdot S \) doesn't depend on \( t \). And we have:

\[
E \simeq u : v : w = (1 - q) g + r h : (1 - r) h + p f : (1 - p) f + q g
\]

This formula complies with the later requirement that: \( u + v + w = f + g + h \).

Remark 25.2.10. When one of the centers \( S, E \) is at infinity, the other is also at infinity.
25.2.3 Pilar point

Definition 25.2.11. The pilar point Ω ≃ E ∈ L is defined by the hard equalities:

\[ f + g + h = u + v + w = \rho + \sigma + \tau \]
\[ S + E + \Omega = (f + g + h)(A + B + C) \]  \hspace{1cm} (25.3)

25.2.4 Parametrization of a LFIT

Parametrization (25.2) has many interesting properties, among them its genericity and its symmetry. Nevertheless, there are many situations where this parametrization induces too burdensome computations. There are several other choices, each of them depending on some assumption about the areas of the variable triangles.

Proposition 25.2.12. The area of triangle abc ∈ L equals the area of ABC times the quantity:

\[ A = \frac{(f + g + h)^2}{fgh} \left( f g (p + q - 1) + g h (q + r - 1) + h f (r + p - 1) \right) + p q r (1 - p) (1 - q) (1 - r) \]
\[ = \frac{(f + g + h)^2}{fgh} \left( f u + 2 f v + g w - h w \right) + p q r (1 - p) (1 - q) (1 - r) \]
\[ = \frac{(f + g + h)^2}{fgh} \left( f u + 2 f v + g w - h w \right) + p q r (1 - p) (1 - q) (1 - r) \]

Proof. First formula is the characteristic property of the determinant. The second one doesn’t apply when \( S ≃ E ∈ L_{\infty} \): in this case, the area is constant, but has to be evaluated using \( E = K S \), and the area is \(-K(K + 1)\). More details at Section 25.13.

Theorem 25.2.13. When \( S, E \) are at finite distance, the area is a second degree quantity and therefore presents an extremum, characterizing the so-called critical triangle. Taking this event as the origin of a shifted time \( T \), we obtain:

\[ a b c (T) = \left( \begin{array}{c} 0 \\ T \frac{f u + 2 f v + g w - h w}{2 (u + v + w) f} \\ -T \frac{f u + 2 f v + g w - h w}{2 (u + v + w) f} \end{array} \right) \]

And then we have:

\[ \frac{\text{area}(T)}{S} = \frac{(f + g + h)}{fgh} T^2 \frac{f + g + h}{2 (u + v + w)^2} \frac{f^2 u^2 + g^2 v^2 + h^2 w^2 - 2 f g w - 2 g h v - 2 h f w u}{4 f g h} \]

Proof. Only the apparent denominators were used to obtain this formula. The \( A \) formula becomes indeterminate if one tries to use it when \( S ∈ L_{\infty} \).

Remark 25.2.14. This parametrization is hard to use, due to its too huge coefficients. Except when the critical triangle is already given. See for more details at Proposition 25.10.14

Theorem 25.2.15. When \( S, E \) are at infinity but aren’t equal, the area is a first degree quantity. We can choose the origin of time at \( A = 0 \), leading to:

\[ a b c = \left( \begin{array}{ccc} 0 & T \frac{u (h + w)}{g w - v h} & T \frac{u (g + v)}{g w - v h} \\ -T \frac{(h + w) v}{g w - v h} & 0 & T \frac{v (f + u)}{g w - v h} \end{array} \right) \]

Here, we have \( g w - v h = h u - f w = f v - g u \) and the area is \( T (h v - g w) / f g h \) (caveat: replacing \( E ∈ kC \) changes the family and multiplies the area by \( k \) !)

Proof. \( S ∩ E ≃ L_{\infty} \); hence the equality. No hidden denominators are implied.

Remark 25.2.16. CAVEAT. The remaining case, i.e. \( S ≃ E ∈ L_{\infty} \), is really special, worth of a specific section (see Section 25.13).
25.2.5 Asymmetric parametrization

**Remark 25.2.17.** As a matter of experiment, the following parametrization is more computation-friendly as any other one, largely enough to justify a lack of symmetry, and the burden of enforcing a binding rule.

**Definition 25.2.18.** The asymmetric parametrization of a LFIT is defined as:

\[
T = \begin{bmatrix}
0 & 1 - t & \frac{w - f}{g} & \frac{t}{h} + \frac{h - v}{h} \\
t & 0 & 1 - \frac{t}{h} + \frac{v - h}{h} & \frac{t}{f} \\
1 - t & \frac{t}{f} & \frac{w - f}{g} & 0
\end{bmatrix}
\]  

where \( f : g : h \) is the slowness center \( S \) and \( u : v : w \) the equicenter \( E \). This amounts to enforce \( a_0 = C \) together with the rule

\[
f + g + h = u + v + w = \rho + \sigma + \tau = 0
\]

**Remark 25.2.19.** In fact, we have to consider \( S : E : \Omega \) or \( f : g : h : u : v : w : \rho : \sigma : \tau \) as an unique projective object. When \( S \notin L_\infty \), the binding relation suffices to synchronize the three columns \( S, E, \Omega \). But when \( S, E \in L_\infty \), we already have \( f + g + h = u + v + w = 0 \) and this relation is no more sufficient for the purpose of synchronization. Caveat: when \( S \in L_\infty \), replacing \( E \) by \( kE \) changes the family (and changes the area of the inscribed triangles).

25.2.6 Gravity centers

**Proposition 25.2.20.** The locus of the gravity centers of the moving triangles of a given LFIT is a straight line, parallel to the tripolar of \( S \).

**Proof.** The alignment property is obvious from \( g_t = (1 - t) g_0 + (t) g_1 \), while premultiplying \( g_t \) by tripolar \( S \) leads obviously to a constant. One can also check directly that

\[
\text{locus}(g) = (3) [gh, hf, fg] - (fp + \sigma g + h\tau) L_\infty
\]

\[\square\]

25.3 Hexagonal graphs

25.3.1 Constructions

**Definition 25.3.1.** The hexagonal points \( \alpha_t, \beta_t, \gamma_t \) are defined by \( \alpha_t = b_t + c_t - A \), etc, so that:

\[
abc = \begin{pmatrix}
0 & 1 - q & r \\
p & 0 & 1 - r \\
1 - p & q & 0
\end{pmatrix} \rightarrow \alpha\beta\gamma = \begin{pmatrix}
r - q & r & 1 - q \\
1 - r & p - r & p \\
q & 1 - p & q - p
\end{pmatrix}
\]

The locus of \( \alpha_t \) is obviously a straight line, noted \( G_a \) and called "the A hexagonal graph". And circularly for the other two.

**Remark 25.3.2.** Since \( \beta_t \) is at intersection of lines \( (a_t, AB) \) and \( (c_t, CB) \), the locus of \( \beta_t \) is nothing but the graph of the correspondence \( BC = AB : a(t) \leftrightarrow c(t) \), drawn using the directions of the sidelines.

**Construction 25.3.3.** When the LFIT is known by triangles \( T_1 \) and \( T_0 \), then \( G_b \) is defined by \( \beta_1 \) and \( \beta_0 \). We draw \( (a_1, AB) \) which cuts \( G_b \) at \( \beta_1 \). And then draw \( (\beta_1, BC) \) which cuts \( AB \) at \( c_1 \). Similarly, \( b_t \) is constructed. And we can check the figure by closing the hexagon and going back to \( a_1 \). See Figure 25.3 When \( T_0 \) and \( S \) are known, draw the graphs using Proposition 25.3.5. When \( S, E, \) etc are known, draw the graphs using 25.8.
Proposition 25.3.4. When \( S, E, \Omega \) are known then, assuming the synchronization rule (25.6), the hexagonal graphs, locus of the \( \alpha_t = b_t + c_t - A \), etc., are (by rows)

\[
\begin{bmatrix}
G_a \\
G_c \\
G_e
\end{bmatrix} = 
\begin{bmatrix}
g + h - u & g - u & h - u \\
f - v & h + f - v & h - v \\
f - w & g - w & f + g - w
\end{bmatrix} = 
\begin{bmatrix}
-\rho & h - \rho & g - \rho \\
h - \sigma & -\sigma & f - \sigma \\
g - \tau & f - \tau & -\tau
\end{bmatrix}
\]

(25.8)

Proof. Direct computation.

Proposition 25.3.5. Directions of the hexagonal graphs are those of the sidelines of the anticevian triangle of the slowness center \( S \).

Proof. Obvious from the preceding proposition. One can also remark that:

\[
\alpha \beta \gamma = \begin{bmatrix}
q - r & r & 1 - q \\
1 - r & p - r & p \\
q & 1 - p & q - p
\end{bmatrix} + t \begin{bmatrix}
h^{-1} - g^{-1} & h^{-1} & -g^{-1} \\
h^{-1} & f^{-1} - h^{-1} & f^{-1} \\
g^{-1} & -f^{-1} & g^{-1} - f^{-1}
\end{bmatrix}
\]

25.3.2 Hexagonal conics

Proposition 25.3.6. Hexagonal conic. A conic \( H_t \) goes through the six points \( \alpha \beta \gamma \) of (25.7). Moreover the matrix

\[
\begin{bmatrix}
H_t^{H} \\
T \cdot H_t \\
T
\end{bmatrix} \simeq \begin{bmatrix}
0 & p + q - 1 & p + r - 1 \\
p + q - 1 & 0 & q + r - 1 \\
p + r - 1 & q + r - 1 & 0
\end{bmatrix}
\]

can be seen as a description of \( H_t \) from the abc frame, or as the description, from the \( ABC \) frame, of another conic, \( H_t^H \), image of \( H_t \) by the collineation \( \phi_t : a, b, c \mapsto A, B, C, L_\infty \mapsto L_\infty \). In any case, the \( H_t^H \) conics form a linear family.

Proof. Computations are obvious, with huge simplifications. Linearity follows from the linearity of the \( p_t \). One can also argue that opposite sides of the hexagon are parallel.

Corollary 25.3.7. The hexagonal conic is a parabola when area \( (a, b, c) = \) area \( (A, B, C) \div 4 \), an ellipse when area \( (abc) \) is greater and an hyperbola otherwise.

Proof. Since \( \phi_t \) is an affinity, conics \( H_t \) and \( H_t^H \) have the same number of points at infinity.

Proposition 25.3.8. The \( H_t^H \) have a fourth fixed point in common, \( X \), whose isotomic is given by:

\[
\text{isotom } X \simeq \begin{bmatrix}
(q - r) gh + (p + q - 1) fg - (r + p - 1) fh \\
(r - p) hf + (q + r - 1) gh - (p + q - 1) gf \\
(p - q) fg + (r + p - 1) hf - (q + r - 1) hg
\end{bmatrix} \simeq \begin{bmatrix}
gh + fu - gw - hw \\
hf + fu + gv - hw \\
gf - fu - gv + hw
\end{bmatrix}
\]

— pldx : Translation of the Kimberling’s Glossary into barycentrics —
Proof. Consequence of the linearity of the family.

**Corollary 25.3.9.** The hexagonal conic degenerates five times: (1) when $a_1 b_2 c_3$ is flat, two occurrences, visible or not and (2) when one of the $p + q + r$ vanishes, three occurrences. In the second case, $\beta \gamma \delta = b_2 c_3 || BC$ while $H_t^H$ degenerates into $AX \cup BC$, etc.

Proof. After simplification, one has $\det H_t = \det T_t \det H_t^H$.

**Fact 25.3.10.** (spoiler) At area’s peak, $H_t$ goes through $E$, while $H_t^H$ goes through $S$.

**Remark 25.3.11.** Both triangles $abc$ and $\alpha \beta \gamma$ have the same area, equal to $s \times S$ where

$$s = pqr + (1 - p)(1 - q)(1 - r) = \det abc = \det \alpha \beta \gamma$$

### 25.3.3 Some collineations

Executive summary: use some collineation to transform $H_t^H$ into $H_t$, and then go back using some other collineation.

**Exercise 25.3.12.** Let $\phi_a$ be the collineation defined by $A, B, C \mapsto a, b, c$, $L_\infty \mapsto L_\infty$. Using $ABC$ as basis, the matrix of $\phi_a$ is $\begin{bmatrix} a \ b \ c \end{bmatrix}$ and its characteristic polynomial is:

$$(\mu - 1)(\mu^2 + \mu + s) = (\mu - 1) \left( \mu - \mu - \frac{1 + W}{2} \right) \left( \mu - \frac{1 - W}{2} \right)$$

where $W$ is defined by: $s = (1 - W^2)/4$. With respect to $ABC$, the coordinates of the fixed point at finite distance are:

$$K_a \simeq \begin{pmatrix} qr - q + 1 \\ rp - r + 1 \\ pq - p + 1 \end{pmatrix} \ (25.9)$$

**Exercise 25.3.13.** Let $\phi_a$ be the collineation: $ABC \mapsto \alpha \beta \gamma$, $L_\infty \mapsto L_\infty$. Using again $ABC$ as basis, the matrix of $\phi_a$ is $\begin{bmatrix} \alpha \beta \gamma \end{bmatrix}$, the characteristic polynomial is the same and, wrt $ABC$, the coordinates of the fixed point at finite distance are:

$$K_a \simeq \begin{pmatrix} -qr + rp + pq - p - q + r + 1 \\ +qr - rp + pq + p - q - r + 1 \\ +qr + rp - pq - p - q + r + 1 \end{pmatrix}$$

**Exercise 25.3.14.** Let $\psi$ be defined by $\begin{bmatrix} \psi \end{bmatrix} = \begin{bmatrix} abc \end{bmatrix}^{-1} \cdot \begin{bmatrix} \alpha \beta \gamma \end{bmatrix}$ and not by $\begin{bmatrix} \alpha \beta \gamma \end{bmatrix} \cdot \begin{bmatrix} abc \end{bmatrix}^{-1}$. The normalized matrix of $\psi$ is:

$$\begin{pmatrix} \psi \end{pmatrix} = \frac{1}{s} \begin{pmatrix} (1 - r)q & (p - 1)(q + r - 1) & (q + r - 1)p \\ (p + r - 1)q & (1 - p)r & (q + r - 1)(p + r - 1) \\ (r - 1)(p + q - 1) & (p + q - 1)r & (1 - q)p \end{pmatrix}$$

$$\chi_{\psi}(X) = (\mu - 1) \left( \mu - \frac{1 + W}{1 - W} \right) \left( \mu - \frac{1 - W}{1 + W} \right) \text{ where } s = \frac{1}{4} (1 - W^2)$$

and its fixed points are:

$$K_1 = \frac{-1}{W^2} \begin{pmatrix} (2p - 1)(q + r - 1) \\ (2q - 1)(r + p - 1) \\ (2r - 1)(p + q - 1) \end{pmatrix} ; \ K_2, K_3 \simeq \begin{pmatrix} 2q + 2r - 2 \\ -2r + 1 + W \\ -2q + 1 - W \end{pmatrix}$$

So what?
25.3.4 Graphs, the general case

Definition 25.3.15. A "delta -triple" is what is obtained by choosing three directions

\[
\delta \doteq \begin{bmatrix}
    p & -1-q & 1 \\
    1 & q & -1-r \\
    -1-p & 1 & r
\end{bmatrix}
\]

and using them to draw the graph of \(b_t \mapsto c_t\), etc, i.e. the locus \(\Delta_\alpha\) of the \(\alpha_t = b_t\delta_c \cap c_t\delta_b\), etc. Quite obviously, these graphs are straight lines.

Proposition 25.3.16. The directions of these graphs (coded the same way as the original ones) are:

\[
[p', q', r'] = \left[ \frac{(r+1)(q+1)h-g}{(r+1)(q-gh)}, \frac{(r+1)(p+1)f-h}{(p+1)(h-fr)}, \frac{g(q+1)(p+1)-f}{(q+1)(f-pg)} \right]
\]

Only the directions of cevian (S) lead to a 1-sized orbit. The 2-sized orbits are characterized by

\[ (uvw + uw + v)gh + (uvw + vw + v)fh + (uvw + vw + w)fg = 0 \]

For a given triple \([u', v', w']\) there are at most two triples \([u, v, w]\).

Proof. Computations are straightforward. Nevertheless, the general formula giving the other antecedent is rather huge.

Example. Starting from \([0, 0, 0]\) (hexagonal conics), we obtain the directions of anticev (S), and conversely. The other antecedent of \([0, 0, 0]\) is given by the directions of \(SA, SB, SC\) (Catalan conics).

Exercise 25.3.17. Study the cevian and anticevian conics.

25.4 Temporal graphs

25.4.1 Pilar point, pilar conic

In the previous section, we have constructed the hexagonal graphs \(G_a, G_b, G_c\) by drawing lines \(ba\) through \(b\), parallel to \(AB\) together with drawing lines \(ca\) through \(c\), parallel to \(AC\). As a result, triangle \(a\beta\gamma(t)\) is the crosstri (see 4.6) of triangles \(abc(t)\) and triangle \(dir_A, dir_B, dir_C\). This method can be repeated, using other auxiliary directions than those of the sidelines \(BC, CA, AB\) and obtaining other graphs than the \(G_j\).

Definition 25.4.1. The pilar point \(\Omega \simeq \rho : \sigma : \tau\) of a LFIT is defined by the hard equality

\[
\Omega + S + E = (f + g + h) \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}, \ \text{i.e.} \ \rho \doteq g + h - u, \ etc
\]

Therefore \(\rho + \sigma + \tau = f + g + h\), so that \(f : g : h : u : v : w : \rho : \sigma : \tau\) is a projective object involving only one factor of proportionality. When \(S \notin L_{\infty}, G = X(2)\) is the ordinary barycenter of points \(S, E, \Omega\). Moreover, the pilar conic \(C\) is defined as the inscribed conic whose perspector is the isotomic conjugate of \(\Omega\), and the center is

\[
\omega = \frac{1}{2} (S + E) = \frac{1}{2} \begin{pmatrix} \sigma + \tau \\ \tau + \rho \\ \rho + \sigma \end{pmatrix} = \frac{1}{2} \begin{pmatrix} f + u \\ g + v \\ h + w \end{pmatrix}
\]

Once again the same caveat: \(S, E, \Omega\) have to be synchronized by (25.6).

Remark 25.4.2. The name can be tracked back to a joke describing a temporal conic \(C_t^*\) as a major general (conique divisionnaire), distinguished among all the hexagonal conics (conique brigadière). In this context, the pilar conic would be something like a lieutenant general in command.
Exercise 25.4.3. Consider a LFIT. The lines drawn by $A$ and parallel to $bc_t$, etc determine a triangle $A'B'C'$.

1. Determine the locus of $A', B', C'$.
2. Compare area $(A'B'C')$ and area $(a_t b_t c_t)$.
3. Consider the collineation $f : \mathcal{L}_\infty \mapsto \mathcal{L}_\infty$, $ABC \mapsto A'B'C'$ and determine a point $X$ such that $f(X)$ is constant.

25.4.2 Temporal graphs

Definition 25.4.4. The graphs $\Delta^*_A$, etc obtained when using the directions of trigoine $a_t b_t c_t$ as auxiliary directions, are called the temporal graphs of the LFIT. Obviously, they are straight lines, while the conics through $a_t b_t c_t$ are called the temporal conics $C^*_t$. This process amounts to observe the LFIT from the fixed triangle that coincides with $abc$ at time $t = s$.

Proposition 25.4.5. The temporal graph $\Delta^*_A$ is the tangent to the pillar conic $\mathcal{C}$ issued from $a_s$ (other than the obvious $BC$).

Proof. Property $a_s \in \Delta^*_A$ is obvious. For the contact: points $\alpha, \beta, \gamma$ are obtained as crossovers of $[abc](t)$, given at (25.2), and

$$\mathcal{D}_s \simeq \begin{pmatrix} (g + h) s + (hw - fh - g) & -f s + (fv - f h) & f s - (f + g - w) f \\ -g s + (gv) & (f + h) s + (-fv) & g s \\ -h s + (fh - hw) & -hs + fh & (-f - g) s + (f + g - w) f \end{pmatrix}$$

which gives the auxiliary directions. Thus, the $ABC$-equation of the $A$-temporal graph is

$$\Delta^*_A \simeq \left[(\rho s^2 - \rho f s ; -(\sigma + \tau) s^2 + f (\sigma + 2 \tau) s - f^2 \tau; -(\sigma + \tau) s^2 + f \tau s\right]$$

Use locuconi, and obtain a line-conic circumscribed to trigone $ABC$, and see that its perspector is $t \Omega$.

Proposition 25.4.6. Consider the temporal graph $\Delta^*_A$. When point $\alpha^*_t$ moves along $\Delta^*_A$, then line $\alpha^*_t a_t$ keeps a constant direction (depending on $s$, but not on $t$).

Proof. For any kind of graphs, directions of $\alpha^*_t b_t$ and $\alpha^*_t c_t$ are constant by the very definition. The $\alpha^*_t a_t$ property is special even if easily checked.

Exercise 25.4.7. Determine the graphs that share this property with the temporal graphs.

25.4.3 Temporal conics

Theorem 25.4.8. The temporal conic $C^*_t$ related to $a_t b_t c_t$ doesn’t depend on the $s$ chosen to construct the graphs (and now will be noted by $C_t$). The centers of the $C_t$ belong to line

$$[(gh - uw) \rho ; (fh - uw) \sigma ; (fg - uw) \tau]$$

that goes through $\omega = S + \mathcal{E}$. Moreover, the matrix

$$C^H \simeq \mathcal{T}_t \cdot \mathcal{C}_t \cdot \mathcal{T}_t \simeq \begin{pmatrix} 0 & h \tau & g \sigma \\ h \tau & 0 & f \rho \\ g \sigma & f \rho & 0 \end{pmatrix}$$

(25.11)

can be seen as a description of $C_t$ from the abc frame, or as the description, from the ABC frame, of another conic, $C^H$, image of $C_t$ by the collineation $\phi_t : a, b, c \mapsto A, B, C$, $\mathcal{L}_\infty \mapsto \mathcal{L}_\infty$. Using the second point of view, we have a fixed circumconic, with perspector the point $S + \Omega$.
Proof. Use the first point of view. Compute \( \frac{a_3 b_4 c_4}{1 - \alpha^2} \cdot \frac{\alpha^2}{1 + \alpha^2} \) and then, according to Proposition 12.7.9, take the wedge of the tripolars of two columns. The result doesn't contain \( s \). Moreover, being equal to \( S \ast \Omega \), this result doesn't depend on \( t \) either. Now use the formula giving center from perspector, go back to \( ABC \)-barycentrics and conclude. 

**Theorem 25.4.9.** The temporal conic \( C_t \) is bi-tangent to the pilar conic \( C \) and the chord through the contact points (that are not necessarily visible) is parallel to line \( SE \).

Proof. A point on \( C \) is \( P(s) \equiv 1/\rho : s^2 - \sigma : (1 + s)^2 \approx \tau \). Thus, using \( SS = s_1 + s_2 \), \( PP = s_1 s_2 \), an equation of the chord \( P(s_1) P(s_2) \) is

\[
\text{chord} \equiv [(g + h - u) (2 PP + SS), (h + f - v) (SS + 2), -(f + g - w) SS]
\]

Then compute \( tP \cdot \left( \frac{a_3 b_4 c_4}{1 - \alpha^2} \cdot \frac{\alpha^2}{1 + \alpha^2} \right) \cdot \left( S \ast \Omega \right) \cdot \left( \frac{a_3 b_4 c_4}{1 - \alpha^2} \cdot \frac{\alpha^2}{1 + \alpha^2} \right) \cdot P \), and substitute \( u = f + g + h - v - w \).

The result factorizes into \( \det \left[ \frac{t}{f} \right] \) times a perfect square. Use the usual formulas for sum and product, and conclude by obtaining:

\[
\text{chord}(t) = \left[ f + v - w, f - h + v, f + g - w \right] - 2t \mathcal{E}_\infty
\]

**Proposition 25.4.10.** When \( S \notin \mathcal{E}_\infty \), the area of \( a_3 b_4 c_4 \) is extremal exactly when the center of \( C_t \) is \( \omega \) (and the chord of contacts is \( SE \)).

Proof. Substitute \( t = \frac{fu + 2fv + gv - hw + T}{2(f + g + h)} \) in the previous results. 

**Definition 25.4.11.** When using the directions of \( SA, SB, SC \) as auxiliary directions, the results are called Catalan conics and Catalan graphs (when doing that, we assume that point \( S \) is at finite distance). Directions \( D \) and point \( \alpha \) are:

\[
D \simeq \begin{pmatrix}
-g - h & f & f \\
g & -h & f \\
h & h & -f - g
\end{pmatrix}; \quad \alpha \simeq \begin{pmatrix}
f + u \\
g - u - v + \frac{hw - gv}{t} + t \frac{f + g + h}{f} \\
\frac{h + v + \frac{gw}{f} - \frac{hw}{f} - t \frac{f + g + h}{f}}{f}
\end{pmatrix}
\]

**Proposition 25.4.12.** The Catalan graphs \( \Delta_s \), etc are the symmetries of the sidelines wrt \( \omega \), the midpoint of \( S, E \). In fact, the Catalan graphs are nothing else than the temporal graphs related to \( s = \infty \). When area of \( abc \) is extremal, the Catalan conic is centered at \( \omega \).

Proof. Make \( s \to \infty \) in \( D_s \).

**25.4.4 Tucker LFIT**

**Definition 25.4.13.** Starting from \( K = X(6) \simeq a^2 : b^2 : c^2 \) and \( t \in \mathbb{R} \), we define \( A' = K + \frac{t}{\overrightarrow{KA}} \), etc.

And then define the 6 points: \( A_b, B_c \) on \( BC \) (intersections with \( A'B' \) and \( A'C' \)), \( B_c, B_a \) on \( CA \) (intersections with \( B'C' \) and \( B'A' \)) and \( C_a, C_b \) on \( AB \) (intersections with \( C'A' \) and \( BC \)). Then

1. The polygon \( A_b A_c B_c B_a C_a B_b \) is called the Lemoine’s hexagon.
2. \([A_b, B_c, C_a]\) and \([A_c, B_a, C_b]\) are called the first and the second Brocard LFIT’s.

**Proposition 25.4.14.** Lemoine’s hexagon is inscribed in the so called Tucker circle. Its center lies on line \( OD \). Equicentre and Slowness center of both LFIT are the Brocard’s points, and the common pilar conic is the Lemoine’s inciconic.

Proof. We have the following coordinates: \( \text{LFIT}_1, \text{LFIT}_2 = \)

\[
\begin{pmatrix}
0 & -ta^2 + a^2 & tb^2 + a^2 + c^2 \\
tc^2 + a^2 + b^2 & 0 & -tb^2 + b^2 \\
-tc^2 + c^2 & ta^2 + b^2 + c^2 & 0
\end{pmatrix}; \quad \begin{pmatrix}
0 & tc^2 + a^2 + b^2 & -ta^2 + a^2 \\
-tb^2 + b^2 & 0 & ta^2 + b^2 + c^2 \\
 tb^2 + a^2 + c^2 & -tc^2 + c^2 & 0
\end{pmatrix}
\]

So that \( S_1 = a^2 b^2 : b^2 c^2 : c^2 a^2 \), etc.
Fact 25.4.15. The flat triangles which belong to the Tucker LFIT’s are the isotropes of the Brocard points.

Definition 25.4.16. The points $\tilde{a}_t, \tilde{b}_t, \tilde{c}_t$ where the temporal conic $C_t$ cuts again the sidelines of $ABC$ form themselves a LFIT, called the **Tucker associate** of the original one.

Proposition 25.4.17. The slowness center and the equicenter are exchanged when passing from the original LFIT to its Tucker associate. Therefore both LFIT have the same pilar conic.

Proof. The coordinates of the Tucker’s associate triangle are:

$$\tilde{T}_t = \begin{bmatrix} 0 & t & t + f \\ t v & -t + f & -w + f \\ -w + f & w & 0 \end{bmatrix} \text{ when } T_t = \begin{bmatrix} 0 & -t + f & t h + v h \\ t g + 1 - w - f g & -t + f & 0 \\ -t + f & w - f g & 0 \end{bmatrix}$$

so that $\tilde{S} = E$ is obvious. This implies $\tilde{E} = S$.

Corollary 25.4.18. When all temporal conics are circles, then $S$ and $E$ are isogonal conjugates.

Proof. Describe the LFIT and its Tucker associate as

$$\tilde{T}_t : \tilde{T}_t = \begin{bmatrix} 0 & t & t + F \\ t m + l & -t + F & t m + l \\ l + H - m & -n + F & l + H - m \end{bmatrix} \text{ and then write that } \tilde{a}_t \text{ belongs to circle } a_t b_t c_t \text{ for all } t.$$

One obtains a polynomial equation, whose leading term is

$$t^4 (l + f) (a^2 (f + g + h) gh - (c^2 fg + b^2 fh + a^2 gh) l)$$

Applied circularly, this shows the necessity of

$$E = \frac{f + g + h}{a^2 gh + b^2 hf + c^2 fg} \left( \begin{array}{c} a^2 gh \\ b^2 hf \\ c^2 fg \end{array} \right)$$

One can see that this is sufficient.

25.5 HH and temporal point of view

25.5.1 Temporal embedding

Definition 25.5.1. Since we are dealing with moving points $M_t$, it makes sense to define a **temporal embedding** of such points by the following projective map:

$$G \left( \begin{array}{c} x(t) \\ y(t) \\ z(t) \end{array} , t \right) \mapsto \left( x t (x+y+z) \right)$$

Proposition 25.5.2. Given a LFIT, the temporal embeddings of the variable vertices belong to a
quadric $\Omega$, whose matrix is:

$$
\begin{bmatrix}
2v(f-w) & -fh + 2fv + hw - vw & v\tau & -f - v + w \\
-fh + 2fv + hw - vw & 2f(v-h) & f\tau & -f - v + h \\
-v\tau & f\tau & 0 & -\tau \\
-f - v + w & -f + h - v & -\tau & 2
\end{bmatrix}
$$

$\Omega^*$

$$
\begin{bmatrix}
0 & \tau & \sigma & \tau f \\
\tau & 0 & \rho & \tau v \\
\sigma & \rho & 0 & fh - hw + vw \\
\tau f & \tau v & fh - hw + vw & 2f\tau v
\end{bmatrix}
$$

And, for these instanciations, we have $\det[\Omega] = \det[\Omega^*] = (\rho\sigma^2)$ (squared coordinates of $\Omega$ in the triangle plane).

**Proof.** Well known property of hyperbolic paraboloids. 

Signature is $+2;-2$. Using the completesquare algorithm, one obtains

$$
\frac{(tz)^2}{(fh - hw + vw)^2} - (twz)^2 + \left(1 - \frac{(tz)^2}{(fh - hw + vw)^2}\right)k
$$

where $t_{xyz}, t_{yz}, tz$ are linear expressions and $K$ a constant.

**Remark 25.5.3.** Matrix $[\Omega^*]$ is symmetric, while matrix $[\Omega]$ is not. One could enforce symmetry by choosing $t = 0$ when the area is extremal. But the price to pay would be a huge size for the coefficients.

**Remark 25.5.4.** The directions of sidelines of $ABC$ are embedded at

$$0 : 1 : -1 : f ; -1 : 0 : 1 : g ; 1 : -1 : 0 : h$$

**Proposition 25.5.5.** The temporal embeddings of the $s$-temporal graphs, i.e. the lines

$$\mathfrak{G}(\Delta_A^s) = \{ \mathfrak{G}(a_t^s, t) \mid t \in \mathbb{R} \}, \text{ etc}
$$

are drawn on the $\Omega$ quartic. Moreover, $\mathfrak{G}(a_t^s, t)|_{t=s} = \mathfrak{G}(a_s, s)$. Therefore $\mathfrak{G}(\Delta_A^s)$ is nothing but the other line drawn on $\Omega$ through $\mathfrak{G}(a_s, s)$.

**Proof.** The first assertion is easily computed. The second is Proposition 25.4.5. And the conclusion follows.

**Proposition 25.5.6.** A point $x : y : z$ on the triangle plane has two temporal embeddings on the quartic $\Omega$ (counting multiplicity and complex values). The critical points of this double coating of the plane are the points of the pillar conic (and the points at infinity).

**Proof.** First assertion is about the degree in $t$. Second assertion is about the discriminant, which is $(x + y + z)^2$ times the equation of the key conic.

**Proposition 25.5.7.** The embedding $\mathfrak{G}(\mathcal{C})$ of the pillar conic lies in the plane:

$$[w-f-v ; h-f-v ; -\tau ; 2]$$

The horizontal plane $[s,s,s,-1]$ cuts this curve in two points that are the embeddings of the contact points of the pillar conic with the temporal conic $\mathcal{C}(s)$. One of the horizontal lines of the polar plane is drawn through points $\mathfrak{G}(\mathcal{S}, t^*)$ and $\mathfrak{G}(\mathcal{E}, t^*)$ where $t^*$ is the parameter related to the extremal area of triangle $a_b t c_c$.

Using a parameter $k$, a point on the pillar conic can be described as: $1 : \rho : k^2 : \sigma : (1 + k)^2 : \tau$. It’s only date of embedding is

$$
t = \frac{f\rho k^2 + \sigma \rho k + v\sigma}{(f+u)\rho k^2 + 2\sigma \rho k + (g+v)\sigma}
$$

and the conclusion follows. One should remark that the plane equation is the last line of $[\Omega]$. 

— pdlx : Translation of the Kimberling’s Glossary into barycentrics ——
Definition 25.5.8. A linear motion $M(t)$ on a given line $\Delta$ is said to be an incident motion to a given LFIT when the temporal parameters at cutting points are the same on the sidelines and on the transversal.

Proposition 25.5.9. A given line $\Delta \simeq [l, m, n]$ is the support of an incident motion if and only if the line is tangent to the pillar conic. And then, the incident motion is embedded along a line drawn on $\Omega$.

Proof. Solving in $t$ the equation $[l, m, n] \cdot [0, p + t/f, (1 - p) - t/f] = 0$ tells us when point $\Delta \cap BC$ is reached. Substituting into $a_t$ and doing the same for the other two sidelines leads to the three embeddings:

$$
\begin{bmatrix}
0 & -n & -m \\
-n & 0 & l \\
m & l & 0 \\
-nf & (l - n)(f - w) + lg & (l - m)v - lh
\end{bmatrix}
$$

Asking for the degeneracy of the plane drawn by these three points gives four equations whose gcd is the tangential equation of $C$.

25.5.2 Menelaüs HH (parallelogy)

Caveat: in this subsection, vectors $P = t[p, q, r, 1]$, etc aren't projective vectors, and $P[4] = 1$ is a hard coded quantity.

Definition 25.5.10. The Menelaüs encoding of the inscribed triangle $a, b, c$ is defined by:

$$
T = (abc) \simeq \begin{pmatrix} 0 & 1 - q & r \\ p & 0 & 1 - r \\ 1 - p & q & 0 \end{pmatrix} \mapsto P = \begin{pmatrix} p \\ q \\ r \end{pmatrix}
$$

while the Menelaüs quadric is defined by the symmetric matrix:

$$
\begin{bmatrix}
0 & 1 & 1 \\
1 & 0 & 1 \\
1 & 1 & 0 \\
-1 & -1 & -1 & 2
\end{bmatrix}
$$

Proposition 25.5.11. (1) the area of an inscribed triangle is given by: area ($T$) = $S \times ^tP \cdot \begin{bmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{bmatrix} \cdot P$;
(2) two non flat inscribed triangles are parallelogic if and only if $^tQ \cdot \begin{bmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{bmatrix} \cdot P = 0$.

Proof. Direct examination.

Proposition 25.5.12. When $S \notin \mathcal{L}_\infty$, the triangles $T$ of a LFIT are parallelogic by pairs. This relation is involutive and is described by the homography:

$$
s = \begin{pmatrix} fu + 2fv + gw - hw \\ 2(h + g + f) t - (fu + 2fv + gw - hw) \end{pmatrix}
$$

Proof. Immediate consequence of $^tP \cdot \begin{bmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{bmatrix} \cdot P_s = 0$.

Proposition 25.5.13. Any flat inscribed triangle verifies $(p + q - 1)(p + r - 1) = p(p - 1)$ and determines two LFIT of flat triangles:

1. Defining $\mu = \frac{p + q - 1}{p} = \frac{p - 1}{p + r - 1}$ leads to

$$
\begin{bmatrix} T_1 \end{bmatrix}(p) = \begin{bmatrix}
0 & (1 - \mu)p & (\mu + 1 - \mu^{-1})p + 1 - \mu^{-1} \\
p & 0 & (1 - \mu^{-1})p + \mu^{-1} \\
1 - p & (\mu - 1)p + 1 & 0
\end{bmatrix}
$$

Then we have $S = \mu - 1 : 1 : -\mu ; \mathcal{E} = -S ; \Omega = \overrightarrow{0}$. All the lines are parallel and directed by $S$.

January 3, 2024 21:08 published under the GNU Free Documentation License
2. Defining \( \lambda \doteq \frac{p + q - 1}{p - 1} \) leads to

\[
\mathcal{T}_2 (p) = \begin{bmatrix} 0 & \lambda + (1 - \lambda) p & 1 + (\lambda^{-1} - 1) p \\ p & 0 & (1 - \lambda^{-1}) p \\ 1 - p & 1 - \lambda + (\lambda - 1) p & 0 \end{bmatrix}
\]

Then we have \( S = \lambda - 1 : 1 : -\lambda ; E = \overrightarrow{0} ; \Omega = -S \). All the lines are tangent to the inscribed parabola through \( S \)–whose focus is isogon (\( S \)):

\[
\mathcal{C}^* \doteq \begin{bmatrix} 0 & -\lambda & 1 \\ -\lambda & 0 & \lambda - 1 \\ 1 & \lambda - 1 & 0 \end{bmatrix}
\]

**Proof.** When \( E = K S \) then area \((\mathcal{T}_1) = -K (K + 1) S \). And so we have either \( K = -1 \) or \( K = 0 \). The fact that the \( \mathcal{C} \) contains two lines through \( P \in \mathcal{C} \) is a general property of a HH.

### 25.6 Miquel circles

**Definition 25.6.1.** Circles \((A, b_i, c_i)\) are called the \( A \)–Miquel circles of the LFIT family. Et circ. for \( B \) and \( C \). Their Veronese columns are:

\[
\text{miq}_A, \text{miq}_B, \text{miq}_C \doteq \begin{bmatrix} 0 \\ c^2 g (h - v + t) \\ b^2 h (u + v - h - t) \\ gh \\ a^2 h t \\ fh \\ fg \end{bmatrix}
\]

**Proposition 25.6.2.** The \( A \)–Miquel circles are going through \( A \) and another fixed point \( O_A \), etc. And we have \( O_A, O_B, O_C \)

\[
\begin{bmatrix} a^2 gh - b^2 hu - c^2 gu \\ b^2 h (u - g - h) \\ c^2 g (u - g - h) \end{bmatrix}
\begin{bmatrix} a^2 h (v - h - f) \\ -a^2 hv + b^2 fh - c^2 f v \\ c^2 f (-h - f + v) \end{bmatrix}
\begin{bmatrix} a^2 g (w - f - g) \\ b^2 f (w - f - g) \\ -a^2 gw - b^2 fw + c^2 fg \end{bmatrix}
\]

(25.12)

**Proof.** The existence is obvious from the linear nature of the Veronese’s (and the computation is straightforward).

**Proposition 25.6.3. Circle of similarity.** For a given \( t \), the three Miquel circles concur at a point \( \mu_t \), called the Miquel point of \( \Gamma_t \). The locus of this point is the circle \( \Gamma_\sigma \) through \( O_A, O_B, O_C \). Moreover, \( \mu_{t=\infty} \) is \( S^* \), the isogonal conjugate of \( S \), i.e. \( S^* \doteq a^2 / f : b^2 / g : c^2 / h \). Therefore, \( S^* \) is the perspector of triangles \( ABC \) and \( O_A O_B O_C \).

**Proof.** Compute the circle orthogonal to the three Miquel circles, i.e. take the wedge of their Veronese. Test that this circle is a point-circle, and that \( \mu_t \) is of second degree in parameter \( t \). Use locusconi to obtain the locus. Check that we have a circle, and obtain its Veronese, namely

\[
\Gamma_\sigma \doteq \begin{pmatrix} b^2 c^2 (g + h - u) f \\ c^2 a^2 (h + f - v) g \\ a^2 b^2 (f + g - w) h \\ a^2 gh + b^2 f h + c^2 fg \end{pmatrix} = \begin{pmatrix} b^2 c^2 f \rho \\ c^2 a^2 g \sigma \\ a^2 b^2 h \tau \\ a^2 gh + b^2 f h + c^2 fg \end{pmatrix}
\]

What remains is straightforward. The name "circle of similarity" is explained at Section 25.9. The \( \mu_t \) coordinates themselves are not really handy, except from the second degree terms:

\[
\mu_t = (a^2 gh + b^2 f h + c^2 fg) t^2 \begin{pmatrix} a^2 gh \\ b^2 fh \\ c^2 fg \end{pmatrix} + \mathbf{O} (t)
\]

—— pldx : Translation of the Kimberling’s Glossary into barycentrics ——
Proposition 25.6.4. **Miquel of a flat triangle.** When \( T_t = abtc_t \) is flat, then \( \mu_t \) belongs to the circumcircle \( \Gamma \) and conversely.

**Proof.** When \( T_t \) is a single line, circles \( \miquel_A, \) etc. are three of the ordinary Miquel circles of the quadrilateral \( ABC, T_t \). So that \( \mu_t \) belongs also to the fourth one, the circumscribed circle of \( ABC \).

Proposition 25.6.5. **Fixed point.** All the lines \( at_M_t \) are going through a fixed point \( P_a \) whose barycentrics are:

\[
P_a, P_b, P_c \approx \begin{bmatrix}
a^2gh & a^2gh - a^2hw + c^2fu & a^2gh - a^2gv + b^2fu \\
b^2fh - b^2hw + c^2gv & b^2fh & a^2gv + b^2fh - b^2fu \\
c^2hw + c^2fg - c^2gv & c^2hw + c^2fg - c^2fu & c^2fg
\end{bmatrix}
\]

Points \( E, O_a, P_a \) are aligned and \( O_a, P_a \) belong to the circle of similarity. Moreover \( S^*P_A \parallel BC \).

**Proof.** Equation of line \( at_M_t \) is

\[
\left[ \left( b^2h + c^2g \right) ft - f \left( b^2fh - b^2hw + c^2gv \right), a^2ghf - a^2ght, -a^2ght \right]
\]

This leads to \( P_a \). One can see directly that \( P_a = fghS^* + \left( b^2hw - c^2gv \right) \delta_{BC} \) and that \( O_a - P_a \) (as written here) is \( \approx E \).

## 25.7 Constructions of the inscribed triangles

**Construction 25.7.1.** Suppose now that \( S \approx f : g : h \) and \( E \approx u : v : w \) are given (with \( f + g + h = u + v + w, \) as ever). We have, inter alia, the following constructions.

1. **Miquel.** Compute the \( O_j \) from \( S, E \). When they are different, chose a point \( \mu_t \) on circle \( \Gamma_\sigma = (O_A, O_B, O_C) \). Draw the circles \( \miquel_A = (\mu_t, A, O_A) \), etc. Then \( \miquel_B, \miquel_C \) concur on \( BC \) to give point \( a_t \), etc. Condition \( O_B = O_C \) induces \( E = S^* = O_a = O_b = O_c \).

2. **Alt-Miquel.** Compute the \( P_j \) from \( S, E \). When they are different, chose a point \( \mu_t \) on circle \( \Gamma_\sigma = (P_A, P_B, P_C) \). Draw the lines \( (\mu_t, P_A) \), etc. They cut the sidelines \( BC \), etc at the required points \( a_t \), etc. Condition \( P_B = P_C \) induces \( E = S^* = O_a = O_b = O_c \) as for the Miquel construction.
3. **Catalan.** When $S$, $E$ are at finite distance, the Catalan graphs are obtained from the sidelines by a symmetry around $(S + E)/2$.

4. **Hexagonal graphs.** The A-graph $G_\alpha$ is $[-\rho, h - \rho, g - \rho]$.

### 25.8 The three similarities theorem

This section has been inspired by Rouché and de Comberousse (1922b, Note III, p. 632-5)

**Definition 25.8.1.** Let be given three similarities $\sigma_a, \sigma_b, \sigma_c$, intended to act on three figures $\Phi_j$ according to

$$
\Psi_a \overset{\sigma_c}{\to} \Phi_b \overset{\sigma_a}{\to} \Phi_c \overset{\sigma_b}{\to} \Psi_a \quad \text{where} \quad \sigma_c \cdot \sigma_b \cdot \sigma_a = \text{Id}
$$

We can consider the $\Phi_j$ as the images of a main figure $\Phi$ by three independent similarities $\sigma_1, \sigma_2, \sigma_3$. One can also introduce three other figures $\Phi'_j$ according to Figure 25.5. In what follows, similarity $\psi$ will be forgotten most of the time.

**Remark 25.8.2.** In this section, the centers (fixed points) $O_a, O_b, O_c$ are the main objects, and will be parametrized by $\rho, \sigma, \tau$ on the unit circle (called the circle of similarity and noted $\Gamma_{\sigma}$), while the $\omega^2$ of the angles of the similarities are noted $\beta/\gamma, \gamma/\alpha, \alpha/\beta$. Thus, triangle $ABC$ is only defined up to a rotation. In the same vein, the ubiquitous quantity $\delta \in \Gamma$

$$
\delta \overset{\text{def}}{=} -\frac{(\sigma - \tau) \alpha + (\tau - \rho) \beta + (\rho - \sigma) \gamma}{\rho (\sigma - \tau) \beta \gamma + \sigma (\tau - \rho) \alpha \gamma + \tau (\rho - \sigma) \alpha \beta}
$$

is defined up a counter-wise rotation, so that only quantities like $\alpha \delta \rho$ are well-defined.

**Remark 25.8.3.** Among the figures, we will take later $\Phi_a = BC$, $\Phi_b = CA$, $\Phi_c = AB$, so that $\sigma_a$ sends $CA$ onto $AB$ and the angle of $\sigma_a$ is $(AC, AB) = -\bar{A}$. And one can check that $(\text{id/conj}) ((\beta - \alpha) / (\gamma - \alpha)) = \beta/\gamma$.

![Figure 25.5: The "three" similarities configuration](image)

**Proposition 25.8.4.** The matrices of the three similarities are:

$$
\begin{pmatrix}
\beta (\alpha \tau - \gamma \rho) (\rho - \sigma) & \rho - \beta (\alpha \tau - \gamma \rho) (\rho - \sigma) & \gamma (\alpha \sigma - \beta \rho) (\rho - \tau) \\
\gamma (\alpha \sigma - \beta \rho) (\rho - \tau) & 0 & \rho (\alpha \sigma - \beta \rho) (\rho - \tau) \\
0 & \frac{1}{\rho} & (\alpha \tau - \gamma \rho) (\rho - \sigma)
\end{pmatrix}
$$

so that equalities $\alpha / \beta = \rho / \sigma$, etc are to be excluded. The Neuberg relation:

$$
\forall M \in \Phi : (f \sigma_1 + g \sigma_2 + h \sigma_3) (M) = \mathcal{E}
$$
defining the equicenter $\mathcal{E}$ is satisfied by:

$$\begin{pmatrix} f \\ g \\ h \end{pmatrix} \simeq i \begin{pmatrix} (\beta \tau - \gamma \sigma) (\beta - \gamma) \\ \beta \gamma (\sigma - \tau) \\ (\gamma \rho - \sigma \tau) (\gamma - \alpha) \\ \alpha \gamma (\tau - \rho) (\alpha - \beta) \\ \alpha \beta (\rho - \sigma) \end{pmatrix} \in \mathbb{R}^3; \quad \mathcal{E} \simeq \begin{pmatrix} \rho (\sigma - \tau) \alpha + \sigma (\tau - \rho) \beta + \tau (\rho - \sigma) \gamma \\ (\sigma - \tau) \alpha + (\tau - \rho) \beta + (\rho - \sigma) \gamma \end{pmatrix} \begin{pmatrix} \beta \gamma + \sigma (\tau - \rho) \gamma \alpha + (\rho - \sigma) \alpha \beta \end{pmatrix}$$

Proof. Write $\sigma_a = \text{simil}(O_a, k, \mu)$, $\sigma_b = \text{simil}(O_b, K, \lambda)$, $\mu^2 = \beta / \gamma$, $\lambda^2 = \gamma / \alpha$ and assume that $O_c$ is the fixed point of $\sigma_c$. This gives the matrices. For the Neuberg relation, one can check that $(f^2/g^2 - g + h^2) = \mathcal{E} \cdot L_z$.

**Proposition 25.8.5.** Exceptional cases are of two kinds

1. Except 1: $\rho : \sigma : \tau \simeq \alpha : \beta : \gamma$ occurs when $\mathcal{E} \in L_z$ (and $\delta$ is undetermined)

2. Except 2: $\rho = \sigma = \tau$ occurs when centers $O_a, O_b, O_c$ are collinear

Proof. Equation $f + g + h = 0$ is equivalent to $\text{num} (\delta) \times \text{denom} (\delta)$ (see (25.14)). These two quantities are conjugate of each other, so they vanish together. Solving gives the property: $ABC$ equal to $O_a, O_b, O_c$, up to a rotation.

**Proposition 25.8.6.** The R-C hodograph. Choose a point $E$ in the plane (the handle of the hodograph) and $M, N$ in figure $\Phi$. The first modular triangle $E_j$ is defined by $E_a = E + \phi_1 MN$, $E_b = E + \phi_2 MN$, $E_c = E + \phi_3 MN$. And the second modular triangle $F_j$ is defined by $EE_a \perp F_b, F_c$, etc. Then:

1. When $E$ is translated to $E'$, the whole R-C hodograph is translated by $EE''$

2. We have $f : g : h \simeq \det (E, E_b, E_c) : \det (E_a, E, E_c) : \det (E_a, E_b, E)$ while the $\omega^2$ of $E_a E_b E_c$ are $\sigma \tau \alpha : \tau \rho \beta : \rho \sigma \gamma$. When $\mathcal{E} \in L_\infty$, triangle $E_a E_b E_c$ is flat.

3. Triangle $F_a F_b F_c$ is similar to triangle $ABC$. Their common $\omega^2$ are $\beta \gamma : \gamma \alpha : \alpha \beta$. When the $O_j$ are collinear, then $F_a = F_b = F_c$.

4. From $E$, you see the $F_j$ at $\omega^2 = \rho : \sigma : \tau$, i.e. we have: $\angle (E F_b, E F_c) = \angle (O_a O_b, O_a O_c)$

Proof. (1) is obvious; (2) is easy to compute, and more powerful than only $(f + g + h) E = f E_a + g E_b + h E_c$; (3) describes why triangle $F_a F_b F_c$ is useful. Incantation for computing the $\omega^2$: \( U \rightarrow \text{reduce}\) (U.mWW[1]);

**Proposition 25.8.7.** Consider the triples $\sigma_1 (M), \sigma_2 (M), \sigma_3 (M)$ where two elements are equal. We have three cases (in column):

<table>
<thead>
<tr>
<th>$\text{to_next_point}$</th>
<th>fixed</th>
<th>adjunct</th>
</tr>
</thead>
<tbody>
<tr>
<td>$A$</td>
<td>$\sigma_a$</td>
<td>$O_a'$</td>
</tr>
<tr>
<td>$B$</td>
<td>$\sigma_a$</td>
<td>$O_a$</td>
</tr>
<tr>
<td>$C$</td>
<td>$\sigma_a$</td>
<td>$O_a$</td>
</tr>
</tbody>
</table>

Pairs $O_j O_j'$ are aligned with $\mathcal{E}$ due to the Neuberg relation. Triangles $O_a' O_b O_c$ are similar with each other and with triangle $O_a, O_b, O_c$.

$$O_a' = \begin{bmatrix} \frac{\tau \sigma (\beta - \gamma)}{\beta \tau - \sigma \gamma} - \frac{(\sigma - \tau) \beta \gamma}{(\tau \beta - \sigma \gamma) \alpha} \\ \frac{1}{\beta \tau - \sigma \gamma} - \frac{(\sigma - \tau) \alpha}{(\tau \beta - \sigma \gamma) \rho} \end{bmatrix}$$

Proof. Straightforward computation. R-C are using angles not so clearly defined, remarking that $\sigma_c$ maps $[O_c, O_a]$ onto $[O_c, O_a]$, so that triangle $[O_c, O_a, O_b]$ is similar to triangle $[E, E_a, E_b]$. □
Definition 25.8.8. Points $O_j'$ are called the adjunct points. The circle $\gamma_a \simeq (O'_a, O_b, O_c)$ is called the adjunct circle. It goes through the equicenter $E$. Its center is called $\eta_a$ and we have $\sigma_c(\eta_a) = \eta_b$ where

$$
\eta_a \simeq \begin{bmatrix}
\gamma - \beta \\
(\sigma + \tau) (\beta - \gamma) \\
(\gamma - \beta) \sigma \tau \\
\beta \tau - \gamma \sigma
\end{bmatrix} ;
\eta_a \simeq \begin{bmatrix}
(\beta - \gamma) \sigma \tau \\
\beta \tau - \gamma \sigma \\
\beta - \gamma
\end{bmatrix}
$$

(25.15)

Corollary 25.8.9. Circle $(E, O_b, O_c)$ is the $A$-adjunct circle, while its second intersection with line $EO_a$ is $O'_a$, the $A$-adjunct point.

Proposition 25.8.10. The trigone formed by three corresponding lines $L_j$ is ever perspective with triangle $O_aO_bO_c$ and their perspector $L_q$ (to be celebrated as $S^*$) belongs to circle $\Gamma$.

Proof. Writing $L_a \simeq [f ; g ; h]$, we obtain $z(L_q) = a\delta (h/f)$, see (25.14).

Proposition 25.8.11. When three corresponding lines $L_a$, $L_b$, $L_c$ are concurrent, their common point $L_q$ lies on the circle $\Gamma$, while each line $L_j$ goes through the second intersection $P_j$ of $EO_j$ and $\Gamma$. And we have: $z(P_a) = \frac{1}{a\delta}$ (see (25.14)) so that triangle $P_aP_bP_c$ is skew similar with $ABC$.

Proof. Obviously, $z(L_q) = a\delta (h/f)$.

Proposition 25.8.12. The triangle formed by three corresponding points $M_j$ is ever perspective with the fixed triangle $P_aP_bP_c$, and the perspector $\mu$ (Miquel) belongs to circle $\Gamma$.

Proof. One obtains: $z(M_q) = (a \delta z_1 - t_1) \div (a \delta t_1 - z_1)$.

Proposition 25.8.13. When three corresponding points $M_j$ are aligned then $M_a$ belongs to adjunct circle $\gamma_a = (O'_aO_bO_c)$, etc. And conversely. Moreover, the $M_j$ are aligned with $E$ (Neuberg property).

Proof. Consider $\det (M_aM_bM_c)$ and obtain the locus. Then identification is easy from (25.15).
25.9 Similarities and Cremona transforms

Now, we describe points $A, B, C$ using the Lubin’s parametrization, i.e. $A \simeq \alpha : 1 : 1/\alpha$ in the $\mathbb{Z} : T : \mathbb{Z}$ complex projective plane. We are giving both equations using $f, g, h, p, q, r$ and using $f, g, h, u, v, w$ (the equicenter).

**Proposition 25.9.1. Three similarities theorem.** Let $O_\alpha, O_\beta, O_\gamma$ be three generic points, not on the sidelines. Define the similarity $\sigma_A$ by its center $O_\alpha$ together with $CA \mapsto AB$, etc. Note $\mu = \sigma_C \cdot \sigma_B \cdot \sigma_A$. Then

1. $\mu$ is an homothety translation that maps $CA$ onto $CA$.
2. $\mu$ is a translation if and only if lines $AO_A, BO_B, CO_C$ concur at some point $S^*$. 
3. $\mu$ is the identity if and only if, moreover, $S^*$ belongs to circle $O_\alpha O_\beta O_\gamma$ (the so-called circle of similarity).

![Figure 25.7: Circle of similarity](image)

**Proof.** Let $A_\alpha, A_\beta$ be the orthogonal projections of $O_\alpha$ on $AB$ and $AC$. We compute $\sigma_A$ as the pointwise collineation $\Omega_x \mapsto \Omega_x, \Omega_y \mapsto \Omega_y, O_A \mapsto O_A, A_\alpha \mapsto A_\beta$. Or as the inverse of the linewise collineation $\mathcal{L}_A \mapsto \mathcal{L}_A, \Omega_x O_A \mapsto \Omega_x O_a, \Omega_y O_a \mapsto \Omega_y O_a, AC \mapsto AB$. We get:

$$
\sigma_A \simeq \begin{pmatrix}
\alpha \beta \zeta_1 - (\alpha + \beta) t_1 + z_1 & \frac{z_1 (\gamma - \beta) (\alpha \zeta_1 - t_1)}{t_1 (\alpha \gamma \zeta_1 - (\alpha + \gamma) t_1 + z_1)} & 0 \\
\alpha \gamma \zeta_1 - (\alpha + \gamma) t_1 + z_1 & \frac{1}{t_1 (\alpha \gamma \zeta_1 - (\alpha + \gamma) t_1 + z_1)} & 0 \\
0 & \frac{\zeta_1 (\gamma - \beta) (\alpha t_1 - z_1)}{(\alpha \gamma \zeta_1 - (\alpha + \gamma) t_1 + z_1) \beta} & \frac{\gamma (\alpha \beta \zeta_1 - (\alpha - \beta) t_1 + z_1)}{(\alpha \gamma \zeta_1 - (\alpha + \gamma) t_1 + z_1) \beta}
\end{pmatrix}
$$

1. Obvious since $\mu$ fixes $\Omega_x, \Omega_y$ and $\delta(AC)$. The factor of homothety is given by $\lambda$, the product of the three $(1,1)$ elements or by the product of the three $(3,3)$ elements of the matrices.
2. Let $A_2 = BC \cap AO_A$. Then $\frac{A_2 B}{A_2 C} = \text{birap}(B, C, A_2, \infty) = \text{birap}(\delta CA, \delta AB, \delta BC, \delta AO_a) = \gamma (\alpha - \beta) (\alpha \beta \zeta_1 - (\alpha + \beta) t_1 + z_1) \\
\beta (\alpha - \gamma) (\alpha \gamma \zeta_1 - (\alpha + \gamma) t_1 + z_1)$

Clearly, $-\lambda$ is the product of these three quantities. And we conclude by the Ceva theorem.
3. Define $S^* = AO_A \cap BO_B, OC = x S^* + (1 - x) C$, obtain $\mu = 1$ as a first degree equation in $x$ and conclude. 

January 3, 2024 21:08 published under the GNU Free Documentation License
Proposition 25.9.2. *Similarities.* Given a LFIT, the correspondence \( CA \rightarrow AB : b \rightarrow c \) induces a similarity \( \sigma_A \) of the whole plane. Center is \( O_A \), the fixed point of the Miquel circles \((25.12)\), while angle and ratio are, respectively, \(-A\) and \(-\alpha \gamma / \beta h\). In the Morley frame, the matrix of \( \sigma_A \) is:

\[
\sigma_A = \begin{pmatrix}
-\frac{g(\alpha - \beta)}{h(\alpha - \gamma)} & \alpha r + (1 - r) \beta + \frac{(\alpha - \beta)(\gamma + (1 - q) \alpha) g}{h(\alpha - \gamma)} & 0 \\
0 & \frac{r}{\alpha} + \frac{1 - r}{\beta} + \frac{1}{(\alpha - \gamma) \alpha \beta h} & \frac{(\alpha - \beta) \gamma h}{(\alpha - \gamma) \beta}
\end{pmatrix}
\]

The equicentric property holds for the whole plane, since:

\[
f[1] + g \sigma_C + h \sigma_B^{-1} = \mathcal{E} \cdot \mathcal{L}_\infty
\]

**Proof.** The quicker is to describe \( \sigma_A \) as the collineation: \( \Omega_x \rightarrow \Omega_z, \Omega_y \rightarrow \Omega_z, b_0 \mapsto c_0, b \mapsto c \) and apply the general formulas. The obvious result \( \sigma_C \sigma_B \sigma_A = 1 \) can be used to check the obtained matrices.

Proposition 25.9.3. *Skew-similarities.* The correspondence \( CA \rightarrow AB : b \mapsto c \) induces a skew similarity \( \sigma_A^* \) of the whole plane. Center is

\[
\sigma_A^* \simeq \begin{pmatrix}
\frac{h(\gamma g - hr) b^2 - g(\gamma g - hr - g + h) c^2}{(\gamma g - hr + h) b^2}
& -\frac{(\gamma g - hr - g + h) b^2}{(\gamma g - hr + h) b^2}
\end{pmatrix}
\]

while \( \sigma_C \sigma_B \sigma_A \) is the involutory affinity that fixes \( AC \) and reverts the direction of \( BH \).

Proposition 25.9.4. *Miquel homography.* Assume that \( S \notin \mathcal{L}_\infty \) and define \( \sigma \) as the homography: \( A \mapsto O_A, B \mapsto O_B, C \mapsto O_C \). The matrix of \( \sigma \) in the upper spherical map \( (\mathbf{Z}, \mathbf{T}) \) is:

\[
\sigma = \begin{pmatrix}
z_e & \sigma_3 z_S + z_e - z_H \\
1 & -z_S
\end{pmatrix} ; \quad \sigma^{-1} = \begin{pmatrix}
z_S & \sigma_3 z_S + z_e - z_H \\
1 & -z_e
\end{pmatrix}
\]

where \( z_e = \frac{u \alpha + v \beta + w \gamma}{u + v + w} \), \( z_S = \frac{f \alpha + g \beta + h \gamma}{f + g + h} \), \( z_H = \sigma_1 \).

This defines a Cremona transform \( \tilde{\sigma} \) of the whole projective plane, whose indeterminacy points are the umbilics and \( S \) while exceptional lines are \( \mathcal{L}_\infty \) and the isotropic lines of \( S \). By \( \sigma^{-1} \), the center \( \omega \) of circle \( O_A O_B O_C \) is mapped to \( 1/\zeta_S \) (symmetric of \( S \) in the circumcircle).

The four fixed points of \( \sigma \) are the foci of the pillar conic and therefore verify:

\[
\tau Z - T_\gamma + \sigma Z - T_\beta + \rho Z - T_\alpha = 0
\]

**Proof.** Direct computation.

Proposition 25.9.5. *Homography \( \sigma \) degenerates when \( S \) and \( E \) are isoconjugates (thus none of them at infinity). In this case, the Miquel circle is reduced to point \( E = S^* = O_\alpha = O_\beta = O_\epsilon \).

**Proof.** Obvious from \((17.5)\) (when assuming \( S \notin \mathcal{L}_\infty \)). Assuming \( S \in \mathcal{L}_\infty \), this would require \((u \alpha + v \beta + w \gamma)(\alpha f + \beta g + \gamma h) = 0\), i.e. \( E \) at origin, instead of \( E \in \mathcal{L}_\infty \).
25.10 Describing a LFIT from its degenerate triangles

25.10.1 General results

**Fact 25.10.1.** In the general case, a LFIT contains two degenerate triangles $T_j$ defining two lines $\Delta_j \simeq [p_j, q_j, r_j]$. Using an adapted timeline, we can enforce $t = \pm 1$ for these triangles and then:

$$
T_t = \frac{1 + t}{2} \begin{bmatrix}
    0 & r_1 & -q_1 \\
    -r_1 & r_1 - p_1 & -q_1 \\
    q_1 - r_1 & 0 & p_1 - q_1 \\
    q_1 - r_1 & -p_1 & 0 \\
\end{bmatrix} + \frac{1 - t}{2} \begin{bmatrix}
    0 & r_0 & -q_0 \\
    -r_0 & r_0 - p_0 & -q_0 \\
    q_0 - r_0 & 0 & p_0 - q_0 \\
    q_0 - r_0 & -p_0 & 0 \\
\end{bmatrix}
$$

Using (25.1), we obtain the following synchronized values:

$$
S = \begin{pmatrix}
    (p_0 q_1 - q_0 p_1)(r_0 p_1 - p_0 r_1)(q_1 - r_1)(q_0 - r_0) \\
    (q_0 r_1 - r_0 q_1)(p_0 q_1 - q_0 p_1)(r_1 - p_1)(r_0 - q_0) \\
    (r_0 p_1 - p_0 r_1)(q_0 r_1 - r_0 q_1)(p_1 - q_1)(p_0 - q_0) \\
    (p_0 q_1 - q_0 p_1)(q_0 r_1 - r_0 q_1)(r_0 - q_0)(p_1 - q_1) \\
\end{pmatrix}
$$

$$
E = \Theta_2 \begin{bmatrix}
    q_0 r_1 - r_0 q_1 \\
    r_0 p_1 - p_0 r_1 \\
    q_0 q_1 \\
\end{bmatrix} ; \quad \Omega = \Theta_1 \begin{bmatrix}
    p_0 p_1(q_0 r_1 - r_0 q_1) \\
    q_0 q_1(r_0 p_1 - p_0 r_1) \\
    r_0 r_1(p_0 q_1 - q_0 p_1) \\
\end{bmatrix}
$$

$$
\mathcal{E} = \begin{bmatrix}
    0 & r_0 r_1(p_0 q_1 - q_0 p_1) & q_0 q_1(r_0 p_1 - p_0 r_1) \\
    r_0 r_1(p_0 q_1 - q_0 p_1) & 0 & p_0 p_1(q_0 r_1 - r_0 q_1) \\
    q_0 q_1(r_0 p_1 - p_0 r_1) & p_0 p_1(q_0 r_1 - r_0 q_1) & 0 \\
\end{bmatrix}
$$

where $\Theta_1 \doteq (p_0(q_1 - r_1) + q_0(r_1 - p_1) + r_0(p_1 - q_1))$

$\Theta_2 \doteq (p_0 q_0 r_1(p_1 - q_1) + q_0 r_0 p_1(q_1 - r_1) + r_0 p_0 q_1(r_1 - p_1))$

**Remark 25.10.2.** One has $\mathcal{L}_\infty \cdot S = \mathcal{L}_\infty \cdot E = \mathcal{L}_\infty \cdot \Omega = \Theta_1 \times \Theta_2$. Then $\Theta_1$ vanishes when the lines are parallel, while $\Theta_2$ vanishes when the reciprocal of the lines are parallel.

**Exercise 25.10.3.** Point $\omega = (S + E) / 2$ is the intersection of lines $cevadiv(T_j, \mathcal{L}_\infty)$. Spoiler: these lines are the Newton axes of quadrilaterals $(ABC, T_j)$.

**Proposition 25.10.4.** The temporal conic relative to a flat triangle $T_j$ is the reunion of two parallels, each of them tangent to the pillar conic $\mathcal{E}$: the line $T_j$ itself (which goes through $E$) and its symmetric wrt $\omega = (S + E) / 2$ (which goes through $S$).

**Proof.** The only safe proof is a direct computation, which is not difficult. Formula (25.11) cannot be used since $\det T_j = 0$. While describing what happens when $t \to 0$ is enlightening... but doesn’t prove anything. $\square$

**Theorem 25.10.5.** The two flat triangles (visible or not) that belongs to a LFIT are the tangents from equicenter $\mathcal{E}$ to the pillar conic $\mathcal{E}$.

**Proof.** This is only a paraphrase of the former proposition. Nevertheless, this is a key result. One can also note that $\mathcal{E}$ satisfies $(u + v + w) \mathcal{E} = f_{a_j} + g_{b_j} + h_{c_j}$ so that $\mathcal{E} \in T_j$ is an obvious requirement. $\square$

**Proposition 25.10.6.** About temporal conics. Applying (25.11), we have

$$
\mathcal{E}^H \simeq \begin{pmatrix}
    0 & h \tau & g \sigma \\
    h \tau & 0 & f \rho \\
    g \sigma & f \rho & 0 \\
\end{pmatrix}
$$

$$
\begin{pmatrix}
    0 & (p_1 - q_1)(p_0 - q_0)r_0 r_1 & (p_1 - r_1)(p_0 - r_0)q_0 q_1 \\
    (p_1 - q_1)(p_0 - q_0)r_0 r_1 & 0 & p_0 p_1(q_0 - r_0)(q_0 - r_0) \\
    (p_1 - r_1)(p_0 - r_0)q_0 q_1 & p_0 p_1(q_0 - r_0)(q_0 - r_0) & 0 \\
\end{pmatrix}
$$

January 3, 2024 21:08 published under the GNU Free Documentation License
whose points at infinity are
\[
\delta_0^H, \delta^H = \begin{pmatrix} p_0 (q_0 - r_0) \\ q_0 (r_0 - p_0) \\ r_0 (p_0 - q_0) \end{pmatrix}, \begin{pmatrix} p_1 (q_1 - r_1) \\ q_1 (r_1 - p_1) \\ r_1 (p_1 - q_1) \end{pmatrix}
\]

Except from the two flat triangles (where, in fact, \( \Gamma^H_i = 0 \)), these points generate the points at infinity of the temporal conic \( \mathcal{C}_i \) itself:

\[
\delta_0 = \begin{vmatrix} T_i \end{vmatrix} \cdot \delta_0^H = \begin{vmatrix} T_i \end{vmatrix} = \begin{pmatrix} q_0 - r_0 \\ r_0 - p_0 \\ p_0 - q_0 \end{pmatrix} ; \delta_1 = \begin{vmatrix} T_i \end{vmatrix} \cdot \delta^H = \begin{vmatrix} T_i \end{vmatrix} = \begin{pmatrix} q_1 - r_1 \\ r_1 - p_1 \\ p_1 - q_1 \end{pmatrix}
\]

**Proof.** Line at infinity is invariant by the collineation described by matrix \( \begin{vmatrix} T_i \end{vmatrix} \). And thus we can avoid the direct computation of the points at infinity of the temporal conics. Once again: this doesn’t apply to the flat triangles. \( \square \)

**Proposition 25.10.7.** Assume that \( \mathcal{E} \) (and thus \( \mathcal{S} \)) remains at finite distance. Then all the following assertions are equivalent:

1. the LFIT contains only one degenerate triangle (and it happens that line \( T_0 \) is tangent to \( \mathcal{C} \) at \( \mathcal{E} \));
2. circumscribed \( \Gamma \) and similarity \( \Gamma_\sigma \) circles are tangent (the contact occurs at \( \mu_0 \)) ;
3. relation \( f^2 u^2 + g^2 v^2 + h^2 w^2 - 2 f g w v - 2 g h w w - 2 h f w u = 0 \) holds;
4. point \( \mathcal{E} \) belongs to \( \mathcal{C} \) (and therefore \( \mathcal{S} \in \mathcal{C} \) also) ;
5. point \( \mathcal{S} \) belongs to the ABC-inconic centered at \((A + B + C - \mathcal{E})/2 \) (and conversely).

**Proof.** (1-2) is from Proposition 25.6.4; (3) is from the discriminant of \( \det \begin{vmatrix} T_i \end{vmatrix} \); (4-5) are straightforward. \( \square \)

**25.10.2 Miscellany**

**Exercise 25.10.8.** All the previous results were obtained ”without circles". But we can identify the adjunct circle \( \gamma_\alpha \) as the circle through \( \mathcal{E}, a_0, a_1 \), etc, point \( O_\alpha \) as the other intersection of \( \gamma_\beta \) and \( \gamma_\gamma \), etc, point \( O'_\alpha \) as the other intersection of line \( \mathcal{E} O_\alpha \) and circle \( \gamma_\alpha \). The similarity circle \( \Gamma_\sigma \) is the circle \( O_\alpha O_\beta O_\gamma \), while fixed points \( \mathcal{P}_\alpha \), etc are the other intersection of line \( \mathcal{E} O_\alpha \) and circle \( \Gamma_\sigma \). And now, the moving part of the system: Miquel circles \( \text{miq}_A = (A, b_t, c_t) \), etc concur at \( \mu_\epsilon \in \Gamma_\sigma \), as well as lines \( \mathcal{P}_\alpha a_t \), etc.

**25.10.3 Critical triangle**

**Definition 25.10.9.** When \( \mathcal{S} \) is not at infinity, \( \text{area}(T_i) \) is second degree in \( t \), and presents an extremum. The corresponding triangle is called the critical triangle of the LFIT, and noted \( T_c \).

**Proposition 25.10.10.** Consider \( T_0, T_1 \) the tangents issued from \( \mathcal{E} \notin \mathcal{L}_\infty \) to the pillar conic \( \mathcal{E} \) (see Figure 25.8). Let \( a_0, a_1 \), etc be the intersections (visible or not) of these tangents with the sideline \( BC \). Then critical triangle \( T_c \) is given by \( a_c = (a_0 + a_1)/2 \), etc.

**Proof.** In the real domain, a second degree polynomial attains its extremum at the middle of its roots. \( \square \)

**Corollary 25.10.11.** Applied to the pedal triangles of the points \( P_t \) of a line \( \Delta \), the critical triangle is obtained at the projection \( P_0 \) of the circumcenter \( O \) onto the given line.

**Proof.** Line \( OP_0 \) is the axis of symmetry of the figure formed by \( \Gamma \) and \( \Delta \). \( \square \)
From $P_a$ to $A_x$ through $\omega$. And then from $A_x$ to $a_c$ through $\mathcal{E}$.

Figure 25.8: Constructing the critical triangle

**Construction 25.10.12.** Construct $\mathcal{T}_{\text{crit}}$ when $S, \mathcal{E}$ are known, but not at infinity. It suffices to construct the middle of a subtangent as described at Construction 17.1.3. Obtain the contacts $P_a$, etc of the pillar conic from its perspector which is isotom $\Omega$. Draw also $\Delta$, the conipolar of $\mathcal{E}$ (see Figure 25.8). Cut this line by line $\omega P_a$ and obtain $A_x$, etc. Then we have

$$a_c = E A_x \cap BC,$$

**Proof.** This works even if $E$ is inside $\mathcal{E}$, i.e. when the flat triangles of the family are not visible. We have the following coordinates:

$$P_a \simeq \begin{pmatrix} 0 \\ (f + u)(u\rho - (v + w)\tau) - (g + v)u(\sigma + \tau) \\ (f + u)(u\rho - (v + w)\sigma) - (h + w)u(\sigma + \tau) \end{pmatrix},$$

$$A_x \simeq \begin{pmatrix} u(\sigma + \tau) \\ u\rho - (v + w)\tau \\ u\rho - (v + w)\sigma \end{pmatrix}; \quad a_c \simeq \begin{pmatrix} 0 \\ u\rho - (v + w)\tau - v(\sigma + \tau) \\ u\rho - (v + w)\sigma - w(\sigma + \tau) \end{pmatrix}.$$

Moreover, a detailed proof has already been given at Construction 17.1.3. Obviously, we re-obtain (25.4).

**Remark 25.10.13.** The point $A_x$ is the conipole of the line $(\mathcal{E}, B C)$.

**Proposition 25.10.14.** When the critical triangle is known as

$$\mathcal{T}_{\text{crit}} = \begin{pmatrix} 0 & 1 - q & r \\ p & 0 & 1 - r \\ 1 - p & q & 0 \end{pmatrix},$$

then the slowness center is constrained to the circumconic $\mathfrak{KS}$ whose perspector and center are:

$$\text{persp} \simeq \begin{pmatrix} 1 - q - r \\ 1 - r - p \\ 1 - p - q \end{pmatrix}; \quad \text{cent} \simeq \begin{pmatrix} (1 - q - r)(1 - 2p) \\ (1 - r - p)(1 - 2q) \\ (1 - p - q)(1 - 2r) \end{pmatrix}.$$
One can parametrize $S, E$ as

$$ S \simeq \left( \frac{K (1 + K) (1 - q - r)}{1 + K (1 - r - p)} \right); \ E = T_{crit} \cdot S $$

and $E$ is constrained to $R_S = T_{crit}^{-1} \cdot R_S \cdot T_{crit}^{-1}$. This conic is nothing else than the hexagonal conic of $T_{crit}$ that goes through the $\alpha_m, etc$ and the $\alpha_m \equiv b_m + c_m - A, etc$.

Proof. One must ensure that the area is a polynomial whose first degree coefficient is null. Caveat: the list $[p, q, r]$ is not to be treated up to a projective multiplier.

25.10.4 Orthologic families

**Proposition 25.10.15.** The lines that support the flat triangles are orthogonal if and only if $S$ and $E$ are conjugate with respect to the polar circle. In such a case, all temporal conics are rectangular hyperbolas... except from the degenerate ones

Proof. Use parametrization (25.1) where $T_0, T_1$ are the flat triangles. Then we have

$$ T \cdot s \cdot cir_{H} \cdot E = \mathcal{L}_0 \cdot \mathbf{M} \cdot T_1 \times \Theta_2 \times \prod_3 (p_0 q_1 - q_0 p_1) $$

where $\Theta_2$ was defined at Remark 25.10.2.

**Proposition 25.10.16.** When two non flat triangles of a LFIT are orthologic, then all triangles of the LFIT are orthologic with each other. Moreover, $S$ and $E$ are conjugate wrt the polar circle (and the flat triangles orthogonal to each other).

Proof. By Proposition 24.4.9, orthology between non degenerate triangles is characterized by trace $\left( \prod_0 T_0 \cdot \prod_1 \cdot \prod_0 \right) = 0$. When using the asymmetric parametrization, this gives:

$$ \frac{t - s}{f g h S} \left( \begin{array}{ccc} S & 0 & 0 \\ 0 & S_b & 0 \\ 0 & 0 & S_c \end{array} \right) \cdot E = 0 $$

**Proposition 25.10.17.** Assume that preceding condition is fulfilled and note $P(t, s)$ the point which sees triangle $T_t$ at right angle to trigone $T_s$. The barycentrics of $P(t, s)$ have degrees in $t$ and $s$ that are, respectively, 1 and 2. The locus $t \mapsto P(t, s)$ is one of the two lines tangent to the pillar conic through the orthocenter of $a_s b_c t_s$, while the locus $s \mapsto P(t, s)$ is the temporal conic $C_s$.

Proof. The linear motion $t \mapsto P(t, s)$ is incident to the linear motions of $a_s b_c t_s$ (see Proposition 25.5.9). And therefore the locus is a line tangent to the pillar conic, while $P(s, s)$ is obviously the orthocenter of $a_s b_c$. On the other hand, $P(t, t) \in C_s$ was granted since this conic is a RH.

25.11 Envelopes of the sidelines (parabolas)

**Proposition 25.11.1.** The envelope of the line $b_c t_s$ is a parabola $\mathcal{P}_A$. Its point at infinity is $g + h : -g : -h, i.e.$ the direction of line $SA$. Its tangential equation is

$$ \mathcal{P}_A \simeq \left( \begin{array}{ccc} 2 (q - 1) g - 2 hr & (1 - q) g - (1 - r) h & hr - gq \\ (1 - q) g - (1 - r) h & 0 & (1 - r) h + gq \\ hr - gq & (1 - r) h + gq & 0 \end{array} \right) \left( \begin{array}{ccc} 2 u & h - u & g - u \\ h - u & 0 & -g - h + u \\ g - u & -g - h + u & 0 \end{array} \right) $$

— pldx : Translation of the Kimberling’s Glossary into barycentrics ——
This parabola is tangent to $AB$ and $AC$ at their intersections with locus $(\alpha)$. When using metric properties, the focus is the already encountered point $O_A$, while its directrix is

$$
\Delta_A \simeq [S_a (g + h) , -S_b g, -S_c h] - u S_a L_\infty \quad \text{where} \ u = \xi_1 = hv - gq + g
$$

**Proof.** The coefficients of line $bc$ have degree 2 in $t$. Therefore, the envelope is a conic. Matrix $[\mathcal{F}_A]$ is obtained by locusconi. Then $L_\infty \cdot [\mathcal{F}_A] = -g - h : g : h$ and we have a parabola. The focus comes from the Plucker method, and the directrix is the polar line of the focus.

And circularly for the other two parabolas.

**Proposition 25.11.2.** Tangents from $E$ to the three parabolas are the same.

**Proof.** The contact points themselves have a terrific expression, involving

$$
W = \sqrt{t^2 u^2 + g^2 v^2 + h^2 w^2 - 2 fg uv - 2 gh wv - 2 fw uv}
$$

Nevertheless, one can see that the three expressions

$$(E \cdot \mathcal{P}_j \cdot \mathcal{E}) (X \cdot \mathcal{P}_j \cdot X) - (E \cdot \mathcal{P}_j \cdot X)^2
$$

giving the pair of tangents issued from $E$ are the same.

\section*{25.12 Special shapes of the inscribed triangles}

\subsection*{25.12.1 LFIT of equilateral triangles}

**Proposition 25.12.1.** For a given triangle $ABC$, there are two LFIT of equilateral triangles. They are described by: $a, b, c \simeq$

$$(S_a + 2 \Sigma) \begin{pmatrix} 0 \\ x \\ 1 - x \end{pmatrix}, \begin{pmatrix} +4 \Sigma - (S_b + 2 \Sigma) x \\ S_a - 2 \Sigma + (S_b + 2 \Sigma) x \\ 0 \end{pmatrix}, \begin{pmatrix} S_a - b^2 + 2 \Sigma + (S_c + 2 \Sigma) x \\ +b^2 - (S_c + 2 \Sigma) x \\ 0 \end{pmatrix}
$$

where $\Sigma$ stands for $\Sigma = S/\sqrt{3}$.

**Proof.** Chose $a \simeq 0 : x : 1 - x$ on $BC$ and consider the rotation $\rho(a,+60^\circ)$, so that:

$$
\rho = \begin{pmatrix}
-a^2 x + S_b + 2 \Sigma & -a^2 x + a^2 & -a^2 x \\
(S_a + 2 \Sigma) x - b^2 & (S_a + 2 \Sigma) x - S_b + 2 \Sigma & (S_a + 2 \Sigma) x \\
(S_b - 2 \Sigma) x + S_a + 2 \Sigma & (S_b - 2 \Sigma) x - S_b - 2 \Sigma & (S_b - 2 \Sigma) x + 4 \Sigma
\end{pmatrix}
$$

Define $c$ as $AB \cap \rho(CA)$ and $b$ as $\rho^{-1}(c)$. Then $abc$ is equilateral direct. It remains to synchronize the normalization of the three columns.

**Proposition 25.12.2.** For these two families, the pair $(S,E)$ is either $X(13), X(15)$ or $X(14),X(16)$, i.e a Fermat center and the corresponding isodynamic center.

**Proof.** From the values of $a, b, c, \xi$, one can read the values of $f, g, h$. And then apply these masses to the variable triangle. This leads to:

$$
S \simeq \frac{1}{S_a + 2 \Sigma} : \frac{1}{S_b + 2 \Sigma} : \frac{1}{S_c + 2 \Sigma}
$$

$$
E \simeq (S_a + 2 \Sigma) a^2 : (S_b + 2 \Sigma) b^2 : (S_c + 2 \Sigma) c^2
$$

Caveat: these two sets of coordinates are to be synchronized in order to enforce $f + g + h = u + v + w$ (leading to huge expressions!).

**Proposition 25.12.3.** The pedal triangle of $X(15)$ is equilateral and belongs to the corresponding family.
The determinant of lines

Proof. Since triangles \(abc\) are "turning around" point \(E\), the triangle of minimal area is obtained by orthogonal projection. The center of this triangle is the middle of \([S,E]\). Moreover, the locus of \(g = (a + b + c)/3\) is directed by:

\[
S_c - S_b : S_a - S_c : S_b - S_a \simeq b^2 - c^2 : c^2 - a^2 : a^2 - b^2
\]

One recognizes \(X(531)\), the orthopoint of \(X(30)\): the locus of \(g\) is the perpendicular bisector of segment \([S,E]\), and therefore orthogonal to the Euler line.

\[\square\]

### 25.12.2 LFIT of similar triangles

**Proposition 25.12.4.** The triangles of a linear family are similar to each other if and only if \(S\) and \(E\) form an isogonal pair. And then \(E\) is the center of similitude. Moreover, \(O_A = O_B = O_C = E\).

**Proof.** We already know that \(E\) is a fixed point of \(\frac{\mathcal{T}_1}{1 - \mathcal{T}_0}\). In order to have a similitude, the other two must be the umbilics of the plane. This leads to:

\[
\begin{pmatrix}
  u \\
  v \\
  w
\end{pmatrix} = \frac{h+g+f}{a^2gh+b^2fh+c^2fg} \begin{pmatrix}
  a^2gh \\
  b^2fh \\
  c^2fg
\end{pmatrix}
\]

\[\square\]

**Proposition 25.12.5.** The pedal triangle of \(E\) belongs to the LFIT, and provides the extremal area of the family

\[
\frac{4fghS^3(f+g+h)}{(a^2gh+b^2fh+c^2fg)^2} = \frac{4S^3(a^2uv + b^2uw + c^2uw)}{a^2b^2c^2(u+v+w)^2}
\]

**Proof.** Direct computation.

\[\square\]

### 25.12.3 LFIT of pedal triangles

**Proposition 25.12.6.** When a LFIT contains the pedal triangles of two different points \(P_0, P_1\) (not at infinity), then each of the other inscribed triangles can be written as \(\mathcal{T}_t = (1 - t)\mathcal{T}_0 + t\mathcal{T}_1\) and is the pedal triangle of \(P_1 = (1 - t)P_0 + tP_1\).

**Proof of the first part.** Consider the LFIT generated by \(\mathcal{T}_0, \mathcal{T}_1\) and apply (9.2). Then check that \(P_1\) is the pedal center of \(\mathcal{T}_t\). The slowness centers are easy to obtain, and \(S \in \Gamma\) follows.

\[\square\]

**Proposition 25.12.7.** Any LFIT whose slowness center \(S\) is not on the circumcircle contains exactly one pedal triangle.

**Proof.** Apply (9.2) to the LFIT. This gives a first degree polynomial in \(t\) whose leading coefficient is \(a^2gh + b^2hf + c^2fg\).

\[\square\]

**Construction 25.12.8.** Embedded pedal triangle \((S \not\in \Gamma)\). Start form a triangle \(\mathcal{T}_t\) and cut the perpendicular to \(AB\) issued from \(c_t\) by the perpendicular to \(AC\) issued from \(b_t\). This gives a point \(BC_t\). The locus \(\Delta_n\) of these points \(BC_t\) is a straight line. Obtain another point by using another triangle \(\mathcal{T}_s\). When \(S \not\in \Gamma\), the three lines \(\Delta_n\), etc. are concurrent, as proven in the previous proposition. This gives the central point of the embedded pedal triangle.

### 25.12.4 Cevian triangles in a LFIT

**Proposition 25.12.9.** A LFIT contains exactly three cevian triangles (up to visibility and multiplicity).

**Proof.** The determinant of lines \(a_tA, b_tC, c_tC\) is a polynomial of degree 3 in \(t\) (its leading coefficient doesn’t vanish).

\[\square\]
Construction 25.12.10. Embedded cevian triangles. Draw the conic $C_A$ through $B, C, G_A = B + C - A$ and the two intersections of the hexagonal graph $G_A$ with lines $AB$ and $AC$. Draw the other two. The three conics concur at the three required centers (in the complex plane). Maybe only one of them is visible.

Proof. Determine point $BC_i$ as the intersection of lines $b_iB, c_iC$. The result is in $t^2$. Thus the locus of $BC_i$ is a conic. Use locussconi to obtain its equation, and check for the five given points. Existence is given by the previous proposition.

Exercise 25.12.11. Consider the LFIT generated by the cevians of the two points $P_0$ (fixed) and $P_2$ (mobile). Let $P_3$ be the center of the third cevian triangle of this family. Then, for quite all $P_0$, the transformation $P_2 \mapsto P_3$ is a Cremona involution. Determine the exceptional locus and the points of indetermination. Study how these points are blown out.

25.13 Families with constant area

25.13.1 Three non concurrent lines (rewritten)

Proposition 25.13.1. Assume that area$(T_t)$ remains constant when $t$ changes. Then $S$ is at infinity while $E \simeq S$, so that we can use the hard equalities $S = [\tau - 1 : 1 : -\tau]$ and $E = KS$ for some values $\tau$ and $K$. The converse holds and leads to:

$$
\begin{vmatrix}
0 & K\tau + \tau - \frac{t}{1} & 1 + \frac{K}{\tau} - \frac{t}{\tau} \\
\frac{t}{\tau - 1} & 0 & -\frac{K}{\tau} + \frac{t}{\tau} \\
1 - \frac{1}{\tau - 1} & 1 - K\tau - \tau + \frac{t}{1} & 0
\end{vmatrix}
; \text{ area}(T_t) = -K(K + 1)S
$$

Proof. The coefficient of $t^2$ in area$(T_t)$ contains $f + g + h$, so that $S \in \mathcal{L}_\infty$. And then we have $u + v + w = f + g + h = 0$, together with $\tau v + w = 0$ from the coefficient of $t$.

Corollary 25.13.2. Special case: when $S \in \mathcal{L}_\infty$ and $K = -1$, the LFIT is the set of flat inscribed triangles whose "sideline" is $[1, -\tau + 1, 0] + t\mathcal{L}_\infty$ i.e. is directed by $S$.

Corollary 25.13.3. Special-special case: when $S \in \mathcal{L}_\infty$ and $K = 0$, the LFIT is the set of flat inscribed triangles whose "sideline" is tangent to the inscribed parabola having $S$ as point at infinity. In this case, $\mathcal{L}_\infty$. S = 0 : 0 : 0 and the equicenter $E$ is not defined.

Proof. Apply locussconi to line $[t(-\tau + 1 + t), (t - \tau)(-\tau + 1 + t), (t - \tau)t]$.

Let us assume that $S = K_2$ (choosing the sign of $W$). One obtains:

$$
K_3 \simeq W \begin{pmatrix}
g - h \\
h - f \\
f - g
\end{pmatrix} - 2\left(\frac{fg + gh + hf}{fhg}\right) \begin{pmatrix}
f \\
g \\
h
\end{pmatrix} t
$$

while the locus $\mathcal{P}$ of the $K_1$ becomes a conic since the $t$-degrees of the barycentrics of that point are now 2, due to $f + g + h = 0$. After computing $\mathcal{P}$ using the procedure locussconi, one can check that $K_2 \in \mathcal{P}$ together with $\mathcal{L}_\infty \in \mathcal{P}^*$: the conic is a parabola.

Moreover, the envelope of line $K_1K_3$ is also a conic, since the $t$-degree of $K_1 \wedge K_3$ is two. Using again locussconi, we obtain the matrix of the tangential equation:

$$
\begin{pmatrix}
(1 - W)f & -Wh & -Wg \\
-Wh & (1 - W)g & -Wf \\
-Wg & -Wf & (1 - W)h
\end{pmatrix}
$$

January 3, 2024 21:08 published under the GNU Free Documentation License
25. Linear Families of Inscribed Triangles

When searching for the common points of $Q$ and $P$, one obtains that these curves are bitangent. A first contact occurs in $K_2$, so that $Q$ is a parabola with the same direction as $P$. The second contact is the center of the hexagonal conic relative to

$$t_0 = \frac{(g-h)(f-h)(f-g)}{6(fg + fh + gh)} W$$

when using the former given parametrization. This value is the arithmetical mean of the dates of the degeneracies.

### 25.13.2 Three concurrent lines

Suppose now that points $a, b, c$ are moving on concurrent lines. We only have to consider this case as the limit of what happens when $K \to 0$ to the figure obtained by applying an homothety of center $G$ and ratio $K$ to the previous results (and replacing $X$ by $X/K$ for $X = p, q, r, W, t$, while slownesses $f, g, h$ are unchanged).

Then the inscribed triangle and the equicenter become:

$$abc \simeq \begin{pmatrix} \frac{1}{3} & 1 & q-t+1 \frac{1}{3} & r+t \frac{1}{3} & g \frac{1}{3} \end{pmatrix}, \ E \simeq \begin{pmatrix} hr-gq+1 \frac{1}{3} & (f+g+h) \\ fp-hr+1 \frac{1}{3} & (f+g+h) \\ gq-fp+1 \frac{1}{3} & (f+g+h) \end{pmatrix}$$

while the areal center is the limit of:

$$\begin{pmatrix} (2f-g-h)K+(f+g+h) \\ (2g-f-h)K+(f+g+h) \\ (2h-f-g)K+(f+g+h) \end{pmatrix}$$

and thus is $G$ when $S \notin \mathcal{L}_\infty$, but is $S$ when $S \in \mathcal{L}_\infty$.  

--- pldx : Translation of the Kimberling’s Glossary into barycentrics ---
The hexagonal conic goes through \(a,b,c\) and \(a' = b + c - G\), etc. When \(t = 0\), its points at infinity are:

\[2q + 2r : -W - 2r : W - 2q\]

where \(W^2 = -4(pq + qr + rp)\) is the limiting value of the relative area. Therefore:

**Proposition 25.13.4.** Given three lines that are concurrent at \(G\) and an inscribed triangle \(abc\) it exists two families of inscribed triangles that share the same area. Their areal center is one of the points at infinity of the hexagonal conic defined by \(a,b,c,G\). Point \(S\) is real if \(W^2 \geq 0\). Then, the family is obtained by constructing strips of equal width (in the \(S\) direction).

### 25.14 Concurrent hexagonal graphs

In this section, we assume that the three hexagonal graphs intersect each other at the same point \(K \simeq p : q : r\).

#### 25.14.1 Assuming that \(S\) is known

**Lemma 25.14.1.** In any case, the line through \(A\) parallel to \(G_a\) cuts \(BC\) at \(P' \simeq 0 : g : -h\), etc. These points \(P', Q', R'\) belong to line \([1/f, 1/g, 1/h]\) (the tripolar of \(S\)). Moreover, the points \(P = B + C - P', \) etc belong to line \([f, g, h]\) (isotomic of the tripolar).

**Proof.** Quite obvious from 25.8.

**Proposition 25.14.2.** When the graphs are concurrent and \(S \simeq f : g : h\) is known, then \(K\) belongs to line \([f, g, h]\) (isotomic conjugate of the tripolar of \(S\)), while \(E \simeq K \ast (2S - G)\), the barycentric product of \(K\) and the anticomplement of \(S\). As a result, \(E\) belongs to the line:

\[\Delta_S \simeq \left[ \frac{f}{g + h - f}, \frac{g}{h + f - g}, \frac{h}{f + g - h} \right]
\]

which is the image of tripolar \((S)\) by recip homot \((S, 1/3)\).

**Proof.** Using 25.8, the concurrence gives an equation, and the normalization gives another. This results into a parametrization of \(E\) and \(K\) by a coordinate of \(E\) (say \(w\)). And the results follow.

#### 25.14.2 Assuming that \(K\) is known

**Proposition 25.14.3.** When the graphs are concurrent at a known point \(K \simeq p : q : r\), then \(S\) belongs to line \([p, q, r]\) while \(E\) belongs to line

\[\Delta_K \simeq [qr (q + r), pr (p + r), p (p + q) q]\]

which is the image of tripolar \((K)\) by homot \((K, 2/3)\). Moreover, a parametrization of \([S, E]\) by a point at infinity is:

\[\begin{bmatrix} S, E \end{bmatrix} \simeq \begin{pmatrix} p + q + r \\ p q \end{pmatrix} \left[ \begin{pmatrix} q r \\ r p \\ p q \end{pmatrix} \begin{pmatrix} 1 \\ \mu \\ -1 - \mu \end{pmatrix} \right] \begin{pmatrix} p \\ q \\ r \end{pmatrix} \begin{pmatrix} \mu \\ -\mu \\ \mu + 1 \end{pmatrix}
\]

\[\begin{pmatrix} -pq (\mu + 1) + \mu pr - qr \\ -pq (\mu + 1) - \mu pr + qr \\ pq (\mu + 1) + \mu pr + pq + qr \end{pmatrix}
\]

**Proof.** From previous proposition, \(fp + gq + hr = 0\). Thus the parametrization \(1 : \mu : -1 - \mu\) of \(L_{\infty}\) induces \(S \simeq qr : \mu rp : -(1 + \mu) pq\). Added to \(G_a \cdot K = 0\), etc and the normalization rule, this leads to the remaining results.

#### 25.14.3 Assuming that \(K\) is the center of gravity

**Fact 25.14.4.** When \(K = X(2)\), then \(S \in L_{\infty}\) while \(E = (-2/3) S\). Moreover, the areas of all the inscribed triangles abc, and of all the \(a\beta\gamma\), are equal to \((2/9) S\).

**Fact 25.14.5.** Then \(M_{\infty} = S^\ast\) belongs to the circumcircle, while \(O_a = (2A + M_{\infty}) / 3\), etc. This provides the circle of similarity. And everything flows from this result.
25.15 When the graphs are given

In the previous sections, the graphs were the result of the pre-existing mappings \( a \leftrightarrow b \leftrightarrow c \leftrightarrow a \). Let us now examine what happens when these graphs are chosen from the beginning.

25.15.1 Catalan graphs

**Proposition 25.15.1.** When three lines \( \Delta_j \) are parallel to the sidelines, they are the Catalan graphs of a LFIT if and only if the bisectors of strips \( (\Delta_A, BC) \), etc are concurrent. And then \( S \) can be chosen at will (outside of \( L_\infty ) \).

**Proof.** The necessity comes from the required symmetry wrt \( \omega = (S + E) / 2 \). Since \( \omega \) is at finite distance, \( S, E \) cannot be chosen at infinity. \( \Box \)

**Proposition 25.15.2.** Define the collineation \( \mu^* \) by \( L_\infty \mapsto L_\infty \), \( BC \mapsto \Delta_A = [p_0 q_0, q_0] \), \( CA \mapsto \Delta_B = [r_1 q_1, r_1] \), \( AB \mapsto \Delta_C = [p_2, p_2, r_2] \). Then its matrix (acting over the lines !) is

\[
\begin{bmatrix}
\frac{q_0 - p_0}{r_1 - q_1} & \frac{q_0 - p_0}{q_1 - r_1} & \frac{q_0 - p_0}{r_1 - q_1} \\
\frac{r_1 - q_1}{p_2} & \frac{r_1 - q_1}{p_2} & \frac{r_1 - q_1}{p_2} \\
\frac{p_2 - r_2}{p_2 - r_2} & \frac{p_2 - r_2}{p_2 - r_2} & \frac{p_2 - r_2}{p_2 - r_2}
\end{bmatrix}
\]

We have \( \chi (X) = (X + 1)^2 (X - \lambda) \) Then \( \lambda = 1 \), i.e. \( \mu \) is a central symmetry, when

\[
2p_0 q_1 r_2 + q_0 r_1 p_2 - p_0 q_1 r_2 - p_0 r_1 q_2 - q_0 r_1 q_2 = 0
\]

In any case, \( \omega \in \ker (\mu - 1) \sim \frac{q_0}{p_0 - q_0} : \frac{r_1}{q_1 - r_1} : \frac{p_2}{r_2 - p_2} \). When \( \omega \) is the center, we have:

\[
E = \mu (S) \sim \frac{f p_0 + g q_0 + h q_0}{p_0 - q_0} : \frac{f r_1 + g q_1 + h r_1}{q_1 - r_1} : \frac{f p_2 + g p_2 + h r_2}{r_2 - p_2}
\]

**Proof.** Computations are straightforward. \( \Box \)

25.15.2 Hexagonal graphs

**Lemma 25.15.3.** When \( S, E \) are known, the hexagonal graphs have the following equations:

\[
\begin{bmatrix}
G_a \\
G_b \\
G_c
\end{bmatrix} \simeq \begin{bmatrix}
g + h - u & g - u & h - u \\
f - v & h + f - v & h - v \\
f - w & g - w & f + g - w
\end{bmatrix}
\]

(25.17)

**Proof.** From the very definition, we have \( \alpha_t = B + C - a_t \), etc. And then, we use the asymmetrical parametrization. \( \Box \)

**Lemma 25.15.4.** When \( G_c \simeq [p_3, q_3, r_3] \) is given, then both relations hold:

\[
g = \frac{r_3 - p_3}{r_3 - q_3}, \quad w = \frac{r_3 - p_3 - q_3}{r_3 - q_3}
\]

**Proof.** Graph \( G_c \) describes \( a_t \mapsto b_t \). Start from \( a_t \) as given. Compute \( \gamma_t \), then use \( a_t + b_t = C + \gamma_t \) and identify the obtained \( b_t \) with its usual parametrization. \( \Box \)

**Proposition 25.15.5.** Let be given three lines \( \Delta_j \simeq [p_j, q_j, r_j] \) in general position and define

\[
\begin{bmatrix}
f \\
g \\
h \\
u \\
w
\end{bmatrix} = \begin{bmatrix}
p_1 - r_1 & q_2 - p_2 & r_3 - q_3 \\
p_1 - r_1 & q_2 - p_2 & r_3 - q_3 \\
p_1 - q_1 & q_2 - p_2 & r_3 - p_3 \\
p_1 - q_1 & q_2 - p_2 & r_3 - p_3 \\
p_1 - r_1 & q_2 - p_2 - r_2 & r_3 - q_3 - p_3
\end{bmatrix}
\]

— pldx : Translation of the Kimberling’s Glossary into barycentrics —
Then the three lines are the hexagonal graphs (in that order) of a LFIT if and only if
\[
\frac{(p_1 - r_1)(q_2 - p_2)(r_3 - q_3)}{(p_1 - q_1)(q_2 - r_2)(r_3 - p_3)} = +1 \quad \text{and} \quad f + g + h = u + v + w
\]

In such a case, \(f, g, h, u, v, w\) are the synchronized barycentrics of the centers \(S, E\) of this LFIT.

**Proof.** The first condition is required by Lemma 25.15.4 if we want that \(x = f\) in the chain \(f \mapsto g \mapsto h \mapsto x\). The second is the required synchronization. The converse is left as an exercise.

**Exercise 25.15.6.** Use sagemath. Use the nine coefficients of matrix:
\[
\left(\begin{array}{ccc}
g + h - u & g - u & h - u \\
f - v & h + f - v & h - v \\
f - w & g - w & f + g - w
\end{array}\right) - \left(\begin{array}{c}
x \\
y \\
z
\end{array}\right) \cdot \left(\begin{array}{ccc}
p_1 & q_1 & r_1 \\
p_2 & q_2 & r_2 \\
p_3 & q_3 & r_3
\end{array}\right)
\]
together with \(f + g + h - u - v - w\) and \(f - 1\). Use these 11 polynomials to generate an ideal \(I\) over \(\mathbb{Q}\). Eliminate \(f, g, h, u, v, w, x, y, z\). This gives an ideal \(J\). Build the ideal \(K\) generated by the two conditions of the proposition. Divide \(J\) by \(K\), and obtain 1 (condition was necessary). Divide \(K\) by \(J\). The result is generated by three polynomials. And now, be more precise about the "in general position" used in the above proposition.

**Proposition 25.15.7.** Let us now suppose that \(S\) is given. This determines the direction of the graphs, and therefore requires the existence of three numbers \(\rho, \sigma, \tau\) such that:
\[
(G) \simeq \begin{bmatrix}
-\rho & h - \rho & g - \rho \\
h - \sigma & -\sigma & f - \sigma \\
g - \tau & f - \tau & -\tau
\end{bmatrix}
\]

The condition for these three lines to be the hexagonal graphs of a LFIT is \(f + g + h = \rho + \sigma + \tau\), i.e. the synchronization rule (25.6). And then, the equicenter \(E\) and the pillar point \(\Omega\) are given by:
\[
E = \begin{bmatrix}
h + g - \rho \\
h + f - \sigma \\
g + f - \tau
\end{bmatrix} \quad ; \quad \Omega = \begin{bmatrix}
\rho \\
\sigma \\
\tau
\end{bmatrix}
\]
so that \(S + E + \Omega = (f + g + h)(A + B + C)\) as required at (25.3).

**Proof.** The directions of the three hexagonal graphs are those of the anti-cevian triangle of \(S\). Let \(\mathcal{T}_S^*\) be the corresponding trigone, i.e. the three sidelines of this triangle. We have:
\[
\mathcal{T}_S^* \simeq \left(\begin{array}{ccc}
f & f & f \\
g & g & g \\
h & h & -h
\end{array}\right)^* \simeq \left(\begin{array}{ccc}
0 & h & g \\
0 & 0 & f \\
h & f & 0
\end{array}\right)
\]
The remaining computations are straightforward.

**Remark 25.15.8.** This can be written as \(S \in \mathcal{D}\) where
\[
\mathcal{D} \simeq \begin{bmatrix}
f - \rho & g - \sigma & h - \tau \\
f & g & h
\end{bmatrix}
\]
but the geometric interpretation of this line is not so clear.

**25.15.3 The marvelous formula**

**Lemma 25.15.9.** When the homologue sidelines of two trigones are parallel, then the two triangles are perspective.
Proof. The sidelines are perspective from \( L_\infty \). Another proof. The triangle \( T^* \) obtained as the dual of trigone 25.18, is ever perspective with its model, the anticevian triangle of \( S \), and the perspector is

\[
\begin{pmatrix}
-\rho f + g + h \tau \\
+\rho f - g + h \tau \\
+\rho f + g - h \tau
\end{pmatrix}
\begin{pmatrix}
f \\
g \\
h
\end{pmatrix}
\]

Proposition 25.15.10. Let \( G \) be a trigone whose sidelines are respectively parallel to the sidelines of the \( S \)-anticevian triangle and consider the collineation \( \mu \) defined by \( L_\infty \mapsto L_\infty, T^*_G \mapsto (G) \). This is ever an homothety. And then \( G \) describes the graphs of LFIT if and only if \( \text{stein}_{S} \cdot \mu^* \simeq \text{dual}_S \) where

\[
\text{dual}_S \simeq [f, g, h] ; \quad \text{stein}_{S} \simeq \text{anticomplem} \text{ of } (\text{dual}_S) \simeq [g + h, h + f, f + g]
\]

Proof. Describing \( \mu \) by its action over the lines, we obtain the matrix :

\[
\begin{pmatrix}
1 & 0 & 0 \\
0 & 1 & 0 \\
0 & 0 & 1
\end{pmatrix}
- \frac{1}{2 fgh} Q \cdot L_\infty
\]

where \( Q \) is the perspector defined just above. Therefore, \( \mu \) is an homothety centered at \( Q \), whose line-ratio is: \( 1/k = 1 - (L_\infty \cdot Q)/(2 fgh) \ldots \) while \( [\mu] = \text{Adjoint} \mu^* \) when acting on columns, describes the point-homothety homot (\( Q, k \)) (see (7.27), which uses \( k \) as point-ratio).

Line \( \text{dual}_S \simeq [f, g, h] \) is the tripolar of the isotomic of this point. Its image by the anticomplem transform is \( \text{stein}_{S} \simeq [g + h, h + f, f + g] \) (using ratio -2 on points, but ratio -1/2 on lines). We call this line \( \text{stein}_{S} \) since this is the conipolar of \( S \) w.r.t. the Steiner out-ellipse. And then one can see that \( \text{stein}_{S} \cdot \mu^* = \text{dual}_S \) if and only if \( f + g + h = \rho + \sigma + \tau \) is satisfied.

Corollary 25.15.11. Consider \( \mu_G \simeq \mu \text{anticomplem} \). This is another homothety (with point-ratio -2k), and we have:

\[
\begin{pmatrix}
1 & 0 & 0 \\
0 & 1 & 0 \\
0 & 0 & 1
\end{pmatrix}
- \frac{1}{2 fgh} Q_G \cdot L_\infty \quad \text{where } Q_G \simeq \begin{pmatrix}
2 fgh + (g - h) (g \sigma - h \tau) - (h + g) f \rho \\
2 fgh + (h - f) (h \tau - f \rho) - (h + f) g \sigma \\
2 fgh + (f - g) (f \rho - g \sigma) - (f + g) h \tau
\end{pmatrix}
\]

And now the requirement is the global invariance of \( \text{dual}_S \) by \( \mu_G \), i.e. \( Q_G \in \text{dual}_S \). We are back with the same question: a geometric interpretation would be great.

25.16 Observers (about perspectivities)

Definition 25.16.1. We say that a (fixed) triangle \( T_a T_b T_c \) observers the LFIT when the fixed triangle is in perspective with all of the triangles of the LFIT.

Proposition 25.16.2. An observer \( O \) is necessarily parallelogic with the ABC triangle. This implies the existence of a point \( M_o \) (the ray source) that sees ABC with rays parallel to the sidelines of \( O \).

Proof. The observer must be in perspective with \( T_\infty \), i.e. with \( (\delta_{BC}, \delta_{CA}, \delta_{AB}) \). This asserts the existence of the other center of parallelogy and, due to the symmetry of the relation, this implies the existence of \( M_o \).

Our intent is to specify some notations and prove the following theorem:

Theorem 25.16.3. There are three kinds of observers, each family being parametrized by a generic point \( M \) in the triangle plane.
1. The metric observers, are using a generic point as $S^*_H$, in order to obtain the $P^*_a, P^*_b, P^*_c$ of Proposition 25.6.5. And then the locus of the perspectors is the conic through the seven $M, P^*_j, O^*_j$ (here $S^*_H$ is the other parallelogy center, not the ray source).

2. The Poulbot’s observers, are using $M$ as ray source in order to obtain a trigone that is tangent to each of the parabolas $\Psi_j$ described at Proposition 25.11.1. And then, the locus of the perspectors is a straight line.

3. The three families of singular observers, using $M$ as ray source in order to obtain a triangle with one side tangent to one of the parabolas, and the opposite vertex on the related sideline of $ABC$.

Conversely, assume that $T_a$ is generic, i.e. $T_a$ is not on $BC$ while $AT_a$ is not tangent to $\Psi_b$ or $\Psi_c$. Then there are five observers that share $T_a$ as first vertex: one metric and four Poulbot. The $B$-vertices of the Poulbot triangles (say $B_j, j=1..4$) are the common points of the tangents to $\Psi_B$ from $P^*_a$ and to $\Psi_A$ from $P^*_b$, etc. for the vertices $C_k$. And then the $B_j$ and the $C_k$ are paired by the fact that $B_j C_k$ is tangent to $\Psi_A$.

In what follows, we are using the asymmetric parametrization (25.5), discarding the case $S = \mathcal{E} \in \mathcal{L}_\infty$ (constant area) and assuming $f + g + h - u - v - w = 0$. Proceeding that way, the case $S \neq \mathcal{E}; S, \mathcal{E} \in \mathcal{L}_\infty$ can be treated like the other cases. Let us recall that $\Omega = g + h - u : h + f - v : f + g - w$.

25.16.1 Metric observer

By itself, the theory of the LFIT is a projective one, and doesn’t necessitate to use a metric or another. Nevertheless, we have encountered interesting properties when using a metric (circle of similarity). In order to reuse these properties, we are now investigating what happens when the metric is taken as a parameter.

25.16.1.1 Forcing the orthocenter

Definition 25.16.4. When an euclidian structure is given on the $ABC$ plane, then triangle $ABC$ receives an orthocenter $H \simeq 1/S_a : 1/S_b : 1/S_c$. Reverting the process, i.e. saying that a given point $H \simeq \rho : \sigma : \tau$ is the orthocenter of $ABC$, determines a metric structure on the $ABC$ plane.

This process is called "forcing the orthocenter". The resulting objects will be super-scripted with a "$H^*$", like in $X^*H$ (except from the isogonal conjugacy, noted $S^*_H$, in order to avoid a tower of superscripts $S^{*H}$).

Proposition 25.16.5. Define $O^H$ as the complement of $H$, and $\Gamma^H$ (the forced circumcircle) as the circumconic whose perspector is $K^H \simeq H \times O^H$. Then

$$O^H \simeq \begin{pmatrix} \sigma + \tau \\ \tau + \rho \\ \rho + \sigma \end{pmatrix}, K^H \simeq \begin{pmatrix} \rho (\sigma + \tau) \\ \sigma (\tau + \rho) \\ \tau (\rho + \sigma) \end{pmatrix}, \Gamma^H \simeq \begin{bmatrix} 0 & \tau (\rho + \sigma) & \sigma (\tau + \rho) \\ \tau (\rho + \sigma) & 0 & \rho (\sigma + \tau) \\ \sigma (\tau + \rho) & \rho (\sigma + \tau) & 0 \end{bmatrix}$$

Moreover $\Gamma^H$ is the center of the conic, which goes through $2H_A - H$, etc where $H_A \simeq AH \cap BC$, etc.

Proof. Direct application of Theorem 12.7.4.

Definition 25.16.6. The forced isogonal transform is defined as the $ABC$-isogonal conjugacy that exchanges $H$ and $O^H$. In other words, $(x : y : z)^*_H \simeq \rho (\sigma + \tau)yz : \sigma (\tau + \rho)zx : \tau (\rho + \sigma)xy$.

25.16.1.2 Forcing the isogonal center

Definition 25.16.7. Forcing the isogonal center of a LFIT is making an arbitrary choice $M \simeq x : y : z$ for the point $S^*_H$. This amounts to force the metric to $a^2 : b^2 : c^2 \simeq xf : yg : zh$. The values $a, b, c$ are not intended to be real, only different from 0.
Proposition 25.16.8. The "fixed points" \( P_a, \) etc (see Proposition 25.6.5) are obtained from \( S^* \) by collineations that doesn’t depend from a choice of metric. Indeed, we have \( P_a = \phi_a \cdot S^* \) where:

\[
\phi_a \simeq \begin{pmatrix}
1 & 0 & 0 \\
0 & \frac{1}{f} & \frac{v}{f} \\
0 & \frac{w}{f} & 1 - \frac{v}{f}
\end{pmatrix}
\]

This is an affinity whose charpoly is \((X - 1)^2 (X - k)\) where \( k = \frac{u - g - h}{f} \). Proper point associated to \( k \) is the direction \( \delta_{BC} \), while the proper line is \([0, -w, v]\) i.e. the line \( AE \).

Proof. Obvious from (25.13) \( \square \)

Proposition 25.16.9. For any point \( M \), the triangle \( \phi_a M, \phi_b M, \phi_c M \) is an observer of the LFIT. The locus (on \( t \)) of the perspector \( K^H_t \) is a conic, that goes through \( M = S^*_H \), the three \( P^H_j \) and the three \( O^H_j \).

Proof. This is only reformulating some already proven properties (see Proposition 25.6.3) \( \square \)

Proposition 25.16.10. The \( O \mapsto ABC \) para-center of a metric observer is \( M \simeq x : y : z \) itself. The \( ABC \mapsto O \) para-center (i.e. the ray center) is

\[
\varphi(M) \simeq \begin{pmatrix}
f(uz - wx)(uy - vx) \\
g(vx - uy)(vz - wy) \\
h(wy - vz)(wx - uz)
\end{pmatrix}
\]

We have \( \varphi(\mathcal{E}) = 0 : 0 : 0 \). Two \( M \) that share the same \( \varphi(M) \) are aligned with \( \mathcal{E} \), while all the \( \varphi(M) \) belong to the circumconic \( \gamma \) with perspector \( S^* \mathcal{E} \). Moreover \( \varphi(M) \) belongs to \( \Gamma^H \), the forced circumcircle. Therefore \( \varphi(M) \) is the forced gudulic center of conic \( \gamma \).

Proof. Existence is Proposition 25.16.2. Computing the value is easy. The result clearly depends on \( M \land \mathcal{E} \), hence the alignment. \( \square \)

25.16.2 Cevenol graphs

Definition 25.16.11. Let \( \alpha'_t \) be the point where the parallel to \( BH \) through \( b_t \) cuts the parallel to \( CH \) through \( c_t \). As time \( t \) flows, the point \( \alpha'_t \) draws a straight line \( cev^H_a \). We call it the \( A \)-Cevenol graph (related to the forced orthocenter \( H \simeq \rho : \sigma : \tau \)).

Proof. Straight line \( cev^H_a \) is obtained as:

\[
[ u - g - h, \frac{(u - g) \rho + u \tau}{\tau + \rho}, \frac{(u - h) \rho + u \sigma}{\rho + \sigma} ]
\]

\( \square \)

Proposition 25.16.12. The line \( cev^H_a \) is nothing but the line that joins \( O^H_a \) and \( Q^H_a \), the centers of the direct and reverse \( H \)-similarities that generalizes the correspondence \( CA \mapsto AB \) : \( b \mapsto c \).

Proof. Obvious from (25.12) and (25.16). \( \square \)

Proposition 25.16.13. The three \( H \)-Cevenol graphs concur at a point \( G_c \) that belongs to the \( H \)-conic of similarity.

\[
\begin{pmatrix}
(\sigma + \tau) ((gv + hw - fu - gh) \rho^2 + (fv - fu - gh) \rho \sigma + (hw - fu - gh) \rho \tau - f \sigma \tau u) \\
(\tau + \rho) ((hw + fu - gv - hf) \sigma^2 + (hw - gv - hf) \sigma \tau + (fu - gv - hf) \rho \sigma - g \rho \tau v) \\
(\rho + \sigma) ((fu - hw - fg) \tau^2 + (fu - hw - fg) \tau \sigma + (gv - hw - fg) \tau \sigma - hw \rho \sigma)
\end{pmatrix}
\]

Proof. Properties of \( G_c \) are easily computed. \( \square \)
The parallelogy condition is described by the existence of the second tangent to the line $O$. Let line $\mathcal{L}_2$ be tangent to $\mathcal{P}_A$ and circ. for $B$ and $C$.

Proof. There is one and only one tangent to a parabola that contains a given direction.

Proposition 25.16.15. The "Poulbot observer" observes the LFIT. The equations of this observing trigone are:

$$O^* \simeq \begin{bmatrix} qr (f - v - w) & r (u - g)q - gr^2 & (u - h)rq - hq^2 \\ r (v - f) p - r^2 f & rp (g - w - u) & (v - h)rp - hp^2 \\ q(w - f)p - q^2 f & (w - g)qp - gp^2 & pq(h - u - v) \end{bmatrix}$$

while the locus of the perspectors is the straight line:

$$loc_K \simeq \begin{bmatrix} ghp^2 - f uq + (h - v) gpr + (g - w) hq \\ h f q^2 - g v p r + (f - w) hpq + (h - u) f q r \\ f g r^2 - h w p q + (g - u) f q r + (f - v) g p r \end{bmatrix}$$ (25.20)

Proof. Write that $M \cap A + \lambda \mathcal{L}_\infty$ is tangent to $\mathcal{P}_A$. Since $\mathcal{L}_\infty$ itself is a tangent, we obtain a first degree equation, leading to $O^*$. Then we take the adjoint and the perspectivity, for all $t$, is easy to check.

Proposition 25.16.16. Assume that triangle $P_aP_bP_c$ is a Poulbot observer and use $P_a = P_a^H$ to force the metric, determining $P_b^H$ and $P_c^H$. Then lines $P_bP_b^H$ and $P_cP_c^H$ are tangent to $\mathcal{P}_A$ (or degenerate !)

Proof. Note $p_1 : q_1 : r_1$ the coordinates of $P_a$ and $\delta_b \equiv t_b : -1 - t_b : 1$ and $\delta_c \equiv t_c : -1 - t_c$ the directions of $P_aP_b$ and $P_aP_c$. Write the conditions $h_b$ and $h_c$ for lines $P_a\delta_b$ and $P_a\delta_c$ to be tangent to the required parabolas. On the other hand, the parallelism center is $t_b : t_b t_c : 1$, and the direction of the third tangent is $-1 - t_b t_c : t_b t_c : 1$. Draw this tangent, say $\Delta$ and obtain the expressions in $p_1, q_1, r_1, t_b, t_c$ of $P_b, P_c$. Obtaining those of $P_b^H, P_c^H$ is obvious. It only remains to write the contact condition (length $\approx 120000$) and take the Euclidean remainder modulo $h_1$ and then modulo $h_2$. This gives 0.

25.16.4 Singular observers

Definition 25.16.17. When the $A$-vertex of an observer $O$ belongs to the sideline $BC$, we say that $O$ is an $A$-singular observer, etc (observers with two vertices on $ABC$ sidelines are to be discarded).

Proposition 25.16.18. Let line $\Delta_a$ be tangent to $\mathcal{P}_A$. Note $P_b$ the intersection of $\Delta_a$ and $\vartheta_B$ the second tangent to $\mathcal{P}_A$ from $B$, etc for $P_c$. Then $P_bP_bP_c$ is an observer for any $P_a \in BC$. Conversely, any degenerate observer is obtained that way.

Proof. Substitute $p_1 = 0$ in the perspectivity equation and assume that all the four coefficients of $t$ are 0. Solving the system leads to triangles with two vertices on the sidelines of $ABC$ and to:

$$O \simeq \begin{bmatrix} 0 & u & u \\ q_1 & gh s & h - u \\ r_1 & g - u & 1/s \end{bmatrix}$$

where $s$ is a parameter. One can check $P_b \in \vartheta_B, P_c \in \vartheta_c$ and $P_bP_c$ tangent to $\mathcal{P}_a$.

25.16.5 Proof of the theorem

The ray center is $M \simeq p : q : r$ and the LFIT is described using the asymmetric parametrization (25.5). The parallelogy condition is described by the existence of $\alpha, \beta, \gamma$ such that:

$$P_c - P_b = \alpha(M - A), P_a - P_c = \beta(M - B), P_b - P_a = \gamma(M - C)$$
Adding member to member, one concludes that \((\alpha, \beta, \gamma) = k(p, q, r), where k \neq 0\). Thus the observer can be written as:

\[
\mathcal{O} \simeq \begin{pmatrix}
1 - r_1 - q_1 & 1 - r_1 - q_1 + rp k & 1 - r_1 - q_1 - q p k \\
q_1 & q_1 + q r k & q_1 + q (p + r) k \\
r_1 & -r (p + q) k + r_1 & r_1 - q r k
\end{pmatrix}
\]

The perspectivity, for all \(t\), with \(\mathcal{T}_t\) leads to three equations, whose sizes are:

<table>
<thead>
<tr>
<th>length</th>
<th>fgh</th>
<th>vw</th>
<th>k</th>
<th>pqr</th>
<th>q_1 r_1</th>
</tr>
</thead>
<tbody>
<tr>
<td>(t^0)</td>
<td>1155</td>
<td>3</td>
<td>2</td>
<td>1</td>
<td>4</td>
</tr>
<tr>
<td>(t^1)</td>
<td>1141</td>
<td>2</td>
<td>1</td>
<td>4</td>
<td>2</td>
</tr>
<tr>
<td>(t^2)</td>
<td>149</td>
<td>1</td>
<td>0</td>
<td>1</td>
<td>4</td>
</tr>
</tbody>
</table>

The elimination of \(q_1, r_1\) leads to a 30837-sized compatibility condition, that splits into a number of factors:

\[
\text{length} \quad fgh \quad vw \quad k \quad pqr \quad k\]

<table>
<thead>
<tr>
<th>length</th>
<th>fgh</th>
<th>vw</th>
<th>k</th>
<th>pqr</th>
<th>k</th>
</tr>
</thead>
<tbody>
<tr>
<td>5</td>
<td>3</td>
<td>0</td>
<td>2</td>
<td>0</td>
<td></td>
</tr>
<tr>
<td>45</td>
<td>1</td>
<td>1</td>
<td>2</td>
<td>1</td>
<td></td>
</tr>
<tr>
<td>45</td>
<td>1</td>
<td>1</td>
<td>2</td>
<td>1</td>
<td></td>
</tr>
<tr>
<td>61</td>
<td>1</td>
<td>1</td>
<td>2</td>
<td>1</td>
<td></td>
</tr>
<tr>
<td>1101</td>
<td>3</td>
<td>1</td>
<td>4</td>
<td>1</td>
<td></td>
</tr>
<tr>
<td>137</td>
<td>2</td>
<td>1</td>
<td>2</td>
<td>0</td>
<td></td>
</tr>
</tbody>
</table>

- factors \((f, g, h, p, q)\) are only mirroring implicit hypotheses.
- the three small factors, being of first degree in \(k\), are giving three values for \(k\) and lead to the following three singular observers:

\[
X_a X_b X_c, \quad Y_a Y_b Y_c, \quad Z_a Z_b Z_c \simeq \begin{pmatrix}
0 & -ur & uq \\
uq r + h q & h q & h q - u q \\
-\frac{p}{u q r} - gr & w r - gr & -gr
\end{pmatrix}
\]

\[
\begin{pmatrix}
-h p - \frac{p r v}{q} & v p - h p & -gp \\
v r & 0 & -pv \\
-p v + \frac{p r v}{q} & f r - v r & f r
\end{pmatrix}
\]

\[
\begin{pmatrix}
-f g r + g v r p + h w p q = 0, \quad M \in \gamma \quad \text{where} \quad \gamma \quad \text{is the circumconic whose perspector is} \quad S \times E.
\end{pmatrix}
\]

**Proposition 25.16.19.** Assume that \(M \in \gamma\) and consider the line \(\Delta_M\) through \(E\) whose tripole is \(M \cong M\). Then \(\Delta_M\) is the locus of the perspectors of \(\mathcal{T}_t\) with the Poulbot observer \(U_a U_b U_c\). Moreover, this line is the locus of all the \(S_v^*\) whose associated metric observer \(\mathcal{O}_v\) admits \(M\) as ray center. And we have

\[
S_v^* \simeq (u : v : w) + \mu (p (g r - h q) : q (h p - f r) : r (f q - g p))
\]

\[
\mathcal{O}_v \simeq \begin{pmatrix}
u & u & u \\
v & v & v \\
w & w & w
\end{pmatrix} + \mu \begin{pmatrix}p (g r - h q) & -p (f r + h q + h r) & p (f q + g q + g r) \\
q (h r + h p) & q (h p - f r) & -q (f p + f r + g p) \\
r (g q + h q + g p) & r (f p + f q + h p) & r (f q - g p)
\end{pmatrix}
\]

**Proof.** Parametrize with \(p : q : r \simeq f u \mu g v / s : -h w / (1 + s)\), and substitute into (25.20) and obtain

\[
\Delta_M \simeq [v w, u w, -u v (1 + s)]
\]

From Proposition 25.16.10, we know that the all metrics observers whose \(S_v^*\) belongs to \(\Delta_M\) share the same ray center \(M_{\Delta}\). The new fact is that \(M_{\Delta} = M\) (direct computation). \(\square\)
25.16.6 Reciprocal

1. Four equations. We start from $P_a \simeq p_1 : q_1 : r_1$, $P_b \simeq p_2 : q_2 : r_2$, $P_c \simeq p_3 : q_3 : r_3$ (not on the sidelines) and we write that $O \simeq P_aP_bP_c$ observes the LFT family. The corresponding determinant is a 3 degree polynomial in $t$. This gives four equations with the following degrees:

<table>
<thead>
<tr>
<th>length</th>
<th>$p_1, q_1, r_1$</th>
<th>$p_2, q_2, r_2$</th>
<th>$p_3, q_3, r_3$</th>
<th>$f, g, h$</th>
<th>$v, w$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$t^0$</td>
<td>823</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>0</td>
</tr>
<tr>
<td>$t^1$</td>
<td>2075</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>2</td>
</tr>
<tr>
<td>$t^2$</td>
<td>1404</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>1</td>
</tr>
<tr>
<td>$t^3$</td>
<td>286</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>0</td>
</tr>
</tbody>
</table>

2. The $t^3$ equation is nothing but the condition of parallelogy:

$$\text{trace } (O) = \frac{p_1}{p_1 + q_1 + r_1} + \frac{q_2}{p_2 + q_2 + r_2} + \frac{r_3}{p_3 + q_3 + r_3} = 1$$

3. Three cubics. Then we solve two of these first degree equations in $p_3q_3r_3$, and substitute in the other two. This leads to the following three cubics:

<table>
<thead>
<tr>
<th>name</th>
<th>solving</th>
<th>length</th>
<th>$f, g, h$</th>
<th>$p, q, r$</th>
<th>$p_1, q_1, r_1$</th>
<th>$p_2, q_2, r_2$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$K_1$</td>
<td>$t^2, t^4$</td>
<td>6398</td>
<td>2</td>
<td>1</td>
<td>3</td>
<td>3</td>
</tr>
<tr>
<td>$K_2$</td>
<td>$t^2, t^4$</td>
<td>7865</td>
<td>2</td>
<td>2</td>
<td>3</td>
<td>3</td>
</tr>
<tr>
<td>$K_3$</td>
<td>$t^0, t^4$</td>
<td>12382</td>
<td>3</td>
<td>3</td>
<td>3</td>
<td>3</td>
</tr>
</tbody>
</table>

4. Excluding. Line $AC$ is asymptote to $K_1$ and $K_2$, but not to $K_3$: one occurrence of $\delta_{AC}$ must be excluded. Moreover, point $p_1 (gq_1 + (g - f) r_1) : g (q_1 + r_1)^2 : fp_1 q_1$ is on $K_1$ and $K_2$ but not on $K_3$.

5. It remains 7 common points to the first two cubics. Eliminating $q_2$, the equation in $p_2, r_2$ splits into four factors whose degrees are: 1,1,1,4. This allows to identify:

(a) Point $P_1$ itself is a possibility for $P_2$.
(b) Point at infinity $\delta_B$ of $AC$ belongs to the three cubics.
(c) Point $P^H_b$. Indeed, once $P_1 \simeq p_1 : q_1 : r_1$ is known, it exists exactly one metric such that $P_1 = P^H_a$. This leads to:

$$\begin{pmatrix}
a^2 \\
b^2 \\
c^2
\end{pmatrix} \approx \begin{pmatrix} f (v + w - f) p_1 \\
g (v - f) q_1 + gw r_1 \\
hwq_1 + h (w - f) r_1
\end{pmatrix}$$

Substituting into $P_b$, we obtain $P^H_b$. This point belongs to all three cubics.
(d) It remains a group of four points that belongs to the three cubics. Since we already know these points, they are the Poulbot observers associated with the metric observer.

25.17 Orthojoin

orthopole of the tripolar of the isogonal conjugate. It is not clear if this concept is really useful.
Chapter 26

Quadrilaterals

26.1 Immortal glory of our ancestors

Many things were summarized in Ripert, Léon (1901). A generation later, a founding overview was given by Clawson (1919), with other notations.

Here, the transversal $L_0$ is described as the tripolar of a point $P \simeq p : q : r$. In other words, $L_0 = l_4 \simeq [qr, rp, pq]$.

<table>
<thead>
<tr>
<th>Here</th>
<th>Ripert</th>
<th>Clawson</th>
<th>name</th>
</tr>
</thead>
<tbody>
<tr>
<td>$L_A, L_0$</td>
<td>$a, b, c, d$</td>
<td>$l_1, l_2, l_3, l_4$</td>
<td>lines</td>
</tr>
<tr>
<td>$ABC$</td>
<td>$ABC$</td>
<td>$A_{23}A_{13}A_{12}$</td>
<td>vertices</td>
</tr>
<tr>
<td>$A'B'C'$</td>
<td>$A'B'C'$</td>
<td>$A_{14}, A_{24}, A_{34}$</td>
<td>vertices</td>
</tr>
<tr>
<td>$O_j, \Gamma_j$</td>
<td>$O_j$</td>
<td>$C_j, C_j$</td>
<td>circumcire</td>
</tr>
<tr>
<td>$AA', BB', CC'$ was $D_A$</td>
<td>$AA' \ldots$</td>
<td>$n_j = A_{14}A_{kn}$</td>
<td>diagonal lines</td>
</tr>
<tr>
<td>$N_a, N_b, N_c$</td>
<td>$G_n$</td>
<td>$D_{12} = n_1 \cap n_2$</td>
<td>diagonal vertices</td>
</tr>
<tr>
<td></td>
<td>$m_j$</td>
<td>$U_n$</td>
<td>X(2) of $l_il_jl_k$</td>
</tr>
<tr>
<td></td>
<td></td>
<td>$B_1B_2B_3$</td>
<td>mid diagonal points</td>
</tr>
<tr>
<td></td>
<td></td>
<td>$G_j$</td>
<td>proj of $F$ on $l_j$</td>
</tr>
<tr>
<td></td>
<td></td>
<td>$F_j$</td>
<td>$2O_j - F \in \Gamma_j$</td>
</tr>
</tbody>
</table>
26.2 Lines only

Definition 26.2.1. A transversal is a moving line $\mathcal{L}_0$ that cuts the sidelines of a fixed triangle $ABC$ in three other points. When the four lines $\mathcal{L}_0$, $\mathcal{L}_A = BC$, $\mathcal{L}_B = CA$, $\mathcal{L}_C = CA$ are assumed to play the same role, this situation is called a quadrilateral.

Notation 26.2.2. Descriptions are easier when using the symmetry of the situation. Therefore, we will often use the Clawson (1919) notations, where the 4 lines are $\mathcal{L}_0$, $\mathcal{L}_1$, $\mathcal{L}_2$, $\mathcal{L}_3$ and the 6 vertices are $A_{jk} = \mathcal{L}_j \cap \mathcal{L}_k$. When using indices $ijkn$, property $\{i,j,k,n\} = \{0,1,2,3\}$ is ever assumed. When an 3-fold object depends on how indices are paired, as in $(ij)(k0)$, the result will be indexed by $k$ (the one paired with 0).

Notation 26.2.3. Without any other notice, computations (barycentrics, etc.) are relative to triangle $ABC$, using $\mathcal{L}_0 \simeq$ tripolar $(P) \simeq \left[qr,rp,pq\right]$, and therefore breaking the symmetry. It will be convenient to introduce the quantities

$$k^3 = (p-q)(q-r)(r-p)$$
$$\Phi_p = a^2qr - (S_bq + S_cr)p,$$

with the following properties:

$$\sum \Phi_p = 0; \sum qr\Phi_p = 2S\mathcal{L}_0 \cdot \mathcal{M} \cdot \mathcal{L}_0$$

Definition 26.2.4. A quadrilateral induces four embedded triangles and we denote $T_j$ the triangle which is associated with the trigone "all the four lines except from $\mathcal{L}_j$".

Proposition 26.2.5. The diagonals $AA', BB', CC'$ define a trigone. It's dual $D_a, D_b, D_c$ is called the diagonal triangle. Seen from $T_0$, this triangle is the anticevian of $P$. Then $(A, A', D_b, D_c) = -1$ so that each diagonal is harmonically divided by the other two.

Proof. One has $A' \simeq 0 : q : -r$ and $D_b \simeq -p : q : r$. And the result follows. □
Proposition 26.2.6. Let $G_d$, etc be the barycenter of triangle $T_d$, etc. Then

$$G_d[1 : 1 : 1] ; G_a[-3p^2 + 2p(q + r) − qr : q(p − r) : r(p − q)]$$

The barycenter $G$ of these four points is also the barycenter of the three $m_j$ or the barycenter of the six $A_{jk}$. And we have:

$$G ≃ [(q − r)(3p^2 − 2p(q + r) + qr) :]$$

Proof. Obvious from the coordinates.

Proposition 26.2.7. Newton axis (lowbrow version). Consider the traces of the transversal $L_0$ on the other three lines, i.e. $A' = L_0 ∩ L_A$, etc. Lines $AA'$, etc are the so-called diagonals, while points $N_a = (A + A')/2$, etc are the so-called mid-diagonal points. These three $N_j$ are aligned on what is called the Newton axis of the quadrilateral. Its barycentrics are:

$$δ = cevadiv(L_0, L_∞) ≃ [rq – p(q + r), pr – q(r + p), pq – r(p + q)]$$

Proof. One has $m_a ≃ q – r : +q : −r$, etc and the result follows.

Proposition 26.2.8. Reciprocal lines. Let $R_j$ be the reciprocal of line $L_j$ wrt triangle $T_j$. They are parallel to each other. Moreover, their equibarycenter is the Newton axis.

Proof. Remember that $R_0$ is the line through $B + C - A'$, etc. The barycentrics of these lines are:

$$R_0 ≃ p, q, r$$

$$R_A ≃ −p, q − 2p, r − 2p$$

$$R_B ≃ p − 2q, −q, r − 2q$$

$$R_C ≃ p − 2r, q − 2r, −r$$

$$δ ≃ p − q − r, q − r − p, r − p − q$$

The first one comes from the very definition. The other three are obtained by the usual conjugacy methods, and adjusted so that all the $R_j ∩ L_∞$ are equal (and not simply proportional), so that computing $R_0 + R_A + R_B + R_C$ makes sense.

26.3 Newton stuff

Proposition 26.3.1. Newton pencil. Consider all of the conics that are tangent to the four lines $L_j$. Their orthoptic cycles (see Section 12.24) belong to a linear pencil of cycles, called the Newton pencil of the quadrilateral. In the $P_C(C^1)$ barycentric space of cycles, the matrix of this pencil is

$$\begin{bmatrix}
0 & (p-q)S_c & (p-r)S_b & −pS_bS_c \\
(q-p)S_c & 0 & (q-r)S_a & −qS_aS_c \\
(r-p)S_b & (r-q)S_a & 0 & −rS_aS_b \\
pS_bS_c & qS_aS_c & rS_aS_b & 0
\end{bmatrix}$$

Proof. Consider one of the other tangents to such a conic, say tripolar $(u : v : w)$. Then we have:

$$\begin{bmatrix}
0 & pv – qu & ru – pw \\
pv – qu & 0 & qw – rv \\
ru – pw & qw – rv & 0
\end{bmatrix}$$

From (12.17) we have

$$Ω ≃ \begin{bmatrix}
fS_a \\
gS_b \\
hS_c \\
f + g + h
\end{bmatrix} ≃ \begin{bmatrix}
(qw – rv)S_a \\
(ru – pw)S_b \\
(pv – qu)S_c \\
(qw – rv) + (ru – pw) + (pv – qu)
\end{bmatrix}$$

Using tripolar $(u’ : v’ : w’)$ leads to $Ω'$. And we can see that $\Omega_6 ≃ (Ω_5 \cap Ω')$ doesn’t depend on the auxiliary tangents.
Definition 26.3.2. The radical axis $h$ of this pencil is called the Steiner axis of the quadrilateral, while its line of centers $\delta$ is the already defined Newton axis of the quadrilateral.

Proposition 26.3.3. The Newton axis is also the locus of centers $\Omega_C$ of the conics tangent to the four lines (see Proposition 26.4.7 for the parabolic case).

Proof. The set of all the lines through $A$ or through $A'$ is one of the involved tangential conics. Its center is obviously $(A + A')/2$, while $[AA']$ is a diameter of the corresponding orthoptic circle $\nu_a$.

$\nu_a, \nu_b, \nu_c \simeq \begin{bmatrix} 0 & -p S_a & +p S_a \\ +q S_b & 0 & -q S_b \\ -r S_c & +r S_c & 0 \\ q - r & r - p & p - q \end{bmatrix}$

(26.1)

And we conclude since a conic and its orthoptic cycle are concentric. $\square$

Exercise 26.3.4. Determine the 4-tangent conic such that $\Omega_C = G$.

Exercise 26.3.5. Determine the locus of the $ABC$ perspector of the conics tangent to the four lines.

26.4 Steiner stuff

Proposition 26.4.1. Steiner pencil. The four polar circles $\gamma_j$ relative to the four embedded triangles form a pencil with $\delta$, the Newton axis. This pencil, called the Steiner pencil, is orthogonal to the Newton pencil. The barycentrics of these cycles are:

$\gamma_0 = \begin{bmatrix} S_a \\ S_b \\ S_c \\ 1 \end{bmatrix}; \gamma_A = \gamma_0 - \frac{p S_a}{(p - q)(p - r)} \delta, \text{ etc} ; \delta \simeq \begin{bmatrix} p - q - r \\ q - r - p \\ r - p - q \\ 0 \end{bmatrix}$

In the $\mathbb{P}_C(\mathbb{C}^4)$ barycentric space of cycles, the matrix of this pencil is

$[\gamma_b] \simeq \begin{bmatrix} 0 & r - p - q & p - q + r & a^2 (q - r) + (c^2 - b^2) p \\ * & 0 & p - q - r & b^2 (r - p) + (a^2 - c^2) q \\ * & * & 0 & c^2 (p - q) + (b^2 - a^2) r \\ * & * & * & 0 \end{bmatrix}$

Proof. From Section 12.24, each orthoptic circle $\Omega \in \mathcal{U}$ is orthogonal to the polar circle of any of the embedded triangles. For $\gamma_0$, see Section 13.7. For $\gamma_A$, make a change of algebraic basis. Another method: compute $\eta_A$, the NPC of triangle $T_A$, and use $\gamma_A = 2\eta_A - \Gamma_A$. Finally, the simplest method to obtain $[\gamma_b]$ is the general formula $[\Omega] \cdot \text{dual}([\gamma_b]) = \frac{[\Omega]}{[\delta]}$ but a direct computation is also possible. $\square$

Proposition 26.4.2. Steiner axis. The orthocenters of the four embedded triangles belong to a same line, the already defined Steiner axis of the quadrilateral. Its barycentrics are:

$h \simeq \text{Steiner_axis} \simeq [(q - r) S_a, (r - p) S_b, (p - q) S_c]$

Proof. From Section 13.7, the polar circle of a triangle is centered at the orthocenter. $\square$

Proposition 26.4.3. The four circumcenters of the four embedded triangles $T_j$ are on a same circle that is called their Miquel circle. The four circumcircles have a common point, called their Miquel point. This point is the isogonal conjugate of $\delta_\infty$. Moreover, the Miquel point belongs to the Miquel circle. The $ABC$-barycentrics of these objects are:

$\Gamma_M \simeq \begin{bmatrix} b^2 c^2 (q - r) \Phi_p \\ c^2 a^2 (r - p) \Phi_q \\ a^2 b^2 (p - q) \Phi_r \\ -8 S^2 k^3 \end{bmatrix}$

$M_q \simeq \begin{bmatrix} a^2 \\ \frac{c^2}{(q - r)} \\ \frac{b^2}{(r - p)} \\ \frac{c^2}{(p - q)} \end{bmatrix}$
Proof. Using wedge, Veronese and wedge3, one obtains the $ABC$-barycentrics of the four circum-circles and the four centers:

$$
\Gamma_0 \simeq \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}; \ \Gamma_A \simeq \begin{bmatrix} 0 \\ p \delta^2 \\ p-q \end{bmatrix}; \ \Gamma_B \simeq \begin{bmatrix} q \delta^2 \\ q-p \\ 0 \end{bmatrix}; \ \Gamma_C \simeq \begin{bmatrix} r \delta^2 \\ r-p \\ r-q \end{bmatrix};
$$

$$
O_0 \simeq \begin{bmatrix} a^2S_a \\ b^2S_b \\ c^2S_c \end{bmatrix}; \ O_A \simeq \begin{bmatrix} 8p(q-p)S^2-a^2\Phi_3-c^2\Phi_p \\ b^2\Phi_r \\ c^2\Phi_q \end{bmatrix}, \text{ etc}
$$

And, some wedge, Veronese and wedge3 later, the result is obtained. Other proofs are possible (Ehrmann, 2004).

**Corollary 26.4.4.** Point $S_n$, the other intersection of $\Gamma_n$ and $\Gamma_M$, belongs to the three lines $O_iA_{jk}$ (remember the convention $\{n, i, j, k\} = \{0, 1, 2, 3\}$).

$$
S_0 \simeq \begin{bmatrix} a^2 \\ b^2 \\ c^2 \end{bmatrix}_p \cdot \begin{bmatrix} \Phi_p \\ \Phi_q \\ \Phi_r \end{bmatrix}
$$

Proof. Compute $S_0$ as the intersection of $O_1A_{23}$ and $O_2A_{13}$. By symmetry, $S_0 \in O_4A_{12}$. Its isogonal conjugate wrt $ABC$ is at infinity. Conclude by checking that $\text{Ver}(S_0) \cdot \Gamma_M = 0$.

**Corollary 26.4.5.** Define $S'_n = 2O_n - S_n$ (on $\Gamma_n$). Then $S'_nA_{jk}$ is tangent to $\Gamma_i$ at $A_{jk}$. As an example, $S'_nA$ is tangent to $\Gamma_0$, $S'_nB'$ is tangent to $\Gamma_c = (A'B'C)$ at $B'$ and $S'_nC'$ is tangent to $\Gamma_b = (A'B'C')$ at $C'$.

$$
S'_0 \simeq \begin{bmatrix} a^2 \\ b^2 \\ c^2 \end{bmatrix}_p \cdot \begin{bmatrix} \Phi_p \\ \Phi_q \\ \Phi_r \end{bmatrix}; \ S'_n \simeq \begin{bmatrix} -a^2(p^2 + qr) + 2S_bpq + 2S_cpr \\ q-r \end{bmatrix}_p \\
S'_n \simeq \begin{bmatrix} pb^2 \\ -pc^2 \end{bmatrix}
$$

**Proposition 26.4.6.** The Miquel point has the same Simson line with respect to all four embedded triangles $T_j$. This line is called the pedal line of the quadrilateral. By Proposition 10.2.1, the image of this line by the homothety $h(M_q, 2)$ contains the orthocenters $H_j$ of the triangles $T_j$ and is, therefore, the Steiner axis $h$ of the quadrilateral.

Proof. A line is defined by two points, and three projections are on each Simson line. One can also use the $ABC$-barycentrics of the orthocenters:

$$
H_0 \simeq \begin{bmatrix} S_bS_c \\ S_aS_c \\ S_aS_b \end{bmatrix}; \ H_a \simeq \begin{bmatrix} S_bb^2r + S_c^2pq - 4S_b^2p^2 - S_bS_crq \\ S_a\Phi_q \\ S_a\Phi_r \end{bmatrix}
$$

**Proposition 26.4.7.** The parabola tangent to the four $\mathbb{L}_j$ has the Miquel point $M_q$ as focus and the Steiner axis has directrix. The pedal line is a tangent.

Proof. Well known property!

**Proposition 26.4.8.** Diagonal triangle. The diagonal triangle $T_{\text{dia}}$ is the dual of the diagonal trisgone $AA', BB', CC'$. This triangle is the anticevian wrt $T_0$ of $P = \text{tripolar} (\mathbb{L}_0)$ and is also the diagonal triangle of the quadrangle of the four contact points with the inscribed parabola.

Its circumcircle $\mathcal{D}$ belongs to the Steiner pencil and one has

$$
\mathcal{D} = \gamma_0 - \frac{S_ap^2 + S_bq^2 + S_cr^2}{(q + r - p)(r + p - q)(p + q - r)} \delta
$$

Proof. Straightforward computation.

— pldx : Translation of the Kimberling’s Glossary into barycentrics —
Exercise 26.4.9. Clawson (1919, prop. 2) Consider the 3 circles \( \vartheta_A \cong (AA'M_q) \). Call \( \vartheta_{K,j} \) the point where \( \vartheta_K \) cuts again line \( L_j \). There are 12 of such points. Each of them characterizes a tangent to one circle \( \Gamma_j \) at one of its eponymous points. More precisely, when \( M \in \Gamma_j \), line \( M\vartheta_{M,j} \) is tangent to circle \( \Gamma_j \) at \( M \). Otherwise, line \( M'\vartheta_{M',j} \) is tangent to circle \( \Gamma_j \) at \( M' \).

Exercise 26.4.10. Clawson (1919, prop. 3) Each of the 6 circles through \( A_{ij}, (A_{ik} + A_{jn})/2, (A_{jk} + A_{jn})/2 \) goes through \( M_q \).

Exercise 26.4.11. Each of the 6 points \( Q_{ij} = med[A_{ij}, A_{jk}] \cap med[A_{in}, A_{jn}] \) belongs to \( \Gamma_M \). Examples: \( Q_{bc} = med[A_{ba}, A_{ca}] \cap med[A_{ba}, A_{ca}] = med[B, C] \cap med[B', C'] \), \( Q_{ab} = med[B'C'] \cap med[B'C'] \).

Exercise 26.4.12. The line joining \( (B + A')/2 \) and \( (B' + A)/2 \) cuts the line joining \( (B + A)/2 \) and \( (B' + A')/2 \) on the Newton axis.

### 26.5 The Lubin cookbook for quadrilaterals

We describe the transversal \( \mathcal{L}_0 \cong [qr, rp, pq] \) by the turns \( \kappa, \nu \) where this line cuts \( \Gamma \). These turns are visible... or not, so that \( |\kappa| = 1 = |\nu| \) is not assumed. One can also use \( \mathcal{L}_0 \cong [1, m, n] \), i.e. \( \kappa\nu = n, \kappa + \nu = -m \). With the rules:

\[
\begin{align*}
\text{conjugate}(n) &= 1/n ; \quad \text{conjugate}(m) = m/n
\end{align*}
\]

Remark 26.5.1. Tentative improvement, but not so convincing at the end.

**Fact 26.5.2.** Barycentrics v/s Lubin and conversely

1. \( \mathcal{L}_0 \cong \frac{\alpha - \nu}{\alpha} \left[ \frac{(\alpha - \nu)(\alpha - \kappa)}{\beta} ; \frac{(\beta - \nu)(\beta - \kappa)}{\gamma} ; \frac{(\gamma - \nu)(\gamma - \kappa)}{\beta} \right] \)

2. \[
\begin{align*}
\begin{cases}
 n \equiv \kappa\nu & \equiv s_3 \frac{(\beta - \gamma)qr + (\gamma - \alpha)rp + (\alpha - \beta)pq}{\alpha(\beta - \gamma)qr + \beta(\gamma - \alpha)rp + \gamma(\alpha - \beta)pq} \\
 m \equiv -\kappa - \nu & \equiv -\frac{1}{s_3} \frac{\alpha(\beta - \gamma)qr + \beta(\gamma - \alpha)rp + \gamma(\alpha - \beta)pq}{\alpha(\beta - \gamma)qr + (\gamma - \alpha)rp + (\alpha - \beta)pq}
\end{cases}
\end{align*}
\]

**Proposition 26.5.3.** When using the Lubin-1 representation, the orthopole transform is:

\[
\Delta = [1, m, n] \mapsto \mathcal{E}_\Delta \cong \frac{1}{2} \left( \begin{array}{ccc}
\frac{s_3}{n} & 1 & -m \\
1 & -\frac{m}{n} & s_3
\end{array} \right) + \left[ \begin{array}{ccc}
\frac{s_1}{s_2} & 1 & 0 \\
1 & s_2 & s_3
\end{array} \right] = \frac{1}{2} (S' + \sigma_\Delta (O)) + \overline{ON}
\]

Here \( S' \cong s_3/n \); 1/n is the isogonal conjugate of the direction of \( \Delta \) (and the antipode of \( S \) in the circumcenter) while \( \sigma_\Delta \) is the symmetry wrt line \( \Delta \). Finally, \( N \) is \( X(5) \), the Euler center.

**Proof.** Direct computation is easy. One can prefer a change of basis applied to (26.9).

**Fact 26.5.4.** Basic objects

1. Newton \( \delta \cong \frac{t}{1} \left( \begin{array}{c}
2n^2 - 2ns_2 - 2ms_3 \\
-n^2s_1 + ns_2 + (m^2 - n)s_3 - ns_2s_3 \\
-2mn - s_3 - 2ns_1s_3 + 2s_3^2
\end{array} \right) \)

2. Steiner \( h \cong \frac{t}{1} \left( \begin{array}{c}
n^2 - ns_2 - ms_3 \\
-n^2s_1 - mns_2 + ms_1s_3 + s_2s_3 \\
mns_3 + ns_1s_3 - s_3^2
\end{array} \right) \)

3. Miquel \( M_q \cong \frac{t}{1} \left( \begin{array}{c}
n^2 - ns_2 - ms_3 \\
-nm - nns_1 + s_3 \\
1
\end{array} \right) ; \quad \Gamma_M \cong \frac{t}{1} \left( \begin{array}{c}
-mn^2 - n^2s_1 + ns_3 \\
-nns_2 - ns_1s_3 + s_3^2 \\
ns_2s_3 + ms_3^2 - ns_2s_3
\end{array} \right) \)

January 3, 2024 21:08 published under the GNU Free Documentation License
4. Centers of circumcircles: \( O_0 \simeq \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} ; \quad O_a \simeq \begin{pmatrix} \alpha n^2 - n a s_2 - m a s_3 \\ n^2 + n a^2 - n a s_1 + a s_3 \\ -m n - n s_1 + s_3 \end{pmatrix} \)

5. Orthocenters: \( H_0 \simeq \begin{pmatrix} s_1 \\ s_3 \\ s_2 \end{pmatrix} ; \quad H_a \simeq \begin{pmatrix} \alpha^3 s_3 + (\alpha s_2 - s_3) m n + n^2 \alpha s_1 \\ \alpha (\alpha \gamma - n) (\alpha \beta - n) \\ n^2 + m (\alpha s_2 - s_3) + \alpha^2 s_2 \end{pmatrix} \)

6. Mean orthocenter \( H_m = \frac{1}{4} \sum H_j \simeq \frac{1}{4} \)

\[
\begin{pmatrix}
0 & 2 s_3 (m n + n s_1 - s_3) & 2 (m n s_2 - m s_1 s_3 + s_1 n^2 - s_2 s_3) & m^2 s_1 s_3 + m n s_1 s_2 + n^2 s_1^2 + m n s_3 - s_3^2 \\
* & 0 & 2 (n^2 - n s_2 - m s_3) & s_3 m^2 + s_1 n^2 - n s_1 s_2 - n s_3 + s_2 s_3 \\
* & * & 0 & m^2 s_2 + m s_1 s_2 + m s_3 - n^2 + s_2^2 \\
* & * & * & 0
\end{pmatrix}
\]

while the Steiner pencil is described by

\[
\mathbb{R}^4 \cong \mathbb{Q} \cdot \text{dual} \left( \mathbb{R} \right) \cdot \mathbb{Q}
\]

6.4. In the \( \mathbb{P}_C (\mathbb{C}^4) \) space of cycles, the Newton pencil is described by the anti-symmetric matrix: \( \mathbb{R} \equiv \)

\[
\begin{pmatrix}
-\frac{s_3}{n} & m (s_3 + n \alpha) + n (\alpha^2 + s_2) \\
\frac{n}{1 - n} & - (\alpha \gamma - n) (\alpha \beta - n) \\
\frac{1}{s_3} & m (s_3 + n \alpha) + n (\alpha^2 + s_2) \\
\end{pmatrix}
\]

\[
\begin{pmatrix}
-\frac{m n s_1 - n s_1 s_3 + s_3^2}{2 n s_3} \\
n^2 - n s_2 - m s_3 \\
\end{pmatrix}
\]

\[
\begin{pmatrix}
m (\alpha^2 + s_2) + 2 n^2 s_1 + n s_1 (3 \alpha^2 - 2 \alpha s_1 + s_2) - s_3 (\alpha^2 - 2 \alpha s_1 + s_2) \\
m (\alpha^2 + s_2) - n^2 \frac{a^2}{s_3} + n s_2 \frac{3 \alpha^2 - 2 \alpha s_1 + s_2}{s_3} + 2 \alpha s_2 \\
\end{pmatrix}
\]

\[
\begin{pmatrix}
f \\ g \\ h \\ u \\ v \\ w
\end{pmatrix} \cong \begin{pmatrix}
\gamma - \beta \\ \alpha - \gamma \\ \beta - \alpha \\ \gamma - \beta \\ \alpha - \gamma \\ \beta - \alpha
\end{pmatrix} \begin{pmatrix}
\alpha \beta + n \\ \alpha \beta + n \\ \gamma \alpha + n \\ \beta \gamma + n \\ \gamma \alpha + n \\ \beta \gamma + n
\end{pmatrix}
\]

Fact 26.5.6. The four pedal LFIT

1. Slowness centers. \( S_0 \simeq \begin{pmatrix} -s_3 \\ n \\ 1 - n \\ s_3 \end{pmatrix} ; \quad S_a \simeq \begin{pmatrix} m (s_3 + n \alpha) + n (\alpha^2 + s_2) \\ - (\alpha \gamma - n) (\alpha \beta - n) \\ m (s_3 + n \alpha) + n (\alpha^2 + s_2) \end{pmatrix} \frac{1}{\alpha} \)

2. Equicenters. \( E_0 \simeq \begin{pmatrix} -m n s_1 + n s_1 s_3 + s_3^2 \\ 2 n s_3 \\ n^2 - n s_2 - m s_3 \\ \end{pmatrix} \begin{pmatrix} m n (\alpha^2 + s_2) + 2 n^2 s_1 + n s_1 (3 \alpha^2 - 2 \alpha s_1 + s_2) - s_3 (\alpha^2 - 2 \alpha s_1 + s_2) \\ 2 n^2 + 2 m \alpha (\alpha - s_1) + 2 \alpha s_3 \\ m (\alpha^2 + s_2) - n^2 \frac{a^2}{s_3} + n s_2 \frac{3 \alpha^2 - 2 \alpha s_1 + s_2}{s_3} + 2 \alpha s_2 \\ \end{pmatrix} \)

Fact 26.5.7. Specific to the pedal LFIT (\( ABC, L_0 \))

1. Synchronized barycentrics for \( (S_0, E_0) \)
Definition 26.6.2. Quadrilateral induced by an ordered quadrangle $(A, B, C, D)$, given in the order $(ABCD)$.

Proof.

26.6 Van Rees cubic

Definition 26.6.1. As defined in (Van Rees, 1829), the Van Rees cubic $vRK$ associated to points $M_1M_2M_3M_4$ is the locus of the points $P$ such that

$$PM_1, PM_3 = PM_4, PM_2$$

26.6.1 Morley point of view

Definition 26.6.2. Quadrilateral induced by an ordered quadrangle. Consider four points $M_i$, given in the order $M_1M_2M_3M_4$. Pretend that lines $M_1M_2$ and $M_3M_4$ are the diagonals. The four remaining lines form a quadrilateral. Then use $M_5 = M_1M_3 \cap M_2M_4$, $M_6 = M_1M_4 \cap M_2M_3$ and note with a star the correspondence $1 \leftrightarrow 2$, $3 \leftrightarrow 4$, $5 \leftrightarrow 6$.

Proposition 26.6.3. The Miquel point and the direction of the Newton line of the induced quadrilateral are:

$$M_q \approx \left( \frac{t_1 t_2 z_2 z_4 - t_3 t_4 z_1 z_2}{t_1 t_2 z_3 z_4 + t_1 t_2 t_4 z_3 - t_1 t_3 t_4 z_2 - t_2 t_3 t_4 z_1}, 1 \right) \approx \left( \frac{z_3 z_4 - z_1 z_2}{z_3 + z_4 - z_1 - z_2}, 1 \right) (26.3)$$

$$\Delta_\infty \approx \left( \frac{t_1 t_2 z_3 z_4 + t_1 t_2 t_4 z_3 - t_1 t_3 t_4 z_2 - t_2 t_3 t_4 z_1}{t_1 t_2 z_3 z_4 + t_1 t_2 t_4 z_3 - t_1 t_3 t_4 z_2 - t_2 t_3 t_4 z_1}, 0 \right) \approx \left( \frac{z_3 + z_4 - z_1 - z_2}{z_3 + z_4 - z_1 - z_2}, 0 \right) \left( \frac{z_3 - z_4 - z_1 + z_2}{z_3 + z_4 - z_1 - z_2}, z_3 + z_4 - z_1 - z_2 \right)$$

Proof. The four triangles are: $[1, 3, 6], [1, 4, 5], [2, 3, 5], [2, 4, 6]$.

Proposition 26.6.4. The Van Rees cubic, also called the isoptic cubic, goes through the six points $M_i$. Points at infinity are both umbilics and $\Delta_\infty$. Moreover, point $M_q$ is a singular focus (and belongs to the curve). Additionally, the cubic goes through the six $U_j$, where $U_j = M_jM_q \cap M_j^* \Delta_\infty$.

When $t_j = 1$ is assumed, the equation of this cubic is:

$$vRK \approx ((\zeta_1 + \zeta_2 - \zeta_3 - \zeta_4) Z - (z_1 + z_2 - z_3 - z_4) Z) Z \bar{Z} + \left( (\zeta_3 \zeta_4 - \zeta_1 \zeta_2) Z^2 + ((z_1 + z_2) (\zeta_3 + \zeta_4) - (z_3 + z_4) (\zeta_1 + \zeta_2)) Z \bar{Z} + (z_1 z_2 - z_3 z_4) \bar{Z} \bar{T} \right) T \ (26.4)$$

$$+ ((z_3 + z_4) \zeta_1 \zeta_2 - (z_1 + z_2) \zeta_3 \zeta_4) Z T^2 + (z_3 z_4 (\zeta_1 + \zeta_2) - z_1 z_2 (\zeta_3 + \zeta_4)) \bar{Z} T^2 + (z_1 z_2 \zeta_3 \zeta_4 - \zeta_1 \zeta_2 z_3 z_4) T^3 = 0$$

Proof. Direct substitutions are easy... even when a direct examination is equally easy!
Proposition 26.6.5. Let $M_j$ be four generic points in the plane, and define

\[
\begin{align*}
    s_1 &= \frac{(z_1 + z_2) \zeta_3 \zeta_4 - (z_3 + z_4) \zeta_1 \zeta_2}{\zeta_3 \zeta_4 - \zeta_1 \zeta_2} \\
    s_2 &= \frac{z_1 z_2 (\zeta_3 + \zeta_4) - z_3 z_4 (\zeta_1 + \zeta_2)}{\zeta_3 + \zeta_4 - \zeta_1 - \zeta_2} \\
    s_3 &= \frac{z_1 z_2 - z_3 z_4}{(\zeta_3 + \zeta_4) - (\zeta_1 + \zeta_2)}; \quad s'_3 = \frac{(z_3 + z_4) - (z_1 + z_2)}{\zeta_1 \zeta_2 - \zeta_3 \zeta_4}.
\end{align*}
\]

Then $s_3 = s'_3$ if and only if $M_q$ belongs to the unit circle $\Gamma$. And then it exists a triangle $\alpha, \beta, \gamma$ inscribed in $\Gamma$ such that the three pairs $(M_{2j+1}, M_{2j+2})$ are isogonal wrt $\alpha \beta \gamma$. Moreover, the Van Rees cubic $vRK_c(M_1 \cdots M_4)$ goes also through the points $\alpha, \beta, \gamma$.

Proof. A simple elimination gives the condition and the values of the $s_j$. This doesn’t imply that all these three points are visible. Concerning the last assertion, convert $nK(\alpha) nK(\beta) nK(\gamma)$ using the symmetric functions $s_j$, substitute (26.5) and check the appearance of the cube of the condition $M_q \in \Gamma$.

26.6.2 Cartesian version

Proposition 26.6.6. In a cartesian frame $X : Y : T$ where $M_q \simeq 0 : 0 : 1$ and $\Delta_\infty \simeq 1 : 0 : 0$, the equation of the cubic can be written as:

\[
vRK_c(X, Y, T) \doteq (X^2 + Y^2)(Y - AT) + T^2XB + T^2YC = 0
\]

where $A, B, C \in \mathbb{R}$ and a parametrization is:

\[
x = \frac{B + \sqrt{B^2 + 4y(A - y)}C - 4y^2(A - y)^2}{2(A - y)}
\]

Proof. Substitute the values of $M_q, \Delta_\infty$ into 26.3 and solve for $z_2, \zeta_2, z_4, \zeta_4$. Substitute the result, together with $Z = X + iY, \overline{Z} = X - iY$ into 26.4. This gives $vRK_c(X, Y, T)$. Then $A, B, C \in \mathbb{R}$ is obvious.
Proposition 26.6.7. When one of these points is on the cubic, so are the other three:

\[ M \doteq \begin{pmatrix} X \\ Y \\ T \end{pmatrix} ; \varnothing(M) \doteq \begin{pmatrix} \frac{BT^2}{AT} - Y - X \\ \frac{X(AT - Y)}{AT} \end{pmatrix} ; \vartheta(M) \doteq \begin{pmatrix} \frac{X(Y - T)}{AT} - Y \\ \frac{Y(AX - T)}{AT} \end{pmatrix} ; (\vartheta \vartheta)(M) = (\vartheta)(M) \]

Moreover, we have the equivalence:

\[ (\vartheta \vartheta) \begin{pmatrix} x_1 \\ y_1 \\ 1 \end{pmatrix} = \begin{pmatrix} x_2 \\ y_2 \\ 1 \end{pmatrix} \in nK \iff \begin{cases} y_1 + y_2 = A \\ x_1y_2 + x_2y_1 = B \\ y_1x_2 - x_1x_2 = C \end{cases} \] (26.6)

Proof. A direct substitution suffices to establish these formula. Points \( M, \varnothing(M) \) are aligned with \( \Delta_\infty \) while points \( M, \vartheta(M) \) are aligned with \( M_q \). These formula were derived in Van Rees (1829) by remarking that \( y_1 + y_2 = A \) leads to the same \( W \). Transformation \( \vartheta \vartheta \) is called conjugacy. One can check that eliminating \( x_2, y_2 \) in 26.6 leads precisely to \( M_1 \in nK \).

Proposition 26.6.8. When \( P, Q \) are conjugates, then

1. Points \( P, Q \) have the same tangential
2. \( |M_qP| \times |M_qQ| = \sqrt{B^2 + C^2} \)
3. The bisectors of \( (M_qP, M_qQ) \) have constant slopes wrt the x-axis, namely \( (C \pm \sqrt{B^2 + C^2}) / B \).

Proof. Direct computation is easy from 26.6.

26.6.3 Barycentric version

Proposition 26.6.9. Consider the quadrilateral \( \mathcal{L}_j \) where \( \mathcal{L}_0 = A'B'C' = [p, q, r] \). Let \( \delta, F \) be resulting Newton line and focus, i.e.

\[ \delta \simeq [qp + rp - qr, \text{ etc}]; \quad F^* \simeq \delta_\infty \simeq p(q - r) : \]

Use \( M_1 = A, M_2 = A', M_3 = B, M_4 = B', M_5 = C', M_6 = C \) (mind the order !) and apply the construction of the former section. Define

\[ U_A \doteq (A \wedge F) \wedge (A' \wedge \delta_\infty), \quad \text{etc}; \quad U'_A \doteq (A' \wedge F) \wedge (A \wedge \delta_\infty), \quad \text{etc}. \]

Then the 12 points \( A, A', \text{etc}, U_A, U'_A, \text{etc} \) are on the vRK whose barycentric equation wrt \( ABC \) is:

\[ \text{vRK}_b \doteq nK \left( \begin{pmatrix} a^2 \\ b^2 \\ c^2 \end{pmatrix}, \begin{pmatrix} qr \\ rp \\ pq \end{pmatrix}, K \right) \simeq qr \left( b^2 z^2 + c^2 y^2 \right) x + 2 \left( \sum S_a qr \right) xyz \]

Moreover, points \( F, \delta_\infty = F^* \) and both umbilics are also on this curve.

Proof. Direct computation.

Exercise 26.6.10. Prove that \( A, \text{etc}, A', \text{etc}, \Omega_2, \Omega_p, F \) are con-cubic with any tenth point in the plane (see Proposition 20.1.4) and are not sufficient to define the vRK. Prove also that, when \( A \) is restricted to the cubic, then \( A' = A' \).

Exercise 26.6.11. Consider two pairs \( M_1 = P, M_2 = P^*, M_3 = Q, M_4 = Q^* \) of isogonal conjugates wrt triangle \( ABC \). Define \( M_5 = M_1M_3 \cap M_2M_4, M_6 = M_1M_4 \cap M_2M_3 \). Then

1. Miquel: circles \( C_{136}, C_{145}, C_{235}, C_{246} \) concur at some \( F \);
2. Newton: midpoints \( m_{12} = (M_1 + M_2) / 2 \) are aligned on some \( \delta \);
3. \( F, \delta_\infty \) are isogonal conjugates, and so are \( M_5, M_6 \);
4. Let $K_aK_bK_c$ be the re-intersections of the sidelines with the cubic $vRK$ through $A, B, C, F, M_j$. The $K_j$ belong to the $ABC$ tripolar of $U$, the root of the cubic, while their reciprocals $L_a = B + C - K_a$, etc are aligned on the line $h$ $(G, -2) (\delta)$.

**Exercise 26.6.12.** Let $ABC$ be the reference triangle, $R_a$ the foot of the $A$-altitude and $A' = 2O - A$. Compute the locus of points $P$ such that $(PB, PR_a) = (PA', PC)$ (1) wrt $ABC$ (and obtain a $pK$ cubic) ; (2) wrt $A'BC$ (and obtain a $nK$ cubic) ; (3) compare the pivot and the root.

26.7 Another cookbook

**Definition 26.7.1.** The four $c_j$ are defined as:

$$
\begin{align*}
  c_3 &= \alpha^2 (\gamma - \beta) p + \beta^2 (\alpha - \gamma) q + \gamma^2 (\beta - \alpha) r \\
  c_2 &= \alpha (\gamma - \beta) p + \beta (\alpha - \gamma) q + \gamma (\beta - \alpha) r \\
  c_1 &= (\gamma - \beta) p + (\alpha - \gamma) q + (\beta - \alpha) r \\
  c_0 &= \frac{1}{\alpha} (\gamma - \beta) p + \frac{1}{\beta} (\alpha - \gamma) q + \frac{1}{\gamma} (\beta - \alpha) r
\end{align*}
$$

where $p : q : r$ are the barycentrics of tripolar $(L_0)$. See Theorem 3.5.5 for more details. Their conjugates are obtained as:

$$
\begin{align*}
  c_k &= -c_3 - k \sigma_3 \\
  c_2 \sigma_1 + c_0 &= c_3 \frac{1}{\sigma_3} + c_1 \frac{\sigma_2}{\sigma_3} \in i\mathbb{R}
\end{align*}
$$

**Lemma 26.7.2.** In the Lubin frame, the transversal points, the Newton line, the Miquel point $M_q$ of the quadrilateral and the Clauson-Schmidt homography $\Psi$ are:

$$
A', B', C' \sim \frac{\beta q - \gamma r}{\alpha} \left( \frac{q - r}{\beta - \gamma} \right), \frac{\gamma r - \alpha p}{\beta - \gamma} \left( \frac{r - p}{\alpha - \gamma} \right), \frac{\alpha p - \beta q}{\beta - \gamma} \left( \frac{p - q}{\alpha - \gamma} \right)
$$

Newton $\sim \frac{1}{z} [2c_2, -s_2 c_1 - c_3, 2 s_3 c_1]$

$$
M_q \sim \frac{1}{z} \left( \begin{array}{c}
  c_2/c_1 \\
  1 \\
  c_1/c_2
\end{array} \right); \Psi \left( \begin{array}{c}
  Z \\
  T
\end{array} \right) \sim \frac{1}{z} \left[ \begin{array}{c}
  Zc_2 - Tc_3 \\
  Zc_1 - Tc_2 \\
  c_1 \bar{Z} - c_0 T \\
  c_2 \bar{Z} - c_1 T
\end{array} \right]
$$
26.8 Simson lines (using barycentrics)

**Proposition 26.8.1. Simson line.** The pedal vertices of a point \( U \) are collinear if and only if the point is on the circumcircle or on the line at infinity. When it exists, this line is called the Simson line of \( U \). When \( U \in \mathcal{L}_\infty \), Simson (\( U \)) = \( \mathcal{L}_\infty \). When \( U \) belongs to the circumcircle of \( ABC \), the barycentrics of it's Simson line are:

\[
U_t \simeq a^2 \cdot \frac{b^2}{t} + \frac{-c^2}{1+t} \Rightarrow \text{Simson (} U \text{)} \simeq \left[ \frac{1}{a^2 t + S_c} \cdot \frac{-t}{S_c t + b^2} \cdot \frac{1 + t}{S_b t - S_a} \right]
\]  
(26.7)

**Proof.** Collinearity condition is the same as for the Steiner line. Everything else is either the result of straightforward computations, or obtained from the Steiner line equation (10.2), right multiplied by \( h(U, 2) \) of (7.27).

**Proposition 26.8.2.** When \( U \) is on the circumcircle, the two intersections of Simson (\( U \)) and the nine points circle \( \gamma \) are (i) the midpoint \( M \) of \([U, X_4]\) (ii) the intersection \( L \) of Simson (\( U \)) and Simson (\( U' \)) where \( U' \) is the \( \Gamma \)-antipode of \( U \).

**Proof.** By definition, Simson \( (U) \) is obtained from Steiner \( (U) \) by the point-transformation \( h(U, 1/2) \).

Since \( X_4 \in \text{Steiner} (U) \), point \( M \) is on Simson \( (U) \). Using now \( h(X_4, 1/2) \), \( U \in \Gamma \) becomes \( M \in \gamma \) and (i) is proved. Therefore midpoint \( M' \) of \([U, X_4]\) is the \( \gamma \)-antipode of \( M \), while \( LM \perp LM' \).

**Proposition 26.8.3.** When \( \Delta \simeq [p, q, r] \) is a Simson line, then we have:

\[
p (q^2 + r^2) S_a + q (p^2 + r^2) S_b + r (p^2 + q^2) S_c - 2 S_w p q r = 0
\]  
(26.8)

In other words, tripole \((\Delta)\) belongs to the Simson cubic K010.

**Proof.** See Section 20.5.2.

26.9 The Steiner deltoid (using Lubin-1)

**Proposition 26.9.1.** When \( \tau \) is a turn, its Simson line wrt triangle \( ABC \) is

\[
\Delta (\tau) \simeq [2 \tau^2, s_3 + \tau s_2 - \tau^2 s_1 - \tau^3, -2 s_3 \tau]
\]

The envelope of all these lines is the so-called Steiner’s deltoid while the contact point of \( \Delta (\tau) \) with its envelope is \( \delta (\tau) = \frac{1}{2} s_1 + \tau + \frac{s_3}{2 \tau^2} \). This deltoid is a bicircular quartic, whose implicit equation (using coordinates \( Z = \frac{1}{2} s_1 + z \), etc) is:

\[
16 Z^2 \overline{Z}^2 - 32 \left( \frac{1}{s_3} Z^4 + s_3 \overline{Z}^3 \right) T + 72 Z \overline{Z} T^2 - 27 T^4 = 0
\]

**Proof.** The contact point comes from \( \delta (\tau) = \Delta (\tau) \wedge \frac{\partial}{\partial \tau} \Delta (\tau) \).

**Proposition 26.9.2.** This deltoid is the roulette created by a point on the circumference of a circle \( \rho = 1/2 \) as it rolls without slipping along the inside of the fixed circle \( \rho = 3/2 \) centered at \( X(5) \), the Euler’s point. Cups are at \( \rho = 3/2 \), innermost points are at \( \rho = 1/2 \).

**Proof.** Let \( \Theta \) be the fixed circle and \( E_\theta \) its center, while \( \theta \) is the moving circle and \( X \) its center. The locus of contact is point \( Y = (3X - E_\theta) / 2 \). Let \( Y_\theta \) and \( Y_\theta \) the material points of both circles that are at place \( Y \) at the moment of contact. As ever, speed of \( Y_\theta \) is zero. But, by composition of movements, speed of \( Y_\theta \) vanishes at the moment of contact, proving the "without slipping" assertion.

**Proposition 26.9.3.** Any line \( \Delta (\tau) \) tangent to the deltoid contains the following points:

\[
d_\tau \simeq \frac{1}{2} s_1 + \tau \pm \frac{s_3}{2 \tau^2} \; ; \; m_\tau, n_\tau \simeq \frac{1}{2} s_1 \pm \sqrt{s_3} \tau \; ; \; h_\tau \simeq \frac{1}{2} s_1 + \frac{1}{2} \tau \; ; \; k_\tau \simeq \frac{1}{2} s_1 - \frac{1}{2} \frac{s_3}{\tau^2}
\]

Points \( d, \delta, m, n \) are the four points in common with the curve (\( \delta \) is the contact point, \( m, n \) are the so called extremities of the tangent). Point \( h_\tau, k_\tau \) are the two points in common with the inscribed circle. Moreover \( h_\tau \) is both the middle of \([m_\tau, n_\tau]\) and of \( [d_\tau, k_\tau] \).
Proof. Obvious computations. \( \square \)

**Proposition 26.9.4.** Use index \( j = 9 \) to save the original triangle as \( A_0B_0C_0 \). Consider four turns \( \tau_j \ (j = 0, a, b, c) \) and the corresponding Simson lines \( \Delta_j \). Note \( A \cdots C' \) their six intersections and use

\[
q_1 = \sum \tau_a, \quad q_2 = \sum \tau_a \tau_b, \quad q_3 = \sum \tau_a \tau_b \tau_c, \quad q_4 = \tau_a \tau_b \tau_c \tau_0
\]

Let \( O_j, H_j, E_j \) be the circumcenters, orthocenters and NPC of triangles \( T_j \). One has:

\[
O_j = \frac{1}{2} (s_1 + q_1) - \frac{1}{2} \tau_j \quad ; \quad H_j = \frac{1}{2} s_1 + s_3 (q_1 - \tau_j) \tau_j + (2q_4)
\]

while the perpendicular bisector of \( [O_jH_j] \) and the perpendicular from \( E_j = (O_j + H_j) / 2 \) to \( \Delta_j \) are:

\[
\text{med}_j \simeq \left[ 2 \tau_j^2 - 2 \tau_j q_1 + 2 q_2, -\tau_j^2 s_1 + (q_1 s_1 - s_2) \tau_j + s_2 q_1 - s_1 q_2, 2 \tau_j s_3 - 2 s_3 q_1 \right]
\]

\[
\text{perp}_j \simeq \left[ 4 q_4 \tau_j, - (s_3 q_2 + 2 s_1 q_4) \tau_j - (q_2 + 2 s_2) q_4, 4 s_3 q_4 \right]
\]

Proof. Use the cookbook, or write down the six intersections \( A = \frac{1}{2} \left( s_1 + \tau_b + \tau_c + \frac{s_3}{\tau_b \tau_c} \right) \), etc and compute directly. \( \square \)

**Theorem 26.9.5.** Morley (1903) The four lines \( \text{med}_j \) concur at \( E_9 = \frac{1}{2} s_1 \), i.e. at the center of the deltoid, while the four lines \( \text{perp}_j \) concur at \( H_m = \frac{1}{2} s_1 + s_3 q_2 / (4q_4) = \frac{1}{4} \sum H_j \), on the Steiner axis.

Proof. As ever, a theorem is a key result, not necessarily something difficult to prove. Indeed, computations are easy! \( \square \)

**Construction 26.9.6.** Construct the deltoid tangent to four given lines. Obtain the \( O_j \) and \( H_j \) and then \( E_j, H_m \) from the Morley theorem. Let \( \omega \) be the center of the Miquel circle and \( \rho \) its radius. Consider the points \( h_j \equiv E_j + \omega - O_j \). For each \( j \), \( h_j \) belongs to \( \mathcal{L}_j \), while the four \( h_j \) are on the circle \( (E_2, \rho) \). Define the turns \( \tau_j = (\omega - O_j) / \rho = (h_j - E_2) / \rho \) and compute the \( q_k \). Then

\[
\alpha \mapsto \delta_\alpha \equiv E_2 + 2 \rho \alpha + \psi \frac{\rho}{\alpha^2}
\]

where \( \psi = \frac{1}{\rho} E_2 H_m \frac{2q_4}{q_2} \)

is the parametric equation of the deltoid tangent to the four lines. The contact points are obtained at \( \alpha = \tau_j \).

Proof. One obtains the following values

\[
W = \sqrt{\frac{1}{n^2 s_3} \left( mn + ns_1 - s_3 \right) \left( ms_3 + ns_2 - n^2 \right)}
\]

\[
\rho = \frac{i \sqrt{n} s_3 W}{n^3 - n^2 s_2 + n s_3 - s_3^2} ; \quad \psi = \frac{i \left(n s_3 + ns_2 - n^2 \right)}{\sqrt{n} W}
\]

\( \square \)

**Remark 26.9.7.** Points \( h_j \) are also the orthocenters of the four triangles \( O_i O_k O_n \), with affixes:

\[
h_0 = \frac{(ns_1 - s_3) (ms_3 - n^2 + ns_2)}{(\alpha \gamma - n) (\alpha \beta - n) (\beta \gamma - n)} ; \quad h_0 = \frac{n (s_1 - \alpha) (ms_3 - n^2 + ns_2)}{(\alpha \gamma - n) (\alpha \beta - n) (\beta \gamma - n)}
\]

Following Proposition 26.9.3, we can also obtain \( k_j \) as the second intersection of \( \mathcal{L}_j \) with the circle, and then the \( \delta_j \) as \( 2h_j - k_j \).
26.10 Pedal LFIT and orthopole

26.10.1 Starting by some computations

Definition 26.10.1. The orthopole of a line $\Delta \neq \mathcal{L}_\infty$, is the Neuberg center of the FLIT provided by the pedal triangles of its points. Some orthopoles are given in Table 1.1.

Proposition 26.10.2. Synchronized values of the orthopole $E(\Delta)$, the slowness center $S(\Delta)$ and the pillar point $\Omega(\Delta)$ associated to line $\Delta$ are given by:

$$
S(\Delta) = \frac{-1}{2S \times \text{isogon}} (\mathbf{M} \cdot \Delta)
$$

$$
E(\Delta) = t \left( \Delta \cdot \mathbf{M} \right) \ast t \left( \Delta \cdot \mathbf{N} \right)
$$

$$
\Omega(\Delta) = t \left( \Delta \cdot \mathbf{M} \right) \ast t \left( \Delta \right)
$$

$$
f + g + h = (\Delta \cdot \mathbf{M}^t \Delta)
$$

where isogon is understood as isogon $(f : g : h) = a^2gh : b^2hf : c^2fg$ while matrix $\mathbf{M}$ is defined by (7.18) and matrix $\mathbf{N}$ is defined by :

$$
\mathbf{N} = 1 - \frac{2}{\mathcal{L}_\infty \cdot O} \cdot \mathcal{L}_\infty = \frac{1}{4S^2}
\begin{pmatrix}
S_bS_c & -a^2S_a & -a^2S_a \\
-b^2S_b & S_aS_a & -b^2S_b \\
-c^2S_c & -c^2S_c & S_aS_b
\end{pmatrix}
$$

while quantity $f + g + h = u + v + w = \rho + \sigma + \tau$ vanishes for isotropic lines, i.e. lines through an umbilic. Aren’t they special lines?

Proof. Denote the line by $\Delta \simeq [p, q, r]$. Describe its points as $M = (0 : -r : q) + (q - r) t (\mathcal{L}_\infty \wedge \Delta)$.
and consider the set of the pedal triangles of $M$ ($t$). Their normalized expression are:

$$
T(t) = \begin{bmatrix}
0 & -t & \frac{rS_c}{(q-r)b^2} & \frac{t}{h} + \frac{S_b q}{(q-r)c^2} \\
\frac{t}{f} - \frac{r}{g} & 0 & -\frac{b^2q + S_a r}{(q-r)b^2} & \frac{t}{h} - \frac{c^2r + S_a q}{(q-r)c^2} \\
\frac{t}{f} + \frac{q - r}{g} & \frac{t}{f} + \frac{q - r}{g} & 0 & 0
\end{bmatrix}
$$

where $f, g, h$ can be identified as the isogonal conjugate of orthocenter ($\Delta$). The $E$ formula follows. 

**Remark 26.10.3.** The normalization of $[\mathcal{M}]$ comes from $[\mathcal{N}]^T \mathcal{N} = [1, 1, 1]$ while the normalization of $[\mathcal{M}]^T \mathcal{M}$ is required to provide synchronized values for $S$ and $E$, the common factor $32S^3$ which appears when computing the quantities of formula (26.9) cannot be avoided. Afterwards, one can only proceed to a global simplification, in order to provide better looking formula.

**Exercise 26.10.4.** Consider the Euler line, using for example $P_1 = X(2)$ and $P_2 = X(4)$. Obtain $S = 110$, $E = \Omega = 125$, $\omega = 5972$. Intersections with circumcircle: $X(1113)$, $X(1114)$. Check the three cevians:

<table>
<thead>
<tr>
<th>pedal</th>
<th>3</th>
<th>4</th>
<th>20</th>
</tr>
</thead>
<tbody>
<tr>
<td>cevian</td>
<td>2</td>
<td>4</td>
<td>69</td>
</tr>
</tbody>
</table>

**Construction 26.10.5.** Construct the orthopole $E$ of $\Delta$ wrt triangle $ABC$. Project $A, B, C$ onto $\Delta$ and obtain $Q_a, Q_b, Q_c$. Draw $\Delta_a$ by $Q_a$ orthogonally to $BC$, etc. Then the three lines $\Delta_a, \Delta_b, \Delta_c$ concur at $E$. This property was at the origin of the name "orthopole".

**Proof.** Straightforward computation.

### 26.10.2 Using Simson lines

It has been already stated that a LFIT is best characterized by its degenerate triangles. And we know that a pedal triangle is flat if and only if its center is on the circumcircle. Thus we introduce points $P_3, P_4$ that are the intersections (visible or not) of line $\Delta = P_0P_1$ and the circumcircle $\Gamma$.

**Proposition 26.10.6.** Let $T_3 = (a_3b_3c_3), T_4 = (a_4b_4c_4)$ be the pedal (flat) triangles of $P_3, P_4$. Then $E$ is the intersection of the Simson lines $T_3$ and $T_4$, while $\omega = (S + E)/2$ is the intersection of the Newton lines $N_3$ and $N_4$ where $N_3$ goes through the aligned points $a_3 + A; b_3 + B; c_3 + C$, etc.

**Proof.** General property of the degenerate triangles of an LFIT, see Section 25.10.

**Proposition 26.10.7.** As already said, point $S$ lies on the circumcircle. Its Simson line is parallel to $\Delta$. Point $S' = 20 - S$ is the isogonal conjugate of the direction of $\Delta$. The Simson line of $S'$ is perpendicular to $\Delta$ and, moreover, this line goes through $E$, the orthopole of $\Delta$.

**Proof.** Everything is obvious, except the last point. And every computation is straightforward. One can also follow the Steiner movie, given at Corollary 10.2.4.

**Proposition 26.10.8.** The equicenter (orthopole) $E$ lies on the reciprocal to the image of $\Delta$ in the circumcenter $O$.

**Proof.** Matrix $[\mathcal{N}]^T$ takes that image. And then, point $pu : qv : rw$ belongs to line $[1/u, 1/v, 1/w]$ when $p + q + r = 1$. And, here, $[\mathcal{M}]^T \Delta \in \mathcal{L}_\infty$. 

**Exercise 26.10.9.** Vertex-Miquel circles. In the pedal-LFIT relative to $\Delta$, the circles $(A, b, c)$ are the circles having $(A,M_A]$ as diameter. Therefore the $Q_A$, etc Points of Proposition 25.6.2 are the projection of the vertices on $\Delta$. The Vertex-Miquel point is simply $M_i$... whose locus is nothing than line $(Q_A, Q_B, Q_C) = \Delta$, whose direction is the orthopoint of $S^*$. This doesn’t contradict the former assumption $S^* \in \text{cycle}(Q_A, Q_B, Q_C)$ since this cycle is $\mathcal{L}_\infty \cup \Delta$! And $S^*$ remains the perspector of triangles $ABC$ and $Q_AQ_BQ_C$. See Figure 26.3.
Exercise 26.10.10. The fixed points (as characterized by concurrent corresponding lines) are \( P_s = \text{dir}(AH) \in \mathcal{L}_s \), etc, and \( P_a, E, O_a, O'_a \) are aligned as ever.

Proposition 26.10.11. The area of the \( P \)-pedal triangle is \( \frac{S}{4} \left( 1 - \frac{|OP|}{R^2} \right) \).

Proof. This is obviously true for \( P = O \). Otherwise, draw the line \( \Delta \perp OP \): It cuts \( \Gamma \) at \( P_3, P_4 \). Then we parameterize the line \( \Delta \) by \( 2P = (1 + t) P_3 + (1 - t) P_4 \), so that \(|t| = |OP|/R\). The family of all the pedal triangles \( T_P \) of \( P \in \Delta \) forms a \( LFIT \) since formula (9.2) is of first degree in \( t \). As a result, \( \text{area}(T_P) \) is a \( t \)-polynomial of degree 2, while our formula is already true for three points (pedal triangles \( T_{P_{3,4}} \) are flat).

26.11 Sister Marie Cordia Karl

Proposition 26.11.1. Let \( \mathcal{L} = [u, v, w] \) and \( \Delta = [p, q, r] \) be two lines. The condition for the orthopole of \( \mathcal{L} \) belongs to \( \Delta \) can be written as

\[
\Delta \cdot \mathfrak{V}(\mathcal{L}) = 0 \quad \text{where} \quad \mathfrak{V}(\mathcal{L}) = \begin{bmatrix}
2a^2 S_a S_{a'} & 2b^2 S_b S_{b'} & 2c^2 S_c S_{c'} \\
2a^2 S_a S_a' & 2b^2 S_b S_b' & 2c^2 S_c S_c' \\
2a^2 S_a S_b' & 2b^2 S_b S_a' & 2c^2 S_c S_b + c^2 S_c^2 \\
-\frac{S_a}{S_c} (a^2 c^2 + S_b^2) & -\frac{S_b}{S_c} (a^2 b^2 + S_c^2) & -\frac{S_c}{S_c} (a^2 b^2 + S_c^2) \\
\end{bmatrix}
\]

Proof. Obvious from (26.9).

Definition 26.11.2. Map \( \mathcal{L} \mapsto \mathfrak{V}(\mathcal{L}) \) is the full-Veronese map (whose signature is: row(3) \( \mapsto \) column(6)), we are building conics here, not circles. For a given \( \Delta \), equation \( \Delta \cdot \mathfrak{V}(\mathcal{L}) = 0 \) is the equation of a tangential conic. We call it the associated parabola of \( \Delta \) and note it: \( \mathfrak{V}(\Delta) \), while we call \( \mathfrak{V}(\Delta) \) the Sister Mary Cordia Karl’s matrix, due to Karl, E. (Sister Mary Cordia), 1932.

Proposition 26.11.3. Conic \( \mathfrak{V}(\Delta) \) is a parabola. Its point at infinity is \( \mathfrak{V}(\infty) = \frac{1}{\mathfrak{V}(\Delta)} \), while its focus \( \mathfrak{V}(\Delta) \) is the isogonal conjugate of \( \mathfrak{V}(\infty) \). Moreover, its directrix (polar of the focus) is homothetic to \( \Delta \) from \( X(4) \) (ratio=2).

Proof. Compute \( \mathfrak{V}(\Delta) \cdot \mathfrak{V}(\mathcal{L}) \), and recognize the orthodir \( \mathfrak{M}(\mathcal{L}) \) of \( \Delta \), proving the contact. Then write that the isotropic lines of the focus belongs to the conic, and obtain the given result. For the last point, you find it by computing the directrix, obtaining a linear formula, extracting its matrix and recognizing homot (\( H, 2 \))... but you prove it simpler by \( \mathfrak{V}(\Delta) \cdot \mathfrak{V}(\mathcal{L}) \) is homothetic to \( \Delta \) from \( X(4) \) (ratio=2) (and 1/2, we are acting on lines, not on points).

Example 26.11.4. Taking the sidelines and \( \mathcal{L}_\infty \) as examples, one obtains:

\[
\mathfrak{V}((BC)) (u, v, w) = (a^2 u + S_c v + S_b v) (S_b S_c y - b^2 S_b v - c^2 S_c y) \\
\mathfrak{V}((CA)) (u, v, w) = (b^2 v + S_a w + S_b w) (S_a S_b y - c^2 S_b w - a^2 S_a y) \\
\mathfrak{V}((AC)) (u, v, w) = (c^2 w + S_b u + S_a u) (S_b S_a y - a^2 S_a u - b^2 S_b v) \\
\mathfrak{V}(\mathcal{L}_\infty) (u, v, w) = (u (S_b - 2iS) + v (S_a + 2iS) - c^2 w) (u (S_b + 2iS) + v (S_a - 2iS) - c^2 w)
\]

the last one being the conic of isotropic lines.

Proposition 26.11.5. The \( \mathfrak{V}(\Delta) \) conic is degenerate if and only if \( \Delta \) is the Simson line of some point \( M \) (on \( \Gamma \), the circumcircle). In such a case, the pivots are \( M \) and its isogonal conjugate (the orthodir of \( \Delta \), on \( \mathcal{L}_\infty \)).

Proof. Compute the determinant and obtain (26.8). Since this equation is invariant by isotomic conjugacy, this tells us that \( \text{tripolar}(\Delta) \) is on \( K010 \), so that \( \Delta \) is a Simson line. Conversely, computing the conic from parametrization (26.7) of Simson (\( M_4 \)), leads to \( M_4 \) and \( M'_4 \in \mathcal{L}_\infty \).
Remark 26.11.6. When points are opposite on $\Gamma$, Simson lines are orthogonal and also reciprocate (characteristic property).

Proposition 26.11.7. Three Simson lines (visible or not) are going through any generic point. By the orthopole of a known line $L$, we have the Simson line orthogonal to $L$, given by:

$$\Delta_1 \approx \left[ \frac{q - r}{S_a r + S_c q - a^2 p}, \frac{r - p}{S_a r + S_c p - q b^2}, \frac{p - q}{S_a q + S_b p - c^2 r} \right]; \quad t = -\frac{p - r}{q - r}$$

and the Simson lines of the points of $\Gamma \cap L$.

Proof. Degree of equation is 3. For an orthopole, the equation splits. The first degree factor leads to $\Delta_1$. The other part is exactly the condition for $\mathcal{M}_t \in \mathcal{L}$.

Construction 26.11.8. Construct the three Simson lines through a given point $P$. Let $Q = A + B + C - 2P$. Draw the hyperbola $\mathcal{H}_A$ having $[A, Q]$ as diameter and $I_A I_c, I_b I_c$ as asymptotic directions. The three hyperbolas $\mathcal{H}_A, \mathcal{H}_B, \mathcal{H}_C$ have four points in common, $Q$ itself and three others $R_j$, which are on the circumcircle. Then the Simson lines of the $R_j \triangleq 2O - R_j$ are going through $P$. Moreover the three $R_j$ are visible when $P$ is inside of the Steiner deltoid, while only one is visible when $P$ is outside of the deltoid.

Proof. Let $P = z : t : \zeta$ be a generic point in the plane, and $M$ a point on the unit circle (turn $\delta$). Then $P$ belongs to the Simson line of $2O - M$ when

$$\Theta \triangleq 2\delta^2 z + (\delta^3 - \delta^2 s_1 - \delta s_2 + s_3) t + 2 \delta s_3 \zeta = 0$$
On the other hand, asymptotes $\delta_B, \delta_C$ of $\mathcal{H}_A$ are the parallels to $AB, AC$ drawn through $(A + Q)/2$. Then $\mathcal{H}_A = \delta_B \times \delta_C - \mu L^2$ where $\mu$ is determined by $A \in \mathcal{H}_A$ while
\[
\delta_B(M) \times \delta_C(M) = \text{dist}(M, \delta_B)^2 - \text{dist}(M, \delta_C)^2
\]
allows the parametrization $A = z_A$, etc. This leads to
\[
\mathcal{H}_A \simeq \begin{bmatrix}
-2t & -2s + (s + \alpha) t & 0 \\
-2s + (s + \alpha) t & 4 \alpha s + 2 (\beta \gamma - \alpha^2) t - 4 s_3 + \beta \gamma + s_2 + 2 \alpha s_3 \zeta & -\alpha (\beta \gamma + s_2) t + 2 \alpha s_3 \zeta \\
0 & -\alpha (\beta \gamma + s_2) t + 2 \alpha s_3 \zeta & 2 \alpha s_3 t
\end{bmatrix}
\]
And we see that $M \in \mathcal{H}_A$ is either $\delta = \alpha$ or $\Theta = 0$. As a result, the three hyperbolas belong to the same pencil. This can be checked by $\sum (\beta - \gamma) \mathcal{H}_A = 0$. The deltoid appears when one computes the discriminant of $\Theta$. \(\blacksquare\)

### 26.12 The four pedal LFIT of a quadrilateral

**Proposition 26.12.1.** Consider the four LFIT created by the pedal triangles of points on line $\mathcal{L}_j$ wrt the trigone $T_j$ of the other three lines. The four slowness centers $S_j$ are on the Miquel circle, while the four Neuberg centers $\mathcal{E}_j$ are on the Steiner axis.

**Proof.** Computation to obtain $S_0 \in \text{Miquel}$ and $\mathcal{E}_0 \in \text{Steiner}$, symmetry to obtain the rest. One can also use the fact that $S_0$ is the perspector of $\Delta ABC$ and $O_aO_bO_c$. \(\blacksquare\)

**Corollary 26.12.2.** Point $S_j$ is therefore the second intersection of the Miquel circle $(O_0, O_a, O_b, O_c)$ and circumscribed circle $(O_j)$, while $\mathcal{E}_j$ is the intersection of the Simson line of $2O_j - S_j$ with the line $(H_0, H_a, H_b, H_c)$.

**Proposition 26.12.3.** When point $M$ moves along the line $\mathcal{L}_0$, the orthopole $\mathcal{E}_0$ has a constant power $\rho^2$ with respect to the pedal circle of $M$ wrt $T_0 = ABC$. The value of $\rho^2$ is ((Gulasekaram, 1941))

\[
\rho^2 = \frac{\left|\mathcal{L}_0 \cdot \begin{bmatrix} a^2 S_a \\ b^2 S_b \\ c^2 S_c \end{bmatrix} \right|^2}{16 S^4 \left|\mathcal{L}_0 \cdot \left[\mathcal{M} \right]^t \mathcal{L}_0 \right|^2}
\]

(26.11)

Moreover $\rho^2$ is $(-2)$ times the algebraic product of dist$(O_0, \mathcal{L}_0)$ and dist$(\mathcal{E}_0, \mathcal{L}_0)$.

**Proof.** Start from the columns of triangle $[T(t)]$, compute the Veronese rows and take their wedge. Then obtain $\rho^2(t)$ ... and see that all the $t$ cancel. Moreover, we have the formulas:

\[
\text{dist} (\mathcal{L}_0, O_0) = \sqrt{2S} \frac{(S_a a^2 p + b^2 S_b q + c^2 S_c r)}{8S^2 \left|\mathcal{L}_0 \cdot \left[\mathcal{M} \right]^t \mathcal{L}_0 \right|}
\]

\[
\text{dist} (\mathcal{L}_0, \mathcal{E}_0) = \sqrt{2S} \frac{\prod (S_b r + S_c q - a^2 p)}{8S^3 \left(\mathcal{L}_0 \cdot \left[\mathcal{M} \right]^t \mathcal{L}_0 \right)^3}
\]

And we can check the homogeneous degrees: in $a$: $(4 + 2 + 2 + 2) - 8 = 2$ (since $\rho$ is a length); in $p$: $4 - 4$ (as required since $\mathcal{L}_0$ is projectively defined). \(\blacksquare\)

**Corollary 26.12.4.** When point $M$ moves along line $\mathcal{L}_A$, the orthopole $\mathcal{E}_A$ has a constant power $\rho^2_A$ with respect to the pedal circle of $M$ wrt trigone $T_A$.

\[
\rho^2_A = \frac{p a^2 (p - q - r) S_a - S_b b^2 p r - S_c c^2 p q + 8 S^2 r q}{4 a S (p - r) (p - q)} \times \frac{p S_b S_c (-a^2 p + S_b r + S_c q)}{a^3 S (p - r) (p - q)}
\]
26. Quadrilaterals

**Proof.** Simply using the product of distances. One can also recompute everything, and/or using an algebraic change of parameters, followed by a collineation. □

**Proposition 26.12.5.** When point \( M \) moves along \( \mathcal{L}_j \), its pedal circumcircle wrt triangle \( T_j \) remains orthogonal to a fixed circle centered at \( \mathcal{E}_j \) and whose radius is \( \sqrt{\rho_j^2} \). Let us call it the \( j \)-orthopedal circle, noted \( \mathcal{G}_j \). This circle is virtual when \( O_j, \mathcal{E}_j \) are on the same side of \( \mathcal{L}_j \).

**Proof.** Immediate from the invariance of \( \rho_j^2 \). See also Thebault, 1946. □

**Proposition 26.12.6.** The four orthopedal circles belong to the Steiner pencil (generated by the polar circles).

**Proof.** When \( T_0 \), the generic point on \( \mathcal{L}_0 \), comes at \( C' \), we have \( C'T_a \perp T_aC \) and \( C'T_b \perp T_bC \). Thus \( T_a, T_b \) are on the circle with diameter \([T, C]\) and the circle \((T_aT_bT_c)\) is the circle with diameter \([C'C]\). This occurs also with the other two circles defining the Newton pencil. One can also use (26.2). □

### 26.13 Exercises

**Exercise 26.13.1.** Use the Lubin-1 representation, and write the transversal as the \( t \)-locus of

\[
-ik \tau + t \tau : 1 : \frac{ik}{\tau} + \frac{t}{\tau}
\]

where \( k \) is real and \( \tau \) is a turn. Recompute everything, especially \( \text{dist}(\Delta, \mathcal{E}) \)... that only depends on \( \tau \) (see Goormaghtigh, 1939).

**Exercise 26.13.2.** Point \( O_0 \) is the perspector of \( S_A, S_B, S_C \) with \( ABC \).

**Exercise 26.13.3.** Let \( P_0, P_a, P_b, P_c \) be the projections of \( M_q \) on the sidelines. There is a similarity \( P_a \mapsto A', P_b \mapsto B', P_c \mapsto C' \). Elaborate further.

**Exercise 26.13.4.** Let be \( Q_0, Q_a, Q_b, Q_c \) the reflections of \( M_q \) about the sidelines. The similarity \( \sigma \) defined by \( Q_a \mapsto A', Q_b \mapsto B' \) is centered at \( M_q \), maps the Steiner line onto \( \mathcal{L}_0 \), sends \( Q_c \) to \( C' \), the circle \( ABC \) onto the Miquel circle and \( A \mapsto S_A \), etc.

**Exercise 26.13.5.** When points \( U_1 \) and \( U_2 \) are antipodes on the circumcircle, the orthopole of Simson (\( U_1 \)) is the intersection of Simson (\( U_2 \)) with Steiner (\( U_1 \)).

**Exercise 26.13.6.** When a line goes through the circumcenter, its orthopole belongs to the nine points circle.

**Exercise 26.13.7.** When a line goes through a fixed point \( P \), its orthopole belongs to the conic centered at \((P + H)/2\) and passing through the projections of \( P \) on the sidelines.

**Proposition 26.13.8.** When \( X \) moves on a line through the centroid \( X_2 \), then orthopole of \( \text{trilipo}(X) \) moves on a line through the orthocenter \( X_A \). For example :

<table>
<thead>
<tr>
<th>( X ) on line</th>
<th>orthopole (trilipo) ((X)) on line</th>
</tr>
</thead>
<tbody>
<tr>
<td>( L(2, 1) )</td>
<td>( L(4, 9) )</td>
</tr>
<tr>
<td>( L(2, 3) )</td>
<td>( L(4, 6) )</td>
</tr>
<tr>
<td>( L(2, 6) )</td>
<td>( L(4, 3) = L(2, 3) )</td>
</tr>
<tr>
<td>( L(2, 7) )</td>
<td>( L(1, 4) )</td>
</tr>
</tbody>
</table>

**Proof.** Suppose \( X \) is not \( X_2 \) and does not lie on a sideline of triangle \( ABC \). Then, using barycentrics, we have :

\[
\text{orthopole} (\text{trilipo}(X)) \cap X_A = \ S_a (y - z) : S_b (z - x) : S_c (x - z) = (S_a : S_b : S_c) \cdot (X \wedge X_2)
\]

□

—I. pldx : Translation of the Kimberling’s Glossary into barycentrics ——
26.14 Diagonal triangle

Notation 26.14.1. In this section, $A, B, C$ is the diagonal triangle of the quadrilateral, with its $a, b, c, S_a, S_b, S_c$. The four lines are

$$L_0, L_a, L_b, L_c \simeq \begin{bmatrix} \rho & \sigma & \tau \\ -\rho & \sigma & \tau \\ \rho & -\sigma & \tau \\ \rho & \sigma & -\tau \end{bmatrix}$$

while the six vertices are noted by:

$$A'', B'', C'', A', B', C' \simeq \begin{bmatrix} 0 \\ \tau \\ 0 \\ \sigma \\ 0 \\ -\tau \\ 0 \\ \rho \\ 0 \end{bmatrix}$$

so that relation $A', B', C' \in L_0$ remains valid.

Fact 26.14.2. Basic objects

1. Newton $\simeq \left[ \rho^2, \sigma^2, \tau^2 \right]$, while $(A'' + A')/2 \simeq -\tau^2 : \sigma^2$
2. Newton pencil $B_2 = b^2 (\rho^2 - \sigma^2) + c^2 (\tau^2 - \rho^2) ; E_x = a^2 (a^2 \rho^2 - \sigma^2 b^2 - \tau^2 c^2)$
3. Steiner pencil $B_2 = \rho^2 ; E_x = 0$
4. Steiner line $[b^2 (\rho^2 - \sigma^2) + c^2 (\tau^2 - \rho^2), \text{etc}]$
5. Miquel $M_2 \simeq \left[ \begin{array}{c} (\sigma^2 - \tau^2) (b^2 (\rho^2 - \sigma^2) + c^2 (\tau^2 - \rho^2)) \\ (\tau^2 - \rho^2) (c^2 (\sigma^2 - \tau^2) + a^2 (\rho^2 - \sigma^2)) \\ (\rho^2 - \sigma^2) (a^2 (\tau^2 - \rho^2) + b^2 (\sigma^2 - \tau^2)) \end{array} \right] ; \text{length}(\Gamma_M) \approx 4000$

6. Centers of circumcircles: length $\approx 1500$
7. Orthocenters:
   $$H_0 \simeq \left( (\sigma - \tau) (b^2 - c^2) + (2 \rho + \sigma + \tau) a^2 \right) \left( (\rho^2 + \sigma \tau a^2 - (\sigma + \tau) (\sigma b^2 + \tau c^2) \right) , \text{etc}$$
8. vanRees cubic: $\sum_{j=1}^3 (-\rho^2 x^2 + \sigma^2 y^2 + \tau^2 z^2) s_a + xyz \sum_{j=1}^3 \rho^3 a^2 = 0$

26.15 Rigby points

Proposition 26.15.1. Given three distinct points $U_j$ on the circumcircle, the following conditions are equivalent:

1. The three Simson lines are concurrent in a point $K$.
2. Simson line of $U_j$ is orthogonal with line $U_{j+1}U_{j-1}$
3. Two sidelines of $U_1U_2U_3$ have the same orthopole $K$ (and therefore the third too).

In such a case, the three points $U_j$ are said to form a "Rigby triangle", and $K$ is their Rigby result. When using Lubin-I representation, the condition is $\tau_1 \tau_2 \tau_3 = \alpha \beta \gamma$, and $z(K) = \frac{1}{2} (\alpha + \beta + \gamma + \tau_1 + \tau_2 + \tau_3)$.

Proof. All three properties lead to the same condition, and the same value of $z(K)$. $\square$

Exercise 26.15.2. Prove that $K$ is the midpoint of $[\alpha + \beta + \gamma ; \tau_1 + \tau_2 + \tau_3]$ and conclude (Honsberger, 1995, p. 136).

Remark 26.15.3. In ETC, the third of a Rigby triangle is called the Simson-Rigby point of the first two, and noted $U_3 = SR(U_1, U_2)$, while their common result is called the Rigby-Simson point and noted $K = RS(U_1, U_2)$.
Corollary 26.15.4. When using barycentrics, and \( U_1 \simeq p : q : r, U_2 \simeq u : v : w \in \Gamma \), we have:

\[
SR(P, U) = isog \left( (P \land U) \land L_\infty \right) = \frac{a^2}{(q + r)u - (v + w)p} : \frac{b^2}{(p + r)v - (u + w)q} : \frac{c^2}{(p + q)w - (u + v)r}
\]

(26.12)

One can also use the Peter Moses (2004/10) expression:

\[
U_3 \simeq \frac{qw - rv}{rwb^2 - qvc^2} : \frac{ru - pw}{c^2up - a^2wr} : \frac{pv - uq}{a^2qv - b^2pu}
\]

Example 26.15.5. Centers X(2677) to X(2770) are examples of SR and RS points. In the following table, the first three of a quadruple is a (sorted) triangle \( U_1U_2U_3 \) and its Rigby point.

<p>| | | | | |</p>
<table>
<thead>
<tr>
<th></th>
<th></th>
<th></th>
<th></th>
<th></th>
</tr>
</thead>
<tbody>
<tr>
<td>74</td>
<td>98</td>
<td>691</td>
<td></td>
<td></td>
</tr>
<tr>
<td>74</td>
<td>99</td>
<td>842</td>
<td></td>
<td></td>
</tr>
<tr>
<td>74</td>
<td>1113</td>
<td>1114</td>
<td>125</td>
<td></td>
</tr>
<tr>
<td>74</td>
<td>1294</td>
<td>1304</td>
<td></td>
<td></td>
</tr>
<tr>
<td>98</td>
<td>110</td>
<td>842</td>
<td>2682</td>
<td>100</td>
</tr>
<tr>
<td>98</td>
<td>843</td>
<td>1296</td>
<td></td>
<td></td>
</tr>
<tr>
<td>98</td>
<td>1379</td>
<td>1380</td>
<td>115</td>
<td>100</td>
</tr>
<tr>
<td>99</td>
<td>110</td>
<td>691</td>
<td></td>
<td></td>
</tr>
<tr>
<td>99</td>
<td>111</td>
<td>843</td>
<td></td>
<td></td>
</tr>
<tr>
<td>99</td>
<td>1379</td>
<td>1379</td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

Proposition 26.15.6. Third point. For each point \( U \) on the circumcircle, it exists exactly one other point \( U_2 \) such that \( SR(U, U_3) = U \). This point is the isogonal conjugate of the orthopoint of line \( U, X(3) \), belongs to line \( UU^* \) and is given by:

\[
\text{third}(U) = \frac{a^2}{a^2(bw + cv)(bw - cv)} : \frac{b^2}{b^2(cu + aw)(cu - aw)} : \frac{c^2}{c^2(aw + bu)(av - bu)}
\]

\[
= \frac{a^2}{(b^2 - c^2)u + (v - w)a^2} : \frac{b^2}{(c^2 - a^2)v + (w - u)b^2} : \frac{c^2}{(a^2 - b^2)w + (u - v)c^2}
\]

Proof. The limit of line \( U_1U_2 \) is orthogonal to line \( U, X(3) \). Everything else follows. Remark: relation \( SR(U, \text{third}(U)) = U \) is not granted for a random point \( U \) in the triangle plane. But this relation obviously holds when restricting \( U \) to the circumcircle.

Proposition 26.15.7. Simson-Moses point. If points \( U_1, U_2 \) are on the circumcircle then, using isoconjugacy wrt pole \( P = X_6 \) (isogonal conjugacy), the intersection of lines \( U_1(U_2)_P \) and \( U_2(U_1)_P \) is point \( SR(U_1, U_2) \). When another isoconjugacy is used, the intersection remains on the circumcircle. Its barycentrics are:

\[
S^*_P(U_1, U_2) = \frac{-w_1v_2 + v_1w_2}{qw_1w_2 - rv_1v_2} : \frac{-w_1u_2 + u_1w_2}{pw_1w_2 - ru_1u_2} : \frac{-v_1u_2 + u_1v_2}{pv_1v_2 - qu_1u_2}
\]

Proof. The barycentrics are straightforward, while parametrization (7.16) leads to the other properties.

Definition 26.15.8. In ETC, the Simson-Moses point is computed using \( P = X_2 \) (isotomic conjugacy) in \( S_P(U_1, U_2) \), and noted SM \( (U_1, U_2) \). Centers X(2855) to X(2868) are examples of Simson-Moses points.
Chapter 27
Combos

27.0.1 More-more combos

Definition 27.0.1. Suppose that \( ppp, qqq \) are two elements of the space \( C_n [a, b, c] \) of homogeneous polynomials of a given degree. Then the \( nT(ppp, qqq) \) triangle

\[
\begin{bmatrix}
    f(a, b, c) & g(a, c, b) & g(a, b, c) \\
    g(b, c, a) & f(b, c, a) & g(b, a, c) \\
    g(c, b, a) & g(c, a, b) & f(c, a, b)
\end{bmatrix}
\]

is what is obtained by the following procedure:

\[
\begin{align*}
xf &:= \text{unapply}(ppp, a, b, c); \\
xg &:= \text{unapply}(qqq, b, c, a); \\
qqtri &:= (\text{nortri} \circ \text{Tr} \circ \text{Matrix})(3, 3, [xf(a, b, c), xg(b, c, a), xg(c, b, a), xg(a, c, b), xf(b, c, a), xg(c, a, b), xg(a, b, c), xg(b, a, c), xf(c, a, b)]);
\end{align*}
\]

Example 27.0.2. Here are some examples:

\( T(-bc; b^2) \) triangle is

\[
\begin{bmatrix}
    -bc & a^2 & a^2 \\
    b^2 & -ca & b^2 \\
    c^2 & c^2 & -ab
\end{bmatrix}
\]

\( T(-1,1) \) Antimedial triangle (also anticomplementary).

\( T(0,b) \) Incentral triangle. Not \( T(a, 0) \)

\( T(-a,b) \) Excentral triangle

27.0.2 Euler triangles

The reflection in the Euler center \( E = X(5) \) has the following barycentric matrix:

\[
\frac{1}{8S^2} \begin{bmatrix}
    -a^2S_a & 4S^2 + S_bS_c & 4S^2 + S_bS_c \\
    4S^2 + S_aS_c & -b^2S_b & 4S^2 + S_aS_c \\
    4S^2 + S_aS_c & 4S^2 + S_bS_a & -c^2S_c
\end{bmatrix}
\]

cperp1 triangle \( T(+a^2, b(c-b)) \). Mid-arc, near the other vertex
cperp2 triangle \( T(-a^2, b(c+b)) \). Mid-arc, away from the other vertex
Medial triangle \( T(0,1) \), cevian of G. Not \( T(1,0) \) as said at 3758!
Orthic triangle \( T(0,S_c/a^2) \), cevian of H.
Euler1 triangle \( T(4S^2 + S_bS_c, S_aS_c) \), midpoints of the \([A, H]\), etc. Reflect of the medial triangle.
Euler2 triangle \( T((4S^2 + S_bS_c), \ (4S^2 - S_c^2) S_b/a^2) \). Reflect of the orthic triangle.
Euler3 triangle \( T((b - c)^2, c^2 - a^2 - bc) \). Is the cperp2 of medial and cperp1 of Euler1.
Euler4 triangle $T\left((b+c)^2, c^2 - a^2 + bc\right)$. Is the cperp1 of medial and cperp2 of Euler-1. Euler3 and Euler4 are reflected to each other.

Euler5 triangle. Its vertices are the re-intersections of the medians an the NPC.
Chapter 28

Using Kimberling’s database into Maple

In this chapter, we give the specifications of our implementation of the Kimberling ETC. At the present moment, this implementation is restricted to a private use. The intent is to participate to a distributed maintenance of the database, together with providing "random search keys", specific to each local copy, in order to allow a distributed method of proof.

The pillar of everything is the tandem ~/etc/bar_igcta.csv and sk that is

1. written once for ever using mysql/phpMyAdmin, and cured until it becomes Maple-compliant
2. read and parsed by Maple at the beginning of each session.

28.1 Sparse version (2019 and after)

The old procedure relire has been split. The new procedure relire deals with the formal values, stored in the file ~ipse/public_html/etc/bar_igtca.csv. On the other hand, procedure reliresk reads the maple archive stored as cat(encyfile[1 .. -9], "relire_sk/pas_toujours/relire_sk.m").

28.1.1 How-to update

1. copy ~ipse/docs/Cherche/Geometry/ETC_2018 towards .../ETC_2023. Suppress everything, except from download.sh and the various links (the *gif are used by the etc files, the other are probably useless).

2. update the number of pages to load, update http:// into https:// and run the batch download.sh.

3. Run the builders for the search keys. They are in relire_sk.mw. See Alg. 6.4, Alg. 6.7

    new_build_sk (non destructive). Uses Geometry/ETC_2023
    newbuild_enc_sort (non destructive).
        Uses “ipse/public_html/etc/sk_plex.csv Each item is 1=jj ; 2,3,4= sk[jj] ; 5=xk[jj]
    exec save (in relire_sk.m)
        fac47, smax, siz_enc, list_collisions
        enc_sort, fk, sk

4. Then the formal values of the coordinates can be loaded.

28.1.2 Duplicates

From now on (2023), it will be assumed that reliresk contains the authoritative values, while their counterparts in relire are no more maintained. Shouldn’t be loaded.

gerdat list of lines going through the point. Skip it!
28.1.3 la table barita

<table>
<thead>
<tr>
<th>j</th>
<th>ff</th>
<th>sk</th>
<th>xyz</th>
</tr>
</thead>
<tbody>
<tr>
<td>6800</td>
<td>0</td>
<td>−1.498509</td>
<td>−2.971024</td>
</tr>
<tr>
<td>6801</td>
<td>0</td>
<td>0.3093201</td>
<td>0.7440262</td>
</tr>
<tr>
<td>6802</td>
<td>0</td>
<td>−0.6812215</td>
<td>0.09831283</td>
</tr>
<tr>
<td>6803</td>
<td>0</td>
<td>0.4724080</td>
<td>0.5413951</td>
</tr>
<tr>
<td>6804</td>
<td>0</td>
<td>0.2233194</td>
<td>0.1687478</td>
</tr>
<tr>
<td>6805</td>
<td>0</td>
<td>0.2628773</td>
<td>0.2279281</td>
</tr>
<tr>
<td>6806</td>
<td>0</td>
<td>0.4146746</td>
<td>0.4550234</td>
</tr>
<tr>
<td>6807</td>
<td>0</td>
<td>−0.3066239</td>
<td>−0.6240702</td>
</tr>
<tr>
<td>6808</td>
<td>0</td>
<td>−2.636019</td>
<td>−4.108946</td>
</tr>
<tr>
<td>6809</td>
<td>0</td>
<td>1.581039</td>
<td>2.199954</td>
</tr>
<tr>
<td>6810</td>
<td>na</td>
<td>−0.03692055</td>
<td>−0.2205824</td>
</tr>
<tr>
<td>6811</td>
<td>na</td>
<td>−0.1950282</td>
<td>−0.4571182</td>
</tr>
<tr>
<td>6812</td>
<td>na</td>
<td>−0.3790703</td>
<td>−0.7324531</td>
</tr>
<tr>
<td>6813</td>
<td>na</td>
<td>−148.1508</td>
<td>−221.8054</td>
</tr>
<tr>
<td>6814</td>
<td>na</td>
<td>−1.684099</td>
<td>−2.684833</td>
</tr>
</tbody>
</table>

28.1.4 Description

When searching for points satisfying some property, we want to use a procedure like

```plaintext
localize:= proc(expr, px := vx) global myfun; local lediv;
  myfun := unapply(evalf(subs(iciK, expr)), op(convert(px, list)))@OP;
  lediv:= evalf(sqrt(add(myfun[sk[j]][j][1]*_facency, j=1..10)/10));
  seq('if'(abs(myfun[sk[j]])/lediv < 1/100000000000, j, NULL), j = 1 .. smax);
end proc; where the index ranges through the smax = 15639 points. Thus we need a full
list of numerical barycentrics (named sk in our code), even where we don’t have created the
corresponding formal values. This is done using two tables.

fk in [0, 1, 2]. Generalizes the flag ff , by introducing fk=2 when the barycentrics
of X(n) have complex values when using a = 6, b = 9, c = 13.

sk generalizes xyz , i.e. contains the numerical values of the barycentrics of X(n) for n ≤ 15639. For the sake of efficiency, this table is mostly stored in $ipse/public_html/etc/t6913.csv
(the real values), while the complex values are stored in $ipse/public_html/etc/sk_plex.csv
(and also corrections, when required).

28.1.5 Rationales

This paradigm shift has been allowed by the fact that, nowadays, the Kimberling (1998-2021)
database provides the searchkeys for the three systems (6, 9, 13), (9, 13, 6) and (13, 6, 9), and not only for the (6, 9, 13). They are buried in the
http://faculty.evansville.edu/ck6/encyclopedia/Search_6_9_13.html, etc. tables. Put
together, these three numbers are, quite all of the time, the normalized trilinear coordinates of the
given point, and we can use

\[ [x, y, z] = [6t_A, 9t_B, 13t_C] \]

To be more specific, the table fk is filled according to the following criteria (where =ε is defined by "less different than 1E − 16"), we have three kinds of points

fk=0 where \(x + y + z = \varepsilon 8\sqrt{35} \): ordinary points at finite distance, then normalized so that \(x + y + z = 1\)
fk = 2
for the other cases (complex points... or errors) 5000, 5001, 5002, 5003, 5374, 8072, 8073, 11065, 11066, 14091

We can check that (for \( n \leq j_{\text{max}} = 6802 \)) all the corresponding normalized values (see Section 6.3) are equal with the tabulated \( xyz \) except from the complex points (expected behavior).

Remark 28.1.1. This gives a better method to exclude the points that are perceived as having a "bad behavior":

1. Eight points non algebraic points, tagged by \( \text{fdat}[n] = \text{"special"} \), namely \( X(n) \) where \( n = 368, 369, 370, 1144, 3232, 5373, 5394, 5626 \).
2. Eight points with "too long" barycentric formulas (> 735), tagged by \( \text{fdat}[n] = \text{"len=..."} \), namely \( X(n) \) where \( X(n) = 5676, 5677, 5678, 5679, 5680, 5681, 5682, 5973 \).

28.1.6 Usage: the new ency procedure

1. We start from \([-288, 1701, -2197]\) or from \([3538, 5293, -8831]\). The normalize algorithm Alg. 6.2 uses the value of

\[
\frac{|x + y + z|}{|x| + |y| + |z|}
\]

to decide if \( x+y+z \neq 0 \) (i.e. \( M \notin L_\infty \)) or \( x+y+z \approx 0 \) (i.e. \( M \in L_\infty \)). And then the column is standardized according its type. In the first case, we divide \( x \) by \(-288+1701-2197 = -784\).

In the second case, we multiply \( x \) by the sum of inverses \((1/3538 + 1/5293 - 1/8831) \approx 0.00036\). This gives the searchkeys 0.3673469 and 1.2677958.

2. The key is compared with the existing ones. The answer is either fail (and provide an interval for the keys) or a line from \( \text{enc_sort} \). First element is a true key, second element is a sequence of integers, i.e. a single integer (best case) or a sequence of several integers

3. In the second case, a subsequent test is made using the three coordinates. More details at Alg. 6.8.

28.2 Older versions (2017 and before)

A typical line of file \( \text{bar_igtca.csv} \) is

"22";"0";"66";"0";"427";"0";
"a^2*(b^4+c^4-a^4)";"1";"2*s3*s1/(-3*s3+s2*s1)";
" -3.0867096808230018465";" -4.7860722764442097098";" 8.87407324452651015563";
" -0.79595873957891762046";" -1.46306950079735158158";" 3.25902824037626920204";
"-24.36344963696945546203";
"2_3=6_251=32_1194=35_612=36_614=\ldots etc... =1602_1626"

this is a text-only, Maple readable, dump of the fields named (in this exact order)
\( \text{num, ff, gg, tt, cc, ac, bary1, dg, lubin, xx, yy, zz, zxx, zyy, zyy, hc, gerade} \) in the database. By order of significance, we have:

\( \text{num} \) gives the identifier, i.e. the \( n \) in the \( X(n) \) value (lines are dumped in random order).

\( \text{fdat} \) in strict Maple syntax (named \( \text{bary1} \) in the database). The first barycentric coordinate without any spaces. The main variables are \( a, b, c \). But quantities \( S \) (area), \( S_a, S_b, S_c \) (the Conway symbols) are also used while \( R \) is deprecated. Some radicals should be specially quoted.

\( \text{dg, lubin} \), in strict Maple syntax. As of now (2019-08) \( \text{dg} \in \{0, 1, 2\} \), where 0 means N/A. The Lubin’s affix of the point (using the corresponding degree). Variables are \( s_1, s_2, s_3, s_4 \), the symmetric functions of \( \alpha, \beta, \gamma \)

\[
\{ \alpha + \beta + \gamma = s_1, \quad \alpha \beta + \alpha \gamma + \beta \gamma = s_2, \quad \alpha \beta \gamma = s_3, \quad (\alpha - \beta) (\beta - \gamma) (\gamma - \alpha) = -i s_4 \}
\]

remark: \( s_4/s_3 \) is real. Uses \( \text{ff}[n] \) to parse as \( z:1:3 \) or as \( z:0:1 \)!

— pdlx : Translation of the Kimberling’s Glossary into barycentrics ——
contains the \([xx,yy,zz]\) of the database. Given up to 20 decimals, the barycentrics of 
\(X(n)\) in the K system provided by \(a = 6, b = 9, c = 13\). Only given for \(n < 6810\). But 
\([0,0,0]\) is used for \(6802 \leq n < 6810\).

reflects the \([zxx,zyy,zzz]\) of the database. The barycentrics of \(X(n)\) in the Z system, 
provided by \(a = 7, b = 11, c = 17\). Only given for \(n < 6810\). But 
\([0,0,0]\) is used for \(6802 \leq n < 6810\).

the Kimberling searchkey.

in \(\{0,1\}\). When the point is at infinity, this boolean is set to 1. Only given for \(n < 6810\).

where 0 means N/A. Respectively, the isogon, isotom, complem, anticomplem of a point. Only given for \(n < 6810\).

is a long-text listing of the lines that contain the point. The syntax used is 
"1_291=2_76=3_6=4_232=5_114=9_978=10_730=36" i.e. ",=" between two lines 
and ",=" between the two points defining the line. This list is sorted according to the 
head point. Caveat: in the ETC, line "1_8" is recorded as "2_8" in the X(1) list. Only given for \(n < 6810\).

The first aim is to obtain certified pairs of searchkeys and Maple parse-able barycentrics. Using 
\texttt{etc} names, we are dealing with triples \((\text{num, bary1, hc})\) until they become compliant.

Towards a Maple compliant syntax

1. Table and file \texttt{barz.csv} are used to rework the syntax of the barycentrics, until the string 
becomes parse-able and gives an expression having the right searchkey.

<table>
<thead>
<tr>
<th>center</th>
<th>barz</th>
<th>alter</th>
</tr>
</thead>
<tbody>
<tr>
<td>num</td>
<td>bnum</td>
<td></td>
</tr>
<tr>
<td>ff</td>
<td>bff</td>
<td></td>
</tr>
<tr>
<td>gg</td>
<td>bgg</td>
<td></td>
</tr>
<tr>
<td>tt</td>
<td>btt</td>
<td></td>
</tr>
<tr>
<td>if(bary1&lt;&gt;**, bary1, &quot;x&quot;)</td>
<td>bbary1</td>
<td></td>
</tr>
<tr>
<td>quote(baryasis)</td>
<td>basis</td>
<td>varchar(1300)</td>
</tr>
<tr>
<td>quote(triliasis)</td>
<td>btrili</td>
<td>varchar(1300)</td>
</tr>
<tr>
<td>replace(if(lg&lt;&gt;0, left(gerade,1500),&quot;none&quot;),&quot; &quot;, &quot;,=&quot;)</td>
<td>bgerade</td>
<td></td>
</tr>
<tr>
<td>if(bary2&lt;&gt;**, bary2, &quot;x&quot;)</td>
<td>bbary2</td>
<td></td>
</tr>
</tbody>
</table>

2. The request starts with DROP TABLE IF EXISTS barz. Then newlines are changed into 
"QQ". And the table is exported into etc/barz.csv (terminated by ';' enclosed by '"').

3. The Maple file \texttt{gerer07.mw} proceeds in several steps, all resulting into a ryba.csv 
containing pairs num, bary1.

(a) Keys already known to Maple (enc.dat) are checked with the keys read from the web 
(skey). Then enc.dat is filled with skey.

(b) An automated process tries to proceed from bary1, firstline of basis, firstline of btrili 
(11 from 21).

(c) An automated process tries to proceed from the gerade field (4 from 21)

(d) Then file ryba.csv is read into ryba table. And, in a second step, this table is 
imported into the center table.
4. To be deciphered

(a) pas-create-fiche

28.3.2 Creating data records from the ETC Web Site

1. In 2012, ETC was made of two parts. Page one covering the 1-2000 part, and page two covering the 2001-4994 part. As of 03/2015, ETC was made of five parts. And now (01-2018) there are 8 of them.

2. Icons: leftri.gif, rightri.gif, s1blue.gif, underbar.gif are useful to read these files.

3. We use create-data-file.bat to split each web file in two parts. General comments are going to a dutu-files.html file. What remains is split on a "one line per item" basis, leading to

<table>
<thead>
<tr>
<th>name</th>
<th>starting</th>
<th>count</th>
<th>bytes</th>
</tr>
</thead>
<tbody>
<tr>
<td>ETCPart1.cut</td>
<td>0001</td>
<td>1000</td>
<td>1608892</td>
</tr>
<tr>
<td>ETCPart2.cut</td>
<td>1001</td>
<td>2000</td>
<td>1549710</td>
</tr>
<tr>
<td>ETCPart3.cut</td>
<td>3001</td>
<td>2000</td>
<td>1530646</td>
</tr>
<tr>
<td>ETCPart4.cut</td>
<td>5001</td>
<td>2000</td>
<td>1865405</td>
</tr>
<tr>
<td>ETCPart5.cut</td>
<td>7001</td>
<td>3000</td>
<td>1991833</td>
</tr>
<tr>
<td>ETCPart6.cut</td>
<td>10001</td>
<td>2000</td>
<td>1759197</td>
</tr>
<tr>
<td>ETCPart7.cut</td>
<td>12001</td>
<td>2000</td>
<td>2106219</td>
</tr>
<tr>
<td>ETCPart8.cut</td>
<td>14001</td>
<td>1639</td>
<td>1521580</td>
</tr>
</tbody>
</table>

4. The data part contains a line per recorded point. The script create-fiche.sh transcripts each line into an SQL request, that is send to the database. Since ETC is often hand-patched, this script is not so straightforward.

5. As of (X=7000), we use a main script create-fiche.sh xxx -x to cut a record (= a line in the *.cut file) into pieces. Part of the job is common to all records, summarized by the tmp_trig filter, only rewritten when needed by create-tmp_trig.sh. Part of the job is specific to a given number X(nnn) and is summarized into the tmp_nxt filter, controlled by create-tmp_nxt.sh (called at each iteration).

6. When one is satisfied with the result, the command is replayed without the -x, writing into the database.

7. All and everything that was a comment is also copied into the tmp_remaining file.

28.3.3 Importing definitions

1. Split EtcPartj.html into a dutu part (global) and a EtcPartj.cut, one line per point.

2. Import some rows using

for ((j=5; j<10;j++)); do ./create-fiche.sh 680$j ; done

does this writes directly into the database.

3. Obviously, some safety rules are used.

(a) Create a line is INSERT ignore INTO center (num, name, residuel) VALUES ($qui, "$myname", "$mycomment") ON DUPLICATE KEY UPDATE name=values(name);

(b) Other fields except from bary1 are set by UPDATE center SET trili_asis="$mytrili_asis", bary_asis="$mybary_asis", WHERE num=$qui;

(c) The barycentrics are saved line by line, into bary1-bary4, with an UPDATE.
28.4 Requirements (all versions)

1. A web server is required. We are using apache2. Should start chrooted (user= wwwrun, 
   #=30) on boot.
   List of active modules: access_compat, actions, alias, auth_basic, authn_core, authn_file, 
   authz_core, authz_groupfile, authz_host, authz_user, autoindex, cgi, core, dir, 
   env, expires, http, include, log_config, mime, mpm_prefork, negotiation, reqtimeout, 
   setenvif, so, ssl, systemd, unixd, userdir, status, info php5
   Module info is not so easy to launch (don't list its name at the end, but at the beginning 
   of the list).
   Inside httpd.conf: DirectoryIndex index.html index.php index.html.var

2. Don't be too protective. For example, directive
   RedirectMatch 301 /*-p[.]*.php /favicon.png
   in server root is over the top (prefix it with "-p" !)

3. A php server is required. Here, we have php5. As said before, php5 must be loaded 
   by the web server. Remember that *.php scripts are run by wwwrun. This user should 
   receive all the required permissions to write or execute (when required).

4. A database server is required. We are using mysql. Should start chrooted (user= mysql, 
   #=60) on boot. Activation by the run-levels.
   At /etc/my.cnf.d/secure_file_priv.cnf, we have "secure_file_priv", to be set 
   to ""

5. A web interface with mysql is required. We are using phpMyAdmin. The config file is 
   /etc/phpMyAdmin/config.inc.php. Many things are ugly there. Many "__" and "_".

6. Everything must be functional before trying to work with the ETC database. A good test is 
   to create and then work with the amarok database.

7. Dare to create security holes! Users mysql and wwwrun are chrooted and must remain 
   chrooted! Acceptable are links... when wwwrun is really chrooted and cannot use links to 
   escape outside.

8. Finally, don’t forget the HAL line when using the Listing environment + prettyref + hyperref 
   in LyX.

\def\arraystretch{1.2}
\makeatletter
\floatstyle{boxed}
\restylefloat{algorithm}
\def\fname@algorithm{{\scshape Listing}}
\def\fnum@algorithm{name@algorithm~\thealgorithm\,!}
\def\theHalgorithm{	healgorithm}
%% sur le modèle : \newcommand\theTable {\arabic{table}}
%% Semble être fixé pour SuSE 42.1
\makeatother

28.5 Building the database

28.5.1 The context

1. The database is called etcetc and created empty.

2. The user is called etcetc. Privileges are:
   (1) everything on etcetc from localhost;
   (2) FILE privilege from localhost.

3. The most important file is the interface with Maple, i.e. the bar_igtca.csv file (loaded 
   each time Maple is launched). The main table in the etc database is called center.
   Its purpose is to be filled by records reformatted from the ETC web page, and then to be 
   processed and cleaned.
28.5.2 From former archive

1. What is archived is how to recreate the main table, i.e. center. Encoding utf-8.
   Using a compressed archive is mandatory (in order to avoid a file too long error. maxsize=2048 Ko).

2. Therefore, archives are to be created using properties
   - Add custom comment into header (\n splits lines)
   - Add IF NOT EXISTS
   - Add AUTO_INCREMENT value
   - Enclose table and field names with backquotes
   - Data: Complete inserts, Extended inserts
   - Maximal length of created query 50000
   - Use hexadecimal for BLOB
   - Export type= insert
   - Dump all rows
   - Save as file "bzipped"

3. Available archives:

<table>
<thead>
<tr>
<th>Archive Name</th>
<th>Date</th>
<th>Size</th>
<th>Queries</th>
<th>Lines</th>
</tr>
</thead>
<tbody>
<tr>
<td>center-2012-02-17.sql.zip</td>
<td>2012-02-17</td>
<td>890225</td>
<td>77 queries</td>
<td>3899 lines</td>
</tr>
<tr>
<td>center-2013-04-08.sql.zip</td>
<td>2015-04-08</td>
<td>1536735</td>
<td>97 queries</td>
<td>4944 lines</td>
</tr>
<tr>
<td>center-2015-03-29.sql.bz2</td>
<td>2015-03-29</td>
<td>856633</td>
<td>97 queries</td>
<td>4944 lines</td>
</tr>
<tr>
<td>center-2017-12-27.sql.bz2</td>
<td>2017-12-27</td>
<td>1607678</td>
<td>97 queries</td>
<td>4944 lines</td>
</tr>
<tr>
<td>center-2019-08-31.sql.zip</td>
<td>2019-08-31</td>
<td>1529921</td>
<td>97 queries</td>
<td>4944 lines</td>
</tr>
</tbody>
</table>

4. In case of a bad loading, a drop is required, since truncate would keep the old structure.
   Encoding utf-8. Actual size = 836.6 Ko. 97 queries. 4994 lines.

5. Caveat: user etcetc can export to file using the phpMyAdmin frontal, but not directly using mysql (and the scripts). Unless receiving the global 'FILE' property.

28.5.3 Encapsulating everything in *.php commands

1. Writing for eternity implies to keep track of each action. Writing explicit programs is the best way for that.

2. A file named index.php is somewhere in a file tree that is mapped into the web server tree ~/public_html/etc. This file contains a link to each command file. These *.php files are only writable by ipse, and stored in a specific directory. Database is somewhere else (in the protected mysql space /var/lib/mysql/etcetc).

3. These scripts create auxiliary tables or files, import files into temporary tables and commit some changes from auxiliary tables into the main table (center).

4. An individual command file is as listed in Alg. 28.1. The core of the file is the mysql line of command (2). When everything is written by *.php, the redirection is in line (4). Otherwise, the mysql messages are directed to the web page. When a post-treatment is required, it can be embedded in an external routine (6).

5. A list of items is build that way:
   drop table if exists lesqui;
   create table lesqui select 1 as xnum ;
   insert into lesqui (xnum) values (829),(1105),(1288),... ;

28.5.4 Main view of the structure

The purpose of the center table is to contain the original records from the ETC web page... together with the clean data obtained after some extra work. This table contains each and every information.

The evolving structure of this main table is described in Table 28.1.
28.5. Building the database

These are the fields that are extracted towards the bar_igtca.csv file, so that these fields are read each time Maple is launched.

1. field num is identifier, i.e. the n in the X(n) value.

2. Today (2015/03/27), the fields extracted are: num, ff, gg, tt, cc, ac, bary1, dg, lubin, gerade, xx,yy,zz,zxx,zyy,zyz, hc.

3. bary1, in strict Maple syntax, is the first barycentric coordinate without any spaces. The main variables are a,b,c. But quantities S (area), S_a, S_b, S_c (the Conway symbols) are also used while R is deprecated. Some radicals should be specially quoted.

4. lubin, in strict Maple syntax, is the Lubin affix while dg is the corresponding degree. Variables are s1,s2,s3,s4

5. boolean : ff= 1 for points at infinity (#229)

6. links are: gg (isogon), tt (isotom), cc (complem), ac (anticomplem), ho (orthopoint).

7. gerade is a longtext listing the lines that contain the point. The syntax used is "1_291=2_76=3_6=4_232=5_114=9_978=10_730=36" i.e. "=" between two lines and "_" between the two points defining the line. This list is sorted according to the head point. In ETC, line "1_8" is recorded as "2_8" in the X(1) list.

8. Quantities xx,yy,zz,zxx,zyy,zyz,zzz are the coordinates in the K and the Z systems. They are given with 20 decimals.

9. Finally, hc is the searchkey.
### Table 28.1: Structure of the database

<table>
<thead>
<tr>
<th>Field</th>
<th>Type</th>
<th>Field</th>
<th>Type</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>* num</td>
<td>int</td>
<td>33</td>
</tr>
<tr>
<td>2</td>
<td>name</td>
<td>varchar</td>
<td>60</td>
</tr>
<tr>
<td>3</td>
<td>* ff</td>
<td>tinyint</td>
<td>1</td>
</tr>
<tr>
<td>4</td>
<td>* gg</td>
<td>smallint</td>
<td>6</td>
</tr>
<tr>
<td>5</td>
<td>* tt</td>
<td>smallint</td>
<td>6</td>
</tr>
<tr>
<td>6</td>
<td>* cc</td>
<td>smallint</td>
<td>6</td>
</tr>
<tr>
<td>7</td>
<td>* ac</td>
<td>smallint</td>
<td>6</td>
</tr>
<tr>
<td>8</td>
<td>* bary1</td>
<td>varchar</td>
<td>500</td>
</tr>
<tr>
<td>9</td>
<td>* dg</td>
<td>int</td>
<td>11</td>
</tr>
<tr>
<td>10</td>
<td>* lucin</td>
<td>varchar</td>
<td>500</td>
</tr>
<tr>
<td>11</td>
<td>sf</td>
<td>tinyint</td>
<td>1</td>
</tr>
<tr>
<td>12</td>
<td>mf</td>
<td>tinyint</td>
<td>1</td>
</tr>
<tr>
<td>13</td>
<td>ho</td>
<td>smallint</td>
<td>6</td>
</tr>
<tr>
<td>14</td>
<td>bary2</td>
<td>varchar</td>
<td>400</td>
</tr>
<tr>
<td>15</td>
<td>bary3</td>
<td>varchar</td>
<td>100</td>
</tr>
<tr>
<td>16</td>
<td>bary4</td>
<td>varchar</td>
<td>100</td>
</tr>
<tr>
<td>17</td>
<td>baryf</td>
<td>varchar</td>
<td>10</td>
</tr>
<tr>
<td>18</td>
<td>triliasis</td>
<td>mediumtext</td>
<td>50</td>
</tr>
<tr>
<td>19</td>
<td>baryasis</td>
<td>mediumtext</td>
<td>51</td>
</tr>
<tr>
<td>20</td>
<td>combo</td>
<td>varchar</td>
<td>100</td>
</tr>
<tr>
<td>21</td>
<td>lg</td>
<td>smallint</td>
<td>6</td>
</tr>
<tr>
<td>22</td>
<td>gerade</td>
<td>varchar</td>
<td>1500</td>
</tr>
<tr>
<td>23</td>
<td>curves</td>
<td>varchar</td>
<td>100</td>
</tr>
<tr>
<td>24</td>
<td>orthj</td>
<td>smallint</td>
<td>6</td>
</tr>
<tr>
<td>25</td>
<td>iccir</td>
<td>smallint</td>
<td>6</td>
</tr>
<tr>
<td>26</td>
<td>ibroc</td>
<td>smallint</td>
<td>6</td>
</tr>
<tr>
<td>27</td>
<td>iortc</td>
<td>smallint</td>
<td>6</td>
</tr>
<tr>
<td>28</td>
<td>iflec</td>
<td>smallint</td>
<td>6</td>
</tr>
<tr>
<td>29</td>
<td>ibevc</td>
<td>smallint</td>
<td>6</td>
</tr>
<tr>
<td>30</td>
<td>inicc</td>
<td>smallint</td>
<td>6</td>
</tr>
<tr>
<td>31</td>
<td>islec</td>
<td>smallint</td>
<td>6</td>
</tr>
<tr>
<td>32</td>
<td>inine</td>
<td>smallint</td>
<td>6</td>
</tr>
</tbody>
</table>

#### 28.5.6 Ancilary fields

1. booleans: sf = 1 for the 6 special points, mf = 1 for the bare and Morley points (#33).
2. ho is orthopoint
3. bary2, bary3, bary4, baryf (num of lines), triliasis, baryasis, combo: raw data from the ETC web pages.
4. lg is the length of gerade
5. curves is a list containing X(n)
6. reflex is a list of xx_yy so that yy is the reflection of X(n) in xx.
7. refs: references
8. residuel: all that doesn’t go somewhere else.
28.5.7 Information fields
Inverses in various circles: [circ, inc, inc, ibr, ibern, ible, if, iuf, ifuhr] are circumcircle, incircle, ncp, Brocard, ibern, Bevan, Lemoine1, Lemoine2, Fuhrman.

orthj orthojoin of
circa antipode in circumcircle (special case of reflex)
cirna antipode in nine-points circle (special case of reflex)
cocj complementary conjugate of
accj anticomplementary conjugate of
cycc cyclocevian conjugate of
reflex reflection of XI in XJ
crosc XI-cross conjugate of XJ
crosp crosspoint of XI and XJ
crosa crosssum of XI and XJ
crosd crossdifference of any
hirst XI-Hirst inverse of XJ
alph XI-aleph conjugate of XJ
beth XI-beth conjugate of XJ
midp midpoint of XI and XJ
ceva cevapoint of XI and XJ
cevc XI-Ceva conjugate of XJ
licon XI-line conjugate of XJ
eigct eigencenter of cevian triangle of XI
eigat eigencenter of anticevian triangle of XI
lam &Lambda;
psi &Psi;
Chapter 29

Some remarks when importing the ETC files

29.1 SQL requests

- modify names in order to replace some &xxx; web characters. The html code must be hidden, otherwise the text of the command itself is parsed and replaying is scrambled.

\[
\begin{align*}
\text{# SELECT num, name, replace(name,concat(char(38 using utf8), "omega;"),"omega") as qname FROM 'center' where 'name' like "%&%"} \\
\text{# SET name=replace(name,concat(char(38 using utf8), "pi;"),"Pi") WHERE 'name' like "%&%"}
\end{align*}
\]

- \(b^2+c^2 = -a^2+2\cdot Sw\)  \\
  \(b^2\cdot c^2 = a^4-2\cdot Sw\cdot a^2+4\cdot S^2+Sw^2\)

29.2 Some items to revisit

3020 past wrong searchkey. \((b-a+c)(b-c)^2(b^2-bc+c^2)^2\).

3272-3283 related to Morley

3310 Barycentric Products of Perpendicular Directions (on the orthic axis)

5000 5000-5005: To be fully rewritten !

Coordinates \(u : v : w\) are not homogeneous tripolar coordinates, but homogeneous triradial coordinates!

X(5002) is the point whose polar distances in the plane of triangle \(ABC\) are proportional to \((a, b, c)\). If the reference triangle \(ABC\) is acute, the actual polar distances are \(ka, kb, kc\). If triangle \(ABC\) is obtuse, then the barycentrics of X(5002) are nonreal complex numbers. Nevertheless, for all triangles ABC, the midpoint of X(5002) and X(5003) is X(858).

A method for converting from homogeneous polar distances to homogeneous barycentrics, found by Peter Moses (March, 2012), depends on finding the point of intersection of the radical axes of radical circles centered at \(A, B, C\).

Write the polar distances for a point \(U\) as \(u : v : w\), and let

\[
2S_a = b^2 + c^2 - a^2, 2S_b = c^2 + a^2 - b^2, 2S_c = a^2 + b^2 - c^2
\]

Then barycentrics \(x : y : z\) for \(U\) are given by

\[
\begin{align*}
x &= a^2S_a + k^2(S_cv^2 + S_bw^2 - a^2u^2) \\
y &= b^2S_b + k^2(S_aw^2 + S_cu^2 - b^2v^2) \\
z &= c^2S_c + k^2(S_bu^2 + S_aw^2 - c^2w^2)
\end{align*}
\]

\[
x + y + z = 8S^2
\]
where \( k^2 \) has two values (as in the quadratic formula): \((-f-g)/h\) or \((-f+g)/h\), where

\[
\begin{align*}
  f &= -a^2 u^2 2S_a - b^2 v^2 2S_b - c^2 w^2 2S_c \\
  g &= 2S \sqrt{(-au + bv + cw)(au - bv + cw)(au + bv - cw)(au + bv + cw)} \\
  h &= 2(a^2(u^2 - v^2)(u^2 - w^2) + b^2(v^2 - w^2)(v^2 - u^2) + c^2(w^2 - u^2)(w^2 - v^2)) \\
  &\quad \times Sa^2u^2 + b^2 Sb v^2 + c^2 Sc w^2 \pm 2 S \sqrt{\ldots} \\
  &\quad \div \left( a^2u^4 + b^2v^4 + c^2w^4 - 2 (Sa w^2 v^2 + Sb u^2 w^2 + Sc u^2 v^2) \right)
\end{align*}
\]

The meaning of \( k \) is as follows: starting with polar distances \( u : v : w \), the actual polar distances are \( ku, kv, kw \). That is,

\[
|UA| = ku, |UB| = kv, |UC| = kw
\]
Chapter 30
Describing ETC

30.1 Various kinds of points

Table 30.1 describe the various species and subspecies that can be found in first 6800 points listed in the Kimberling's database.

1. 85 in the "out of pattern" class
2. 6092 are in $Q[a, b, c, S]$. 
   (a) Among the 6092, there are 5691 of them in $Q[a, b, c]$, i.e. that don’t depend on the orientation of the plane.
   (b) Among the 6092, there are 1307 of them in $Q[a^2, b^2, c^2, S]$, i.e. invariant by the Lemoine's conjugacies.
3. 339 are using various surds: $\sqrt{3}$, $\sqrt{5}$, $\sqrt{2}$ or even $\sqrt{5 + \sqrt{5}}$
4. 284 are using radicals involving $a, b, c$

30.2 Special points

The following six points have no explicit barycentrics, but are defined as the solution of some special equation.

30.2.1 X(368)

1. Situated on the second Brocard cubic, and on the anticomplement of the Kiepert hyperbola $CC(X_{523})$. Append $x + y + z = 1$.
2. Substitute ency (=local solution). Eliminate $x, y$. Obtain a six degree equation in $z$, with $3z - 1$ in factor (barycenter $G$).
3. Four of the roots are complex, and $X_{368}$ is the remaining one.

30.2.2 X(369), X(3232) trisected perimeter points

There exist points $A', B', C'$ on segments $BC$, $CA$, $AB$, respectively, such that:

$$AB' + AC' = BC' + BA' = CA' + CB' = (a + b + c)/3$$

Lines $AA', BB', CC'$ concur in X(369). Yff found that barycentrics $p : q : r$ for X(369) can be obtained in terms of the unique real root, $K$, of the cubic polynomial equation:

$$2K^3 - 3(b + c + a)K^2 + (a^2 + b^2 + c^2 + 8(bc + ca + ab))K$$
$$- (b^2c + c^2a + a^2b) - 5(bc^2 + ca^2 + ab^2) - 9abc = 0$$

433
and gives at the geometry conference held at Miami University of Ohio, 2004/10/02, a symmetric form for these barycentrics, namely:

\[ p = K^2 - (2c + a)K - a^2 + b^2 + 2c^2 + 2bc + 3ac + 2ab \]

and cyclically. In the reference triangle, we have:

\[ X_{369} = (0.4090242897, 0.3561467951, 0.2348289151) \]

On the other hand, there exist points \( A', B', C' \) on segments \( BC, CA, AB \), respectively, such that \( B'C + C'B = C'A + A'C = A'B + B'A = (a + b + c)/3 \), and the lines \( AA', BB', CC' \) concur in \( X(3232) \), the isotomic conjugate of \( X(369) \).

### 30.2.3 \( X(370) \)

From http://pagesperso-orange.fr/bernard.gibert/Tables/table10.html, point \( X_{370} \) lies on conic:

\[
(xy(a^2 - b^2 + c^2) + xz(a^2 + b^2 - c^2) + 2zya^2) - x(y + z + 2x)\frac{abc}{R\sqrt{3}} = 0
\]

and on the other two obtained cyclically. Two such conics have in common a vertex, \( X_{370} \) and other two (may be not real) points. In any case, \( X_{370} \) is inside triangle \( ABC \).

<table>
<thead>
<tr>
<th>species</th>
<th>subspecies</th>
<th>total</th>
<th>6800</th>
</tr>
</thead>
<tbody>
<tr>
<td>rational</td>
<td>( a^2, b^2, c^2 ) only</td>
<td>784</td>
<td>1075</td>
</tr>
<tr>
<td></td>
<td>( a^2, b^2, c^2 ) and ( S )</td>
<td>241</td>
<td>1025</td>
</tr>
<tr>
<td>2nd deg</td>
<td>( a, b, c ) only, but not rational</td>
<td>3593</td>
<td>4616</td>
</tr>
<tr>
<td></td>
<td>( a, b, c ) and ( S ), but not rational</td>
<td>90</td>
<td>169</td>
</tr>
<tr>
<td>surds</td>
<td>( \sqrt{2} ) (all ( a^2, S ))</td>
<td>#8</td>
<td>8</td>
</tr>
<tr>
<td></td>
<td>( \sqrt{5} ) (all ( a^2, S ))</td>
<td>#18</td>
<td>20</td>
</tr>
<tr>
<td></td>
<td>(among them, #12 ( \sqrt{5} + \sqrt{5} ))</td>
<td>(14)</td>
<td>4616</td>
</tr>
<tr>
<td></td>
<td>( \sqrt{3} ) (#125a2S, #39 aS)</td>
<td>(225, 86)</td>
<td>164</td>
</tr>
<tr>
<td>radicals</td>
<td>( a(b + c - a) )</td>
<td>#50</td>
<td>68</td>
</tr>
<tr>
<td></td>
<td>( \sqrt{(a + b + c) \div abc} ) used in ( \cos \frac{\alpha}{2} )</td>
<td>(#14)(18)</td>
<td>68</td>
</tr>
<tr>
<td></td>
<td>(among them 2 ( \sqrt{a} ))</td>
<td>#18</td>
<td>18</td>
</tr>
<tr>
<td></td>
<td>( \sqrt{a^2b^2 + \cdots} = \sqrt{4S^2 + S_w^2} )</td>
<td>#66</td>
<td>68</td>
</tr>
<tr>
<td></td>
<td>( \sqrt{a^4 - 2a^2c^2 + \cdots} = \sqrt{So^2 - 12S^2} )</td>
<td>#62</td>
<td>72</td>
</tr>
<tr>
<td></td>
<td>( a^8 - b^4c^2 \ldots + 3a^4b^2c^2 = 9a^4b^2c^2 - 32S^2S_w ) Euler</td>
<td>#52</td>
<td>52</td>
</tr>
<tr>
<td></td>
<td>( \frac{OJ}{R} = \sqrt{a^2 - 3abc + 3abc} ) (mixed #2)</td>
<td>#12</td>
<td>12</td>
</tr>
<tr>
<td></td>
<td>( \sqrt{a^2 + s^2} = \sqrt{a^2 + abc} ) Spieker circle</td>
<td>#8</td>
<td>8</td>
</tr>
<tr>
<td></td>
<td>( \sqrt{2bc - a^2} \ldots ) Poncelet In/circum-circle</td>
<td>#2</td>
<td>2</td>
</tr>
<tr>
<td></td>
<td>( \sqrt{2abc(a + b + c) (2bc - a^2 \cdots} )</td>
<td>#6</td>
<td>6</td>
</tr>
<tr>
<td></td>
<td>#4</td>
<td>4</td>
<td></td>
</tr>
<tr>
<td></td>
<td>#2</td>
<td>2</td>
<td></td>
</tr>
<tr>
<td></td>
<td>#1</td>
<td>11</td>
<td></td>
</tr>
<tr>
<td></td>
<td>#1</td>
<td>2</td>
<td></td>
</tr>
<tr>
<td>angles</td>
<td>( \text{trig}(A/3) ) etc</td>
<td>33</td>
<td>58</td>
</tr>
<tr>
<td></td>
<td>bare ( A &amp; \text{weird} )</td>
<td>10</td>
<td>10</td>
</tr>
<tr>
<td>special</td>
<td># means among (i.e. not additive)</td>
<td>4992</td>
<td>6800</td>
</tr>
</tbody>
</table>

Table 30.1: The various species of points in the Kimberling’s database
30.2.4 X(1144)
Suppose P is a point inside triangle ABC. Let Sa be the square inscribed in triangle PBC, having
two vertices on segment BC, one on PB, and one on PC. Define Sb and Sc cyclically. Then X_{1144}
is the unique choice of P for which the three squares are congruent. L(a, b, c) is the common length
of the sides of the three squares. The function L(a, b, c) is symmetric, homogeneous of degree 1,
and satisfies 0 < L(a, b, c) < min (a, b, c). L is the smallest root of :
\[\frac{a^2}{a - L} + \frac{b^2}{b - L} + \frac{c^2}{c - L} - 2 \frac{S}{L} = 0\]
and point X_{1144} is obtained as :
\[\frac{a^2}{a - L} : \frac{b^2}{b - L} : \frac{c^2}{c - L}\]
leading to :
X_{1144} = [0.2373201571, 0.3224457580, 0.4402340851]
This point lies on the hyperbola \{A, B, C, X(1), X(6)\} = CC (X_{649}). Indeed, X(1144) lies on
the open arc from X(1) to the vertex of ABC opposite the shortest side. (Jean-Pierre Ehrmann,
12/16/01)

30.2.5 X(5373) Equiareality Center
Incenter of the intersections of K002 (the Thomson cubic) and the circumcenter (Gibert, 2012).

30.2.6 X(5394) Congruent Incircles Point
Point such that incircles of (A, B, M), etc have the same radius.

30.2.7 X(5626) Center of Electrostatic Potential
The point of maximal electrostatic potential inside a triangle ABC having a homogeneous surface
charge distribution (Abraham and Kovac, 2015).

30.3 Pending to shorten
5676 len = 736, 598
5677 len = 757, 729
5678 len = 705, 609
5679 len = 2323, 2140
5680 len = 810, 667
5681 len = 912, 823
5682 len = 911, 822
5973 len = 851
6802 len = 601, 232

30.4 orthopoint, points on circles
Remark 30.4.1. Orthopoint. Only 6 orthopoint pairs are registered, but 48 can be computed
(involving 96 points on L_{\infty}, that contains 229 named points). Has been abbreviated into hortopoint
since "or" is a reserved word.
Remark 30.4.2. Inverse in a given circle. Points on L_{\infty} are not considered here.
1. circumcircle. There are 352 named points whose inverse in circumcircle is named too. Among them, 258 are on the circumcircle itself, and there are 47 pairs of "true" inverse pairs. Among these 94 points, 29 aren't listed.
On the circumcircle, 220 named points have a named isogonal conjugate (among the 229 points of $L_{\infty}$).
62 antipodal pairs are listed.

2. incircle. There are 53 named points whose inverse in incircle is named too. Among them, 39 are on the incircle itself, and there are 7 pairs of "true" inverse pairs. Among these 14 points, 3 aren't listed.

3. nine points circle. There are 55 named points whose inverse in the nine points circle is named too. Among them, 37 are on the nine points circle itself, and there are 9 pairs of "true" inverse pairs. Among these 18 points, 4 aren't listed.

5. Brocard circle : 98, 4, 47 (pairs). Among 94 points, only 89 are listed.
6. Spieker circle : 8, 8, 0 (pairs)
7. Orthocentroidal circle : 82, 2, 40 (pairs). Among the 80 points, 20 aren't listed.
8. Fuhrmann circle : 12, 2, 5 (pairs). Among the 10 points, only 4 are listed (and belong to 4 different pairs).
10. Second Lemoine : 12, 2, 5 (pairs).

Remark 30.4.3. Antipode in circumcircle

1. From a total of 258 named points on the circle, 124 have a named antipode.
2. Antipode is mostly given as a member of the list "reflection", that appears as I_3 in the corresponding field. 10 are missing.

Remark 30.4.4. Antipode in nine points circle

1. Among 37 named points on the nine points circle, 24 have a named antipode (12 pairs).
2. Among 37 named points on $\gamma$, 33 have a named anticomplement

30.5 points on lines

Remark 30.5.1. Bad extractions. Due to lack of regularity (may be human modifications), 50 lists of lines were badly extracted (mostly : HTML tags, reflection, extra spaces).

SELECT num, gerade FROM center WHERE gerade like "\%\_ \%"
Lack of pattern for these :

<p>| | | | | | | | | | | | | |</p>
<table>
<thead>
<tr>
<th></th>
<th></th>
<th></th>
<th></th>
<th></th>
<th></th>
<th></th>
<th></th>
<th></th>
<th></th>
<th></th>
<th></th>
<th></th>
</tr>
</thead>
<tbody>
<tr>
<td>505</td>
<td>680</td>
<td>1154</td>
<td>1180</td>
<td>1300</td>
<td>1573</td>
<td>1593</td>
<td>1689</td>
<td>1944</td>
<td>1951</td>
<td>2051</td>
<td></td>
<td></td>
</tr>
<tr>
<td>2173</td>
<td>2388</td>
<td>2442</td>
<td>2503</td>
<td>2708</td>
<td>2719</td>
<td>2723</td>
<td>2725</td>
<td>2727</td>
<td>2758</td>
<td>2776</td>
<td></td>
<td></td>
</tr>
<tr>
<td>2974</td>
<td>3037</td>
<td>3110</td>
<td>3111</td>
<td>3113</td>
<td>3308</td>
<td>3340</td>
<td>3354</td>
<td>3363</td>
<td>3501</td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

Remark 30.5.2. There are 32322 quotations, involving 15988 lines.

1. Lines are referred by two other points, so that none of the 169 lines quoted as going through $X_1$ are quoted under their "true" name.
2. Here are the statistics:

<p>| | | | | | | | | | | | | | | | | | |</p>
<table>
<thead>
<tr>
<th></th>
<th></th>
<th></th>
<th></th>
<th></th>
<th></th>
<th></th>
<th></th>
<th></th>
<th></th>
<th></th>
<th></th>
<th></th>
<th></th>
<th></th>
<th></th>
<th></th>
<th></th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>9436</td>
<td>9</td>
<td>43</td>
<td>17</td>
<td>3</td>
<td>25</td>
<td>2</td>
<td>43</td>
<td>1</td>
<td>184</td>
<td>1</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>2</td>
<td>2700</td>
<td>10</td>
<td>32</td>
<td>18</td>
<td>4</td>
<td>26</td>
<td>2</td>
<td>47</td>
<td>1</td>
<td>229</td>
<td>1</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>3</td>
<td>2207</td>
<td>11</td>
<td>24</td>
<td>19</td>
<td>5</td>
<td>27</td>
<td>1</td>
<td>62</td>
<td>1</td>
<td>312</td>
<td>1</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>4</td>
<td>725</td>
<td>12</td>
<td>19</td>
<td>20</td>
<td>2</td>
<td>29</td>
<td>2</td>
<td>75</td>
<td>1</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>5</td>
<td>375</td>
<td>13</td>
<td>14</td>
<td>21</td>
<td>2</td>
<td>30</td>
<td>1</td>
<td>76</td>
<td>1</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>6</td>
<td>151</td>
<td>14</td>
<td>7</td>
<td>22</td>
<td>5</td>
<td>35</td>
<td>2</td>
<td>78</td>
<td>1</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>7</td>
<td>110</td>
<td>15</td>
<td>4</td>
<td>23</td>
<td>1</td>
<td>36</td>
<td>2</td>
<td>81</td>
<td>1</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>8</td>
<td>60</td>
<td>16</td>
<td>10</td>
<td>24</td>
<td>3</td>
<td>38</td>
<td>1</td>
<td>100</td>
<td>1</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

This has to be read as: there are 9446 lines that are quoted only once, and so on.

3. In fact, most of the lines involving only three points are quoted only at their third point (except from 120 of them that are not quoted at the third point). For example, $X_1$ is involved in 294 quoted lines whereas only 169 of them are quoted in the $X_1$ page. When taking this fact into account, we obtain:

<p>| | | | | | | | | | | | | | | | | | |</p>
<table>
<thead>
<tr>
<th></th>
<th></th>
<th></th>
<th></th>
<th></th>
<th></th>
<th></th>
<th></th>
<th></th>
<th></th>
<th></th>
<th></th>
<th></th>
<th></th>
<th></th>
<th></th>
<th></th>
<th></th>
</tr>
</thead>
<tbody>
<tr>
<td>9</td>
<td>44</td>
<td>17</td>
<td>3</td>
<td>25</td>
<td>2</td>
<td>43</td>
<td>1</td>
<td>184</td>
<td>1</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>10</td>
<td>34</td>
<td>18</td>
<td>4</td>
<td>26</td>
<td>2</td>
<td>47</td>
<td>1</td>
<td>229</td>
<td>1</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>12998</td>
<td>11</td>
<td>25</td>
<td>19</td>
<td>5</td>
<td>27</td>
<td>1</td>
<td>62</td>
<td>1</td>
<td>312</td>
<td>1</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>1773</td>
<td>12</td>
<td>20</td>
<td>20</td>
<td>2</td>
<td>29</td>
<td>2</td>
<td>75</td>
<td>1</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>580</td>
<td>13</td>
<td>14</td>
<td>21</td>
<td>2</td>
<td>30</td>
<td>1</td>
<td>76</td>
<td>1</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>220</td>
<td>14</td>
<td>7</td>
<td>22</td>
<td>4</td>
<td>35</td>
<td>2</td>
<td>78</td>
<td>1</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>119</td>
<td>15</td>
<td>4</td>
<td>23</td>
<td>2</td>
<td>36</td>
<td>2</td>
<td>81</td>
<td>1</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>69</td>
<td>16</td>
<td>10</td>
<td>24</td>
<td>3</td>
<td>38</td>
<td>1</td>
<td>100</td>
<td>1</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

4. There are 1195 lines that goes through at least five named points.
Bibliography

▷ 1 citation located at section 30.2.7.

▷ 0 citations located at sections .

▷ 2 citations located at sections 1 and 1.

▷ 1 citation located at section 17.2.

▷ 1 citation located at section 17.3.

▷ 1 citation located at section 19.

▷ 0 citations located at sections .

▷ 0 citations located at sections .

▷ 0 citations located at sections .

▷ 1 citation located at section 8.

▷ 0 citations located at sections .

▷ 0 citations located at sections .


> 2 citations located at sections 12.9.4 and 12.9.4.


> 0 citations located at sections.


> 1 citation located at section 19.


> 0 citations located at sections.


> 0 citations located at sections.


> 0 citations located at sections.


> 2 citations located at sections 21.1 and 21.1.8.


> 0 citations located at sections.


> 0 citations located at sections.


> 4 citations located at sections 26.1, 26.2.2, 26.4.9, and 26.4.10.


> 0 citations located at sections.


> 0 citations located at sections.


> 0 citations located at sections.


> 0 citations located at sections.


> 1 citation located at section 17.3.

—– pldx : Translation of the Kimberling’s Glossary into barycentrics —–
▷ 1 citation located at section 23.8.

http://www.numdam.org/item?id=PMIHES_1998__88__43_0.
▷ 0 citations located at sections.

▷ 1 citation located at section 1.

Court N.A. *College Geometry* (Barnes & Noble) (1952).
▷ 0 citations located at sections.

▷ 0 citations located at sections.

▷ 0 citations located at sections.

▷ 0 citations located at sections.

▷ 0 citations located at sections.

▷ 1 citation located at section 19.9.15.

▷ 1 citation located at section 3.13.7.

▷ 0 citations located at sections.

▷ 1 citation located at section 12.12.5.

▷ 1 citation located at section 17.4.1.

▷ 1 citation located at section 22.15.3.

▷ 2 citations located at sections 1 and 1.

▷ 1 citation located at section 17.3.2.


— pldx : Translation of the Kimberling’s Glossary into barycentrics —
▷ 4 citations located at sections 1.4.11, 20, 20.4.11, and 20.5.11.

▷ 1 citation located at section 20.1.

▷ 0 citations located at sections .

▷ 0 citations located at sections .

▷ 0 citations located at sections .

▷ 0 citations located at sections .

▷ 0 citations located at sections .

▷ 0 citations located at sections .

▷ 0 citations located at sections .

Fuhrmann W. *Synthetische Beweise Planimetrischer Sätze* (Leonhard Simion, Berlin) (1890), xxiv, 190 pp. https://ia700404.us.archive.org/19/items/synthetischebew00fuhrgoog/synthetischebew00fuhrgoog.pdf.  
▷ 0 citations located at sections .

▷ 1 citation located at section 11.2.2.

▷ 0 citations located at sections .

▷ 0 citations located at sections .

▷ 0 citations located at sections .

▷ 0 citations located at sections .
▷ 0 citations located at sections.

▷ 0 citations located at sections.

▷ 0 citations located at sections.

▷ 1 citation located at section 10.7.1.

▷ 6 citations located at sections (document), 20, 20.4.26, 20.4.55, 20.5.16, and 20.5.27.

▷ 1 citation located at section 20.4.26.

▷ 0 citations located at sections.

▷ 1 citation located at section 30.2.5.

▷ 1 citation located at section 14.10.

▷ 0 citations located at sections.

▷ 0 citations located at sections.

▷ 1 citation located at section 26.13.1.

▷ 0 citations located at sections.

▷ 0 citations located at sections.

▷ 0 citations located at sections.

▷ 1 citation located at section 13.22.2.

— pldx : Translation of the Kimberling's Glossary into barycentrics —
> 1 citation located at section 23.8.

> 1 citation located at section 11.5.1.

> 1 citation located at section 11.2.4.

> 0 citations located at sections.

> 0 citations located at sections.

> 2 citations located at sections 13.18 and 13.18.

> 1 citation located at section 26.12.3.

> 0 citations located at sections.

> 0 citations located at sections.

> 1 citation located at section 8.4.7.

> 1 citation located at section 11.5.1.

> 1 citation located at section 26.15.2.

Jackiw N. *The geometer’s sketchpad(r): Dynamic geometry(r) software for exploring mathematics*. $70.00 (2001).
> 0 citations located at sections.

> 0 citations located at sections.

> 0 citations located at sections.

> 1 citation located at section 26.11.2.

January 3, 2024 21:08 published under the GNU Free Documentation License


--- pldx : Translation of the Kimberling’s Glossary into barycentrics ---


January 3, 2024 21:08 published under the GNU Free Documentation License
▷ 0 citations located at sections.

▷ 1 citation located at section 26.9.5.

▷ 0 citations located at sections.

▷ 0 citations located at sections.

▷ 0 citations located at sections.

▷ 0 citations located at sections.

▷ 0 citations located at sections.

▷ 0 citations located at sections.

▷ 2 citations located at sections 25.1.6 and 25.1.6.

▷ 0 citations located at sections.

▷ 0 citations located at sections.

▷ 1 citation located at section 14.10.6.

▷ 1 citation located at section 17.1.3.

▷ 1 citation located at section 9.1.2.

▷ 1 citation located at section 14.10.2.

▷ 4 citations located at sections 3.2.2, 14.5, 18, and 18.4.1.

▷ 1 citation located at section 1.
▷ 0 citations located at sections.

▷ 1 citation located at section 14.1.1.

▷ 1 citation located at section 14.1.1.

▷ 1 citation located at section 14.1.2.

▷ 0 citations located at sections.

▷ 0 citations located at sections.

▷ 1 citation located at section 26.1.

▷ 0 citations located at sections.

▷ 0 citations located at sections.

▷ 1 citation located at section 25.8.

▷ 0 citations located at sections.

▷ 0 citations located at sections.

▷ 1 citation located at section 17.5.8.

▷ 1 citation located at section 3.2.2.


† 1 citation located at section 17.3.


† 1 citation located at section 12.27.2.


‡ 2 citations located at sections 26.6.1 and 26.6.2.


† 0 citations located at sections .


† 0 citations located at sections .


† 1 citation located at section 7.7.1.


† 1 citation located at section 14.2.8.


† 1 citation located at section 14.2.16.


† 0 citations located at sections .


† 0 citations located at sections .


† 0 citations located at sections .


† 1 citation located at section 14.10.


† 0 citations located at sections .


† 0 citations located at sections .


† 0 citations located at sections .

January 3, 2024 21:08 published under the GNU Free Documentation License