Pencils of Cycles in the Triangle Plane

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Abstract

We examine how to describe pencils and bundles of cycles (i.e. circles or lines or points) in the context of the Triangle Geometry. When dealing with orthogonality, the best description is given by a 4D projective space, where the usual 3D projective space (the Triangle Plane) is represented by a paraboloid. When dealing with tangency, the best description is given by a Lie sphere, embedded into a 5D space and obtained by a double coating of the former 4D space (oriented cycles).

The paper shows that these representations are leading to handy computational formulae. Several examples are worked out to illustrate these results. A special attention is devoted to objects at infinity –especially the umbilics– since all these projective spaces are tailored to implement the Poncelet's continuity principle.

1 Introduction

1.1 Aims of the study

Many changes have occurred in the way we are doing geometry since the old ancient times of Euclid and Apollonius. Most of them are related to new methods for "automated" computing of properties, rather than relying on intuition to find "beautiful geometric proofs". Many individuals have contributed to this long process, and attributing a specific discovery to a specific individual is not an easy task (Coolidge, 1940). In fact, the emerging milestones in this long road are rather the individuals that have summarized the discoveries of their time into an efficient way of writing the questions to solve, so that the writing literally thinks for you and leads towards the required result by a least action property.

Writing numbers and their computations in a tractable manner is associated with Al-Khwarizmi and his *Algebra* (825). Using coordinates (x, y) to describe points and compute their geometrical properties (as well as the exponent notation for polynomials) is associated with Descartes and his *Géometrie* (1637). Using homogeneous coordinates $x : y : z$ to implement the principle of continuity when dealing with objects escaping to infinity is associated with Moebius and his *barizentrische Calcul* (1827). More recently, the idea to stamp a barcode on each outstanding point involved in the Triangle Geometry and then perform some kind of computer aided inventory management (Kimberling, 1998-2009) has changed the practice of geometers.

Efficient notations are powerful, poor notations can be confusing. In the opinion of the author, this kind of problem arises when studying pencils and bundles of circles in the context of the Triangle Geometry. On the one hand, the usual context for studying pencils and bundles of circles is Riemann sphere $\mathbb{P}_{\mathbb{C}}(\mathbb{C}^2)$ where $(z_1, z_2) \simeq (\lambda z_1, \lambda z_2)$ for any non-zero $\lambda \in \mathbb{C}$ (Poncelet, 1822, 1865). On the other hand, a Triangle point lives in $\mathbb{P}_{\mathbb{R}}(\mathbb{R}^3)$ i.e. is described as $x : y : z$ in a projective space where $x : y : z = kx : ky : kz$ for any non-zero $k \in \mathbb{R}$.

Obviously, both points of view are reducing to the same elementary Cartesian coordinates when restricted to the finite domain. But they are conflicting *where they are the most useful*, i.e. where they are implementing the continuity principle for objects at infinity. An ordinary line must be completed in a way or another to become a "circle with infinite radius" and having a clear definition of this completion is required in order to unify the three concepts of circle $(0 < \rho < \infty)$, point $(\rho = 0)$ and line $(\rho = \infty)$ into a single concept of cycle.

In the Riemann sphere $\mathbb{P}_{\mathbb{C}}(\mathbb{C}^2)$, there exists only one point at infinity (noted ∞). In this context, a "circle with infinite radius" is an ordinary line Δ completed by point ∞, i.e. $\overline{\Delta} = \Delta \cup \{\infty\}$, while point-circles are either circles with radius 0 around an ordinary point or $\{\infty\}$.

In the Triangle Plane $\mathbb{P}_{\mathbb{R}}(\mathbb{R}^3)$, there exists a whole line $\mathcal{L}_\infty$ of points at infinity, verifying $x+y+z=0$. Here, barycentrics are used. Using trilinears would only change some formulae, but not the very nature of the underlying space. In this context, the barycentric equation of an ordinary circle leads to define a cycle as a second degree curve (a conic) that goes through the umbilics $\Omega^\pm = abc\, X_{512} \pm iR\, X_{511}$. Therefore, the equation of a completed line becomes $(x+y+z)(ux+vy+wz) = 0$ so that we must define $\overline{\Delta}$ as $\Delta \cup \mathcal{L}_\infty$, while the role of $\{\infty\}$ in $\mathbb{P}_{\mathbb{C}}(\mathbb{C}^2)$ is played now by the horizon circle $\mathcal{C}_\infty$ defined by $(x+y+z)^2 = 0$, i.e. as an object having the same points as $\mathcal{L}_\infty$ but each of them counted twice.

Moreover, it will be seen that a cycle has to be represented by a point $u : v : w : t \in \mathbb{P}_{\mathbb{R}}(\mathbb{R}^4)$ rather than by a "circle function" $u : v : w \in \mathbb{P}_{\mathbb{R}}(\mathbb{R}^3)$ that would be some kind of point, with a Kimberling number. On the contrary, it will be seen that these $u : v : w$ are only useful as shadows of the "true" representatives, as in the Socrates' myth.

1.2 Organization of the paper

This paper is organized as follows. Section 2 specifies notations, and introduces cycles and umbilics. Section 3 puts emphasis on properties involving lengths, radiuses and centers. Section 4 introduces the fundamental quadric $\mathcal{Q}$ and reduces orthogonality of cycles into polarity with respect to $\mathcal{Q}$, leading to an efficient representation of the pencils of cycles. These properties are illustrated in Section 5, where relations between the Euler pencil and the incircle are considered.

Section 6 shows how tangency of cycles can be described in space $\mathbb{P}_{\mathbb{R}}(\mathbb{R}^4)$. Computational formulae are given for various problems, including the eight Apollonian cycles tangent to three given cycles, in the general case and also in the special case of three cycles through the same point. A short description of the associated Lie space $\mathbb{P}_{\mathbb{R}}(\mathbb{R}^5)$ is also given, that allows to consider oriented cycles. The paper ends with some concluding remarks.

1.3 Before to start

Letters used are intended to denote : P, Q some flat true points in the Triangle Plane, X a Kimberling-named triangle center ; Γ the circumcircle (and nothing else), Ω a cycle ; U the representative of a circle-point ; V the representative of a cycle in $\mathbb{P}_{\mathbb{R}}(\mathbb{R}^4)$; Y the representative of an oriented cycle in $\mathbb{P}_{\mathbb{R}}\left(\mathbb{R}^5\right)$; G a Gram matrix with diagonal w^2 and non diagonal W coefficients. Abbreviation etc. will stand for "et cyclically".

When possible, computed proofs are given that use formal computing tools. This kind of proof is sometime deprecated, but these proofs are together the easiest (all the messy job is done by computer) and the safest. A construction that sounds like a "beautiful geometrical proof" is too often invalid due to an hidden exception. In a computer proof, exceptions are appearing as multiplicative factors, according to the following polynomial model :

$$\mathit{conclusion} \times \mathit{exceptions} = \mathit{hypothesis}$$

To quote the Knuth's foreword to Petkovsek et al. (1996) :

> Science is what we understand well enough to explain to a computer. Art is everything else we do. During the past several years important parts of mathematics has been transformed from an Art to a Science.

2 Cycles and umbilics

Definition 2.1. A **point** P is what is described by a proportional *column* of three reals (the so-called barycentrics), and a **line** Δ is what is described by a proportional *row*. Typically :

$$P \simeq \left(\begin{array}{c} p \\ q \\ r \end{array}\right) \simeq \left(\begin{array}{c} k\,p \\ k\,q \\ k\,r \end{array}\right), \ \Delta \simeq \left(\begin{array}{ccc} \rho & \sigma & \tau \end{array}\right) \simeq \left(\begin{array}{ccc} \lambda\rho & \lambda\sigma & \lambda\tau \end{array}\right)$$

where k, λ are any real except from 0.

Notation 2.2. The "semi-colon" notation $P = p : q : r$ will be used when writing inline equations. This handy shortcut will ever imply both proportionality **and** column. Everywhere else, the equal sign asserts a rock-solid (component-wise) equality while equalities "up to a proportionality factor" are asserted using the simeq ($\simeq$) symbol.

Definition 2.3. The (projective) Triangle Plane is the set of all points P, seen as generated by barycentric combination from points $A\,(1,0,0)$, $B\,(0,1,0)$, $C\,(0,0,1)$. Points $P = p : q : r$ such that $p+q+r=0$ are said to be "at infinity" and written as $P \in \mathcal{L}_\infty$.

Notation 2.4. A circle will be denoted either by a Greek letter –e.g. Ω, without parentheses– or by a pair center/radius, leading to $\Gamma = (X_3,\, R)$. Using parentheses around a single Roman letter –e.g. (P)– will be reserved to denote circle $(P,\, 0)$ i.e. the circle whose unique real point is P itself.

Proposition 2.5. *Four points at finite distance belong to the same cycle (aka circle or straight line) when their barycentrics $p_i : q_i : r_i$ are such that :*

$$\det_{i=1}^{i=4}\left[\frac{1}{p_i+q_i+r_i}\left(q_i r_i a^2 + p_i r_i b^2 + p_i q_i c^2\right),\, p_i,\, q_i,\, r_i\right] = 0 \tag{1}$$

Computed Proof. Denominators are a reminder of the fact that only finite points are involved here. To obtain the required result, write the Cartesian equation of the cycle as :

$$\Delta_{cart} \doteq \det_{i=1}^{i=4}\left[\xi_i^2+\eta_i^2,\, \xi_i,\, \eta_i,\, 1\right]=0$$

where ξ, η are the Cartesian coordinates of the points. Substitute these coordinates by :

$$\xi=\frac{x\xi_A+y\xi_B+z\xi_C}{x+y+z},\ \eta=\frac{x\eta_A+y\eta_B+z\eta_C}{x+y+z}$$

and obtain another determinant $\Delta'(x,y,z)$. Then compute $F\,.\,\Delta'\,.\,T^{-1}\,.\,G$ where F is the diagonal matrix $diag\,(p_i+q_i+r_i)$ and

$$T=\left[\begin{array}{cccc} 1 & 0 & 0 & 0\\ 0 & \xi_a & \eta_a & 1\\ 0 & \xi_b & \eta_b & 1\\ 0 & \xi_c & \eta_c & 1 \end{array}\right],\qquad G=\left[\begin{array}{cccc} -1 & 0 & 0 & 0\\ \xi_a^2+\eta_a^2 & 1 & 0 & 0\\ \xi_b^2+\eta_b^2 & 0 & 1 & 0\\ \xi_c^2+\eta_c^2 & 0 & 0 & 1 \end{array}\right]$$

Matrix F acts on rows and kills quite all denominators, T acts on the last three columns and goes back to barycentrics while G acts on the first column to kill all square terms. After what everything simplifies nicely and leads to (1). □

Corollary 2.6. *Equation of the circumcircle is* $\Gamma(x,y,z)=0$ *where :*

$$\Gamma(x,y,z)=a^2\,yz+b^2\,zx+c^2\,xy \tag{2}$$

Definition 2.7. A **cycle** Ω is the locus of points $X=x:y:z$ that satisfies equation :

$$t\,\Gamma(x,y,z)-(x+y+z)\,(ux+vy+wz) \tag{3}$$

where $V=u:v:w:t\in\mathbb{P}_{\mathbb{R}}\left(\mathbb{R}^4\right)$ is called the **representative** of Ω. For example, the representative of circumcircle Γ is $0:0:0:1$.

Remark 2.8. When $t\neq 0$, cycle Ω is the (ordinary) circle whose standardized equation is :

$$\Gamma(x,y,z)-(x+y+z)\,(\widehat{u}x+\widehat{v}y+\widehat{w}z) \tag{4}$$

with $\widehat{u}:\widehat{v}:\widehat{w}:1=u:v:w:t$. Cycle $\mathcal{C}_\infty$ represented by $1:1:1:0$ has to be understood as the line at infinity $\mathcal{L}_\infty$ described **twice**, and will be called the **horizon circle**. Otherwise, the cycle represented by $u:v:w:0$ is the union of an ordinary line and $\mathcal{L}_\infty$, and will be called a completed line.

Remark 2.9. Assuming that representatives are living in $\mathbb{P}_{\mathbb{R}}\left(\mathbb{R}^4\right)$ has many advantages. The top one could be to enforce the fact that a representative is not a point in the Triangle Plane. Indeed, the triple $(\widehat{u},\,\widehat{v},\,\widehat{w})$ appearing in the standardized equation (4) is ***not*** defined up to a proportionality factor. The same remark applies to the so-called "circle function" $(\widehat{u}\div bc,\,\widehat{v}\div ca,\,\widehat{w}\div ab)\in\mathbb{R}^3$ that appears when using trilinears as in Weisstein (1999-2009).

Definition 2.10. The **umbilics** Ω^+, Ω^- are points at infinity that belong to the circumcircle (and therefore to any other circle). A possible choice is :

$$\Omega^{\pm} \simeq abc \begin{pmatrix} a^2\left(b^2-c^2\right) \\ b^2\left(c^2-a^2\right) \\ c^2\left(a^2-b^2\right) \end{pmatrix} \pm iR \begin{pmatrix} \left(a^2\left(b^2+c^2\right)-b^4-c^4\right)a^2 \\ \left(b^2\left(c^2+a^2\right)-c^4-a^4\right)b^2 \\ \left(c^2\left(a^2+b^2\right)-a^4-b^4\right)c^2 \end{pmatrix} \simeq abc\, X_{512} \pm iR\, X_{511} \quad (5)$$

where i is the imaginary unit and R the radius of the circumcircle.

Theorem 2.11. *Choosing an Euclidean structure on the Triangle Plane is either : (a) choosing the ratio between the sidelengths a, b, c (b) deciding which circum-ellipse $\alpha yz+\beta zx+\gamma xy = 0$ is* ***the*** *circumcircle (c) choosing the umbilics on the line at infinity. Indeed, umbilics are determining the Euclidean structure of the Triangle Plane in these three (equivalent) ways :*
(i) fixing the ratios of sidelengths by their barycentric product :

$$\Omega^+ *_b \Omega^- = a^2 : b^2 : c^2$$

(ii) defining the orthopoint transform as the linear application whose matrix becomes :

$$\begin{pmatrix} 0 & 0 & 0 \\ 0 & -i & 0 \\ 0 & 0 & +i \end{pmatrix}$$

when expressed in the basis X_4, Ω^+, Ω^-
(iii) characterizing a cycle as a conic that goes through the umbilics.

Proof. Circumcircle and infinity line are image of each other by isogonal conjugacy. Their intersection is globally invariant, and no other umbilical point can lead to a real result, proving (i). Property (ii) comes from direct computation since X_{511} and X_{512} are orthopoint of each other –cf Postnikov (1982, 1986) for better insights on real-complex spaces. Direct (iii) is obvious from (3). To prove the fact that a conic through the umbilics is a cycle, we have to consider the rank of the values taken by x^2, y^2, z^2, xy, yz, zx at the five points A, B, C, Ω^+, Ω^-. This gives a 5×6 matrix where the first three lines are $1,0,0,0,0,0$ etc. and it remains only to show that rank of submatrix 4..5,4..6 is two. A direct inspection shows that critical factors are $a^2+b^2-c^2$ (straight angle, that can occur only once) and $a^4+b^4+c^4-b^2c^2-a^2b^2-a^2c^2$ (condition of equilaterality). In such a case, the property remains when umbilics are written as $1:j:j^2$ and $1:j^2:j$. □

3 Center of a circle

Proposition 3.1. *Let a, b, c be the sidelengths of reference triangle ABC. Then distance $|PQ|$ between points P, Q described by their ABC-barycentrics $p:q:r$ and $u:v:w$ is obtained by :*

$$(\rho,\sigma,\tau) \doteq \frac{u}{u+v+w}-\frac{p}{p+q+r}, \frac{v}{u+v+w}-\frac{q}{p+q+r}, \frac{w}{u+v+w}-\frac{r}{p+q+r} \quad (6)$$

$$|PQ|^2 = a^2\sigma\tau+b^2\tau\rho+c^2\rho\sigma \quad (7)$$

Computed Proof. Let (ξ_X, η_X) be the Cartesian coordinates of point X. We have :

$$\begin{cases} |PQ|^2 & = (\xi_Q - \xi_P)^2 + (\eta_Q - \eta_P)^2 \quad \text{where} \quad \xi_P = \frac{p\xi_A + q\xi_B + r\xi_C}{p+q+r}, \text{ etc} \\ |BC|^2 & = (\xi_C - \xi_B)^2 + (\eta_C - \eta_B)^2 = a^2 \quad \text{etc} \end{cases}$$

This looks like a rather complicated system of four equations, but we know that all the ξ, η must cancel, due to invariance under displacement. Nevertheless, better use a formal computing tool ! □

Remark 3.2. Information conveyed by a triple like (6) is multiple. A first part is the direction of line PQ, described by the point $\rho : \sigma : \tau \in \mathcal{L}_\infty$. Another part is the squared length $|PQ|^2$ given by (7). In this formula, circumcircle appears as a tool that defines how lengths are computed in each direction. As it should be, $|AB|^2 = c^2$ etc.

Corollary 3.3 (Heron)**.** *Center and radius of the circumcircle are :*

$$\begin{aligned} X_3 &= a^2\left(b^2 + c^2 - a^2\right) : b^2\left(c^2 + a^2 - b^2\right) : c^2\left(a^2 + b^2 - c^2\right) \\ R^2 &= \frac{a^2 b^2 c^2}{(a+b+c)(a+b-c)(b+c-a)(c+a-b)} \end{aligned}$$

Computed Proof. Direct elimination from $\left\{|XA|^2 = R^2, \text{ etc}\right\}$ and (6,7). □

Theorem 3.4. *Let $P = p : q : r$ be a finite point. Then barycentric equation of circle (P, ω) is :*

$$\widetilde{t}\left(a^2\, yz + b^2\, zx + c^2\, xy\right) - (\widetilde{u}x + \widetilde{v}y + \widetilde{x}z)(x+y+z) + \widetilde{t}\,\omega^2\,(x+y+z)^2 = 0 \tag{8}$$

where the 4-tuple $\left(\widetilde{u}, \widetilde{v}, \widetilde{w}, \widetilde{t}\right)$ is defined by :

$$\begin{aligned} \widetilde{u} &= c^2q^2 + b^2r^2 + \left(b^2 + c^2 - a^2\right) qr \\ \widetilde{v} &= a^2r^2 + c^2p^2 + \left(c^2 + a^2 - b^2\right) rp \\ \widetilde{w} &= b^2p^2 + a^2q^2 + \left(b^2 + a^2 - c^2\right) pq \\ \widetilde{t} &= (p+q+r)^2 \end{aligned} \tag{9}$$

Computed Proof. Expand $|PX|^2 - \omega^2$ using (7). Obtaining only an expression like circumcircle plus infinity line times a first degree factor is only Proposition 2.5. The added value here is the emphasis put on center and radius. □

Definition 3.5. Pencil. When Ω_1, Ω_2 are distinct cycles (with non proportional representatives), all curves $\lambda_1\Omega_1 + \lambda_2\Omega_2 = 0$, where $(\lambda_1, \lambda_2) \neq (0, 0)$, are cycles and the set of all these cycles is called the pencil generated by Ω_1, Ω_2. It is clear that representatives of the cycles of a given pencil are on the same projective line in $\mathbb{P}_\mathbb{R}\left(\mathbb{R}^4\right)$ –called the representative of the pencil.

Example 3.6. Formula (4) describes circle Ω as a member of the pencil generated by the circumcircle and a completed line. Therefore, the ordinary line $ux + vy + wz = 0$ is the radical axis Δ of both circles Ω and Γ. That's another way to see that knowing $u : v : w$ is not enough to determine a circle.

Example 3.7. Formula (8) describe circle (P, ω) as a member of the pencil generated by the point-circle (P) and the horizon circle, i.e. the pencil of all circles concentric with $(P, 0)$. Here the horizon circle $\mathcal{C}_\infty$ is understand as a circle "whose center is everywhere and circumference nowhere" (Empedocles). This leads to :

$$u : v : w : t = \left(\widetilde{u} - \omega^2\right) : \left(\widetilde{v} - \omega^2\right) : \left(\widetilde{w} - \omega^2\right) : \widetilde{t} \tag{10}$$

Definition 3.8. The point representative of P is another name for the representative of the point-circle (P), as given by (9). Points at infinity are excluded from this definition.

Remark 3.9. Here again, the triple $u : v : w$ is not sufficient to specify P, and $u : v : w : t$ must be used. It can be checked that representative is well specified, i.e. doesn't depends on whichever triple (kp, kq, kr) is chosen as barycentrics of point P.

Example 3.10. Here are some point representatives :

P	u	v	w	t
$1:0:0$	0	c^2	b^2	1
$0:1:1$	$2\,b^2+2\,c^2-a^2$	a^2	a^2	4
1	$bc\,(b+c-a)$	$ac\,(c+a-b)$	$ab\,(b+a-c)$	$a+b+c$
2	$2\,b^2+2\,c^2-a^2$	$2\,a^2+2\,c^2-b^2$	$2\,b^2+2\,a^2-c^2$	9
3	R^2	R^2	R^2	1
4	$R^2a^2\left(b^2+c^2-a^2\right)^2$	$R^2b^2\left(c^2+a^2-b^2\right)^2$	$R^2c^2\left(a^2+b^2-c^2\right)^2$	$a^2b^2c^2$
6	$b^2c^2\left(2\,b^2+2\,c^2-a^2\right)$	$a^2c^2\left(2\,c^2+2\,a^2-b^2\right)$	$a^2b^2\left(2\,a^2+2\,b^2-c^2\right)$	$\left(a^2+b^2+c^2\right)^2$
∞	1	1	1	0

The fact that formula (9) *would* give $1 : 1 : 1 : 0$ for each point on $\mathcal{L}_\infty$ is the reason of their exclusion from the definition of the point representatives. On the contrary, point *Sirius* $= 1 : 1 : 1 : 0 \in \mathbb{P}_\mathbb{R}\left(\mathbb{R}^4\right)$ represents the horizon circle $\mathcal{C}_\infty$. In the usual $\mathbb{P}_\mathbb{C}\left(\mathbb{C}^2\right)$ model, $\mathcal{L}_\infty$ is "in the South plane" while the horizon circle $\mathcal{C}_\infty$ is nothing but the point-circle $\{\infty\}$.

4 The fundamental quadric

Proposition 4.1. *Any point representative $U = u : v : w : t$ belongs to the quadric $\mathcal{Q}$:*

$$\sum a^2u^2 - \sum \left(b^2+c^2-a^2\right)\left(vw + a^2ut\right) + a^2b^2c^2\,t^2 \quad = \quad 0 \tag{11}$$

Using Conway symbols $S_a = \left(b^2+c^2-a^2\right)/2$ etc, quadric $\mathcal{Q}$ can be rewritten as ${}^tU\boxed{\mathcal{Q}}U = 0$ where :

$$\boxed{\mathcal{Q}} = \begin{bmatrix} a^2 & -S_c & -S_b & -a^2S_a \\ -S_c & b^2 & -S_a & -b^2S_b \\ -S_b & -S_a & c^2 & -c^2S_c \\ -a^2S_a & -b^2S_b & -c^2S_c & a^2b^2c^2 \end{bmatrix} \tag{12}$$

Proof. Condition (11) appears when trying to invert (9). After what, a simple substitution proves the result. □

Proposition 4.2. *A point $V = u : v : w : t$ is the representative of a (real) cycle if only if V is outside of $\mathcal{Q}$ (i.e. on the same side as $0 : 0 : 0 : 1$) characterized by $\mathcal{Q}(u, v, w, t) \geq 0$ when (11) is used.*

Proof. Obvious from (10), that states that representative of (P, ω) is "below" representative of (P), while representatives of completed lines are at infinity in $\mathbb{P}_{\mathbb{R}}(\mathbb{R}^4)$ and therefore outside of paraboloid $\mathcal{Q}$. □

Theorem 4.3. *Consider two cycles Ω_1, Ω_2 with representatives V_1, V_2. When V_2 belongs to the polar plane of point V_1 wrt the fundamental quadric then cycles Ω_1 and Ω_2 are orthogonal –and conversely.*

Computed Proof. Begin with two circles. Write representative V_j as in (10) from representative U_j of point-circle (P_j), compute ${}^tV_1 \,.\, \boxed{\mathcal{Q}} \,.\, V_2$ and –using (7)– obtain :

$$ {}^tV_1 \,.\, \boxed{\mathcal{Q}} \,.\, V_2 = \left(\omega_1^2 + \omega_2^2 - |P_1P_2|^2\right) \times (p_2 + q_2 + r_2)^2 \,(p_1 + q_1 + r_1)^2 \, 8S^2 \tag{13} $$

Compute now ${}^tV_1 \,.\, \boxed{\mathcal{Q}} \,.\, V_3$ where $V_3 = u_3 : v_3 : w_3 : 0$ and obtain :

$$ {}^tV_1 \,.\, \boxed{\mathcal{Q}} \,.\, V_3 = -\,(u_3p_1 + q_1v_3 + r_1w_3) \times (p_1 + q_1 + r_1)\; 8S^2 \tag{14} $$

In both cases, the result is the orthogonality condition times a non vanishing factor. Finally, when the representatives of two lines are involved, the conclusion follows directly from Fact 4.4. □

Fact 4.4. *The orthopoint of a line $\Delta \simeq [u, v, w]$ is given by $\mathcal{M} \,.{}^t\Delta$ where :*

$$ \mathcal{M} = \left[\begin{array}{ccc} a^2 & -S_c & -S_b \\ -S_c & b^2 & -S_a \\ -S_b & -S_a & c^2 \end{array} \right] $$

Corollary 4.5. *The locus of the representatives of the points of a given cycle Ω is the intersection between $\mathcal{Q}$ and the polar plane –wrt $\mathcal{Q}$– of the representative of Ω.*

Proof. By definition, point P belongs to cycle Ω if and only if Ω and (P) are orthogonal. □

Corollary 4.6. *When $V = u : v : w : t$ is the representative of a circle (i.e. $t \neq 0$), the corresponding radius is given by :*

$$ \omega^2 = \frac{1}{16\,S^2} \, \frac{{}^tV \boxed{\mathcal{Q}} V}{t^2} \tag{15} $$

Theorem 4.7. *The only point at infinity of the quadric $\mathcal{Q}$ is Sirius $= 1 : 1 : 1 : 0$, the representative of the horizon circle. When $t \neq 0$ and U belongs to $\mathcal{Q}$, then U represents the point $P = p : q : r$ where :*

$$ p : q : r : \theta = \boxed{\mathcal{Q}} \,.\, U \tag{16} $$

Proof. Using a star to denote $1:1:1:0$ is from Kimberling (1998-2009). Using that specific star is from (Voltaire, 1752). It is clear that Sirius belongs to $\mathcal{Q}$. To see that $\mathcal{Q}$ is a paraboloid and that Sirius is the only (real) point at infinity of $\mathcal{Q}$, substitute $t=0$, then compute the discriminant with respect to w and obtain $-(u-v)^2 a^2 b^2 c^2/R^2$. This requires $u=v$, etc.

Equation (16) must hold for any U that is the representative of a point $p:q:r$. Indeed, for any cycle representative $V=u:v:w:t$, ${}^tV\boxed{\mathcal{Q}}U=0$ implies $pu+qv+rw-\theta t=0$ where $\theta=\left(qra^2+rpb^2+pqc^2\right)/(p+q+r)$. The result follows from rank reason (and can be checked directly). Conversely, starting from any U and applying (16) and then (9) leads to an expression that can be recognized as :

$$64\,t\,S^4 * U + 4\,{}^tU\boxed{\mathcal{Q}}U S^2 * Sirius$$ □

Theorem 4.8 (Classification). *Pencils of cycles fall in three classes, depending on the way their representative line $\mathcal{P}$ intersects –in $\mathbb{P}_{\mathbb{R}}(\mathbb{R}^4)$– the fundamental quadric $\mathcal{Q}$.*

$\mathcal{Q}$, $\mathcal{P}$ tangent : $\mathcal{P}$ is the tangent pencil of all the cycles containing a given point ω_0 and tangent at ω_0 to a line Δ_1 containing ω_0. Archetype : $\omega_0=\infty$ and $\mathcal{P}$ is "all the lines parallel to a given line Δ_1".

*$\mathcal{Q}$, $\mathcal{P}$ secant : $\mathcal{P}$ is the isotomic pencil generated by two different point-circles $\{\omega_1\}$ and $\{\omega_2\}$ (ω_i are the **limit** points of $\mathcal{P}$). Archetype : $\omega_2=\infty$ and $\mathcal{P}$ is, apart from $\{\infty\}$, "all the circles centered at a finite point ω_1".*

*$\mathcal{Q}$, $\mathcal{P}$ disjoint : $\mathcal{P}$ is the isoptic pencil of all the cycles going through two different points ω_1 and ω_2 (the **base** points). Archetype : $\omega_2=\infty$ and P is "all the lines through a finite point ω_1".*

When $\mathcal{P}$ is a tangent pencil, so is $\mathcal{P}^{\perp}$ (using ω_0 and $\Delta_1^{\perp}$ orthogonal to Δ_1 at ω_0). When $\mathcal{P}$ is isoptic(ω_1,ω_2) then $\mathcal{P}^{\perp}$ is isotomic(ω_1,ω_2) and conversely. In all cases, representative of $\mathcal{P}$ and $\mathcal{P}^{\perp}$ are conjugate lines wrt $\mathcal{Q}$.

Proof. Everything goes as in (Pedoe, 1970) –using another paraboloid– or (Douillet, 2009) –using a sphere. The only striking thing is that the usual point at infinity of the complex plane, namely $\infty\in\mathbb{P}_{\mathbb{C}}(\mathbb{C}^2)$, has to be replaced by the horizon circle $\mathcal{C}_\infty$: $(x+y+z)^2=0$. □

Proposition 4.9. *A pencil of cycles that contains two lines is a pencil of lines. A concentric pencil contains the horizon cycle. All other pencils (i.e. all the non archetypal pencils) contain exactly one straight line (the radical axis of the pencil).*

Proof. The representative of $\mathcal{P}$ ever intersects the plane at infinity of $\mathbb{P}_{\mathbb{R}}(\mathbb{R}^4)$. □

Proposition 4.10. *Let $P=p:q:r$ be a point in the Triangle Plane. Define its shadow in the Triangle Plane as point $S=u:v:w$ where u,v,w are defined in (9). Then S is not outside the inconic $IC(X_{76})$. Any point on the border of $IC\,(X_{76})$ is the shadow of exactly one point on the circumcircle, while a point inside $IC\,(X_{76})$ –except from X_2– is the shadow of exactly two points. Moreover, these points are inverse in the circumcircle.*

Remark 4.11. Figure 1 shows the shadows of all the named points in ETC (Kimberling, 1998-2009), using the standard values $a=6$, $b=9$, $c=13$. One can see two lines of points : $L\,(X_2,\,X_6)$ and $L\,(X_2,\,X_{39})$ containing the shadows of points from $L\,(X_3,\,X_2)$ –Euler line– and $L\,(X_3,\,X_6)$ –Brocard axis– respectively.

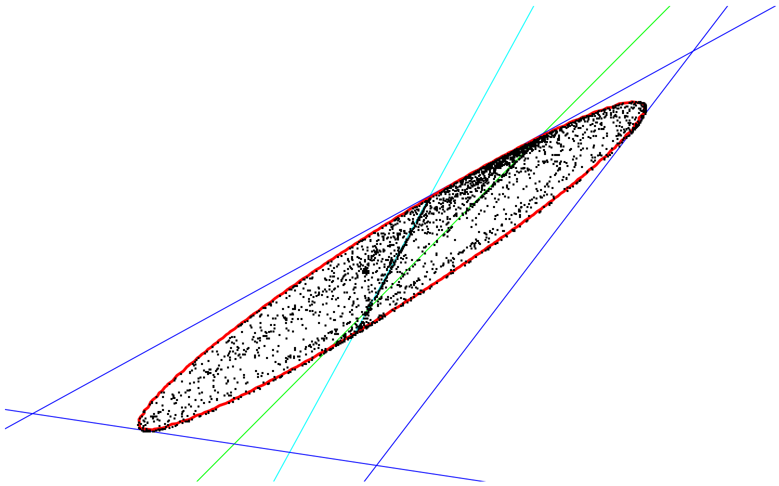


Figure 1: No point-shadow fall outside of the IC(X76) inconic

Proof of Proposition 4.10 . The locus of representatives of the points P_0 that belongs to Γ is the intersection of quadric $\mathcal{Q}$ and the polar plane $\prod$ of $0:0:0:1$, namely :

$$ua^2\left(b^2+c^2-a^2\right)+vb^2\left(c^2+a^2-b^2\right)+wc^2\left(a^2+b^2-c^2\right)-2\,ta^2b^2c^2=0$$

Extracting t and substituting in $\mathcal{Q}$ leads (apart from a constant non-zero factor) to :

$$u^2a^4+b^4v^2+c^4w^2-2\,uva^2b^2-2\,vwb^2c^2-2\,wuc^2a^2=0$$

i.e. the equation of $IC\left(X_{76}\right)$.

When two points P_1, P_2 in the Triangle Plane share the same shadow, then points U_1, U_2 and $0:0:0:1$ are collinear in $\mathbb{P}_{\mathbb{R}}\left(\mathbb{R}^4\right)$ so that cycles (P_1), (P_2) and Γ belongs to the same pencil. Therefore P_1, P_2 are inverse in the circumcircle. Moreover $U_0 \doteq U_1U_2 \cap \prod$ is inside $IC\left(X_{76}\right)$ –otherwise U_0 would be the representative of a real circle belonging to pencil $\left(P_1\right),\left(P_2\right)$ and orthogonal to Γ. □

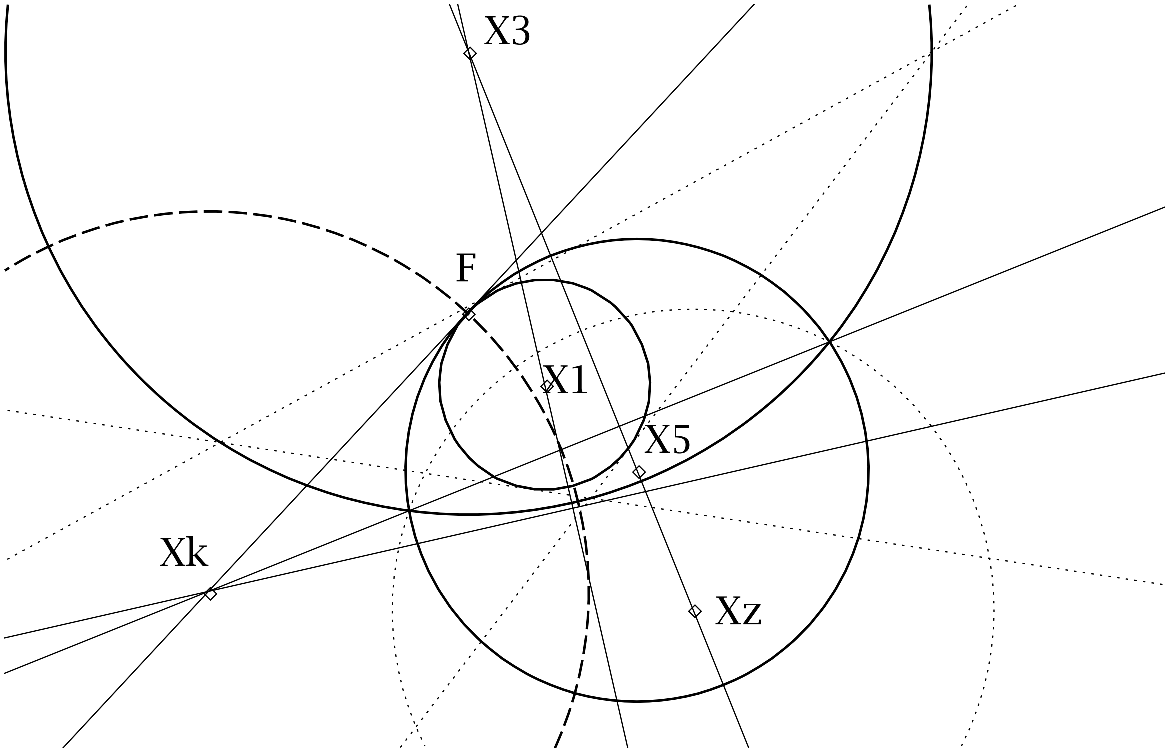
Theorem 4.12. *Let Ω_0 be a fixed cycle with representative V_0 and Ω_1 another cycle with representative V_1. Assume that Ω_0 is not a point-circle and call $\widehat{V_1}$ the intersection of line V_0V_1 with the polar plane of Ω_0. Construct V_3 such that division $V_0, \widehat{V_1}, V_1, V_3$ is harmonic. Then V_3 represents the cycle Ω_3 inverse of Ω_1 wrt cycle Ω_0 (inverse wrt a straight line is the ordinary symmetry wrt this line) and is given by :*

$$V_3 \simeq \sigma\left(V_1\right) \doteq V_1 - 2\,\frac{{}^tV_1 \boxed{\mathcal{Q}}\, V_0}{{}^tV_0 \boxed{\mathcal{Q}}\, V_0}\, V_0 \tag{17}$$

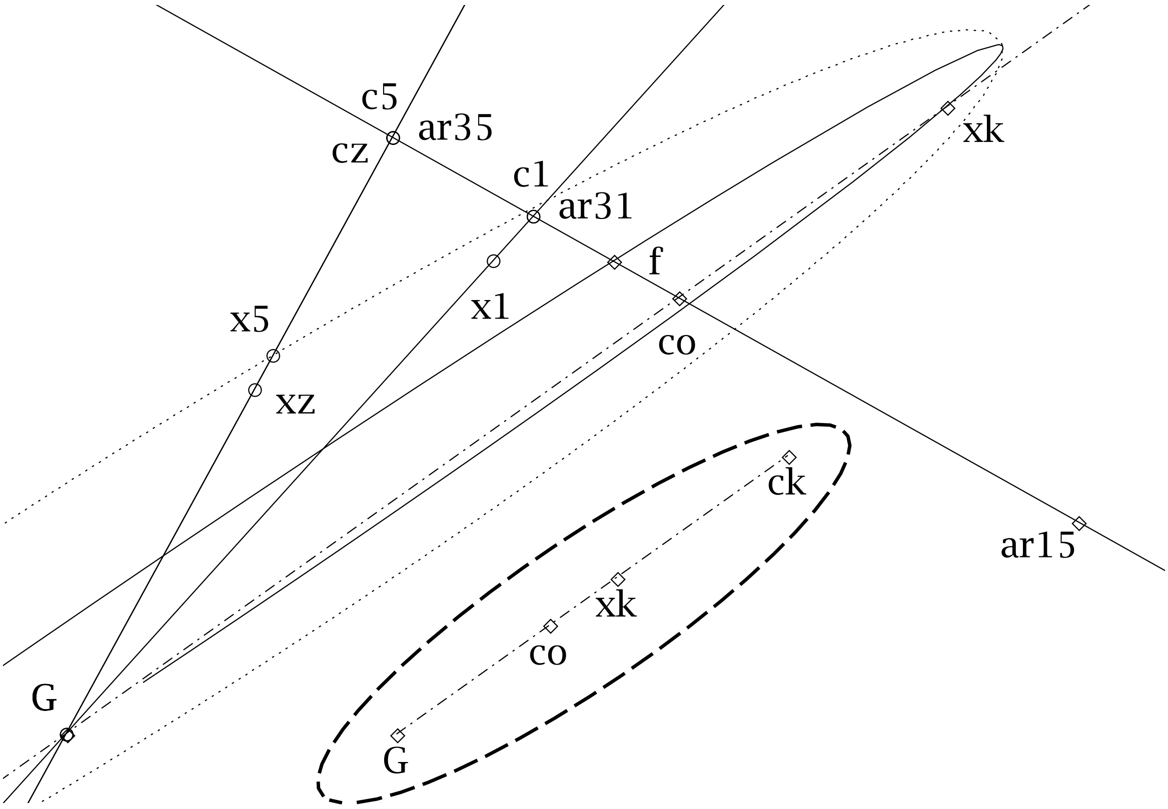
Moreover, when V_2 is yet another cycle representative, we have the conservation law :

$${}^t\sigma\left(V_1\right) \boxed{\mathcal{Q}}\, \sigma(V_2) = {}^tV_1 \boxed{\mathcal{Q}}\, V_2 \tag{18}$$

Proof. Write $\widehat{V_1}$ as $\alpha_1V_0+V_1$ in ${}^tV_0 \boxed{\mathcal{Q}}\, \widehat{V_1}=0$ and then obtain V_3 as $2\alpha_1V_0+V_1$ since division $(\infty,1,0,2)$ is harmonic. Equation (18) is obvious from (17), and shows that σ preserves orthogonality. Moreover, (17) shows that cycles orthogonal to Ω_0 are invariant while cycles concentric with Ω_0 are transformed into cycles concentric with Ω_0 : all together, this proves that σ is the inversion wrt cycle Ω_0. □



(a) lines and circles



(b) shadows

Figure 2: Euler pencil and incircle

Proposition 4.13. *Points inside of $\mathcal{Q}$ are representative of imaginary circles (real center, imaginary radius). The reason to imagine such circles is that inversion wrt such a circle is a real transform. Moreover a real cycle Ω is orthogonal to $(X,\, i\omega)$ when Ω cuts $(X,\, \omega)$ along a diameter.*

Proof. Straightforward computation. □

Proposition 4.14. *It exists exactly one cycle Ω_4 simultaneously orthogonal to three cycles Ω_j (j=1,2,3) that don't belong to the same pencil. Its representative can be computed as :*

$$V_4 = Adjoint\,(\mathcal{Q})\,\dot{\cdot}^{t}(V_1 \wedge V_2 \wedge V_3) \tag{19}$$

where $\wedge$ is the universal factorization of $\det(V_1, V_2, V_3, V)$, aka the row of cofactors involving the V_j (j=1,2,3).

Remark 4.15. The nature of Ω_4, real, point or imaginary fixes the nature of the bundle defined by $\Omega_1, \Omega_2, \Omega_3$.

5 Euler pencil and incircle

Consider $\mathcal{C}_1 = (X_1,\, r)$, $\mathcal{C}_3 = (X_3,\, R)$, $\mathcal{C}_5 = (X_5,\, R/2)$ and $\mathcal{C}_z = (X_z = X_{389},\, |GH|/2)$ i.e., respectively, the in-, circum- nine points and orthocentroidal circles (Figure 2(a)). Let U_j, V_j, x_j, c_j be the respective representatives of centers and circles, together with their respective shadows (Figure 2(b)). Then :

1. Circles $(X_1), \mathcal{C}_1, \mathcal{C}_\infty$ are concentric so that U_1, V_1, *Sirius* are aligned and therefore x_1, c_1, G are aligned too. The same happens for $j = 5$ and $j = z$.
2. Cycles $\mathcal{C}_3$, $\mathcal{C}_5$, $\mathcal{C}_z$ belong to the same (Euler) pencil, together with their radical axis $AR_{3,5}$, so that representatives V_3, V_5, V_z, $V_{3,5}$ are aligned and therefore c_3, c_5, c_z, $ar_{3,5}$ are aligned too. Since c_3 is "far below the paper sheet", we have $c_5 = c_z = ar_{3,5}$. For the same reason, $c_1 = ar_{3,1}$.
3. Circles $\mathcal{C}_1$ and $\mathcal{C}_5$ are tangent at $F \doteq X_{11}$, the Feuerbach point and cycles $(F), \mathcal{C}_1, \mathcal{C}_5$ belong to the same pencil, together with their common tangent $AR_{1,5}$. Representatives U_f, $V_1, V_5, V_{1,5}$ are aligned and so are x_f, c_1, c_5, $ar_{1,5}$.
4. Cycles $AR_{1,3}$, $AR_{1,5}$, $AR_{3,5}$ are on the same pencil (they concur in the radical center X_k) and their shadows $ar_{1,3}$, $ar_{1,5}$, $ar_{3,5}$ are aligned.
5. In fact line c_1c_5 is not representative of a specific pencil, but rather of the bundle generated by $\mathcal{C}_1$, $\mathcal{C}_3$, $\mathcal{C}_5$. We have :

$$V_1 \simeq \begin{bmatrix} (b+c-a)^2 \\ (c+a-b)^2 \\ (a+b-c)^2 \\ 4 \end{bmatrix},\; V_5 \simeq \begin{bmatrix} b^2+c^2-a^2 \\ c^2+a^2-b^2 \\ a^2+b^2-c^2 \\ 4 \end{bmatrix},\; V_3 \simeq \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}$$

and therefore :

$$V_k \simeq \left[\begin{array}{c} (c-b)\,(b+c-a)\left(b^2+c^2-a^2\right)\left(b^2+c^2-ab-ac\right) \\ (a-c)\,(c+a-b)\left(c^2+a^2-b^2\right)\left(c^2+a^2-bc-ba\right) \\ (b-a)\,(a+b-c)\left(a^2+b^2-c^2\right)\left(a^2+b^2-ca-cb\right) \\ 4\,(a-b)\,(b-c)\,(c-a)\,(a+b+c) \end{array}\right]$$

From V_k, well known result $X_k = X_{676}$ and obvious $r_k = |X_k X_f|$ can be re-obtained.

6. As it should be, x_k, c_k, G are aligned (small insert, at the bottom of Figure 2(b)).

7. Consider W_k at intersection between *line* $(V_k Sirius)$ and *plane* (V_1, V_3, V_5). This points represents a circle that is both concentric and orthogonal to $\mathcal{C}_k$.This circle is therefore $(X_k,\, i\, r_k)$ and its shadow co belongs to both Gx_k and c_5c_1. Moreover, division G, x_k, co, c_k is harmonic.

6 The Apollonius configuration

In the general case, it exists eight cycles Ω tangent to three given cycles Ω_1, Ω_2, Ω_3 (not from the same pencil). A survey of this question is Gisch and Ribando (2004), while the usual disjunction into ten cases is Wiki Contributors (2008). The best space where this Apollonius problem can be discussed is $\mathbb{P}_{\mathbb{R}}\left(\mathbb{R}^5\right)$ (cf Subsection 6.4). Nevertheless, most of the results can be formulated in $\mathbb{P}_{\mathbb{R}}\left(\mathbb{R}^4\right)$... and it will appear that only one situation is really special (cycles through the same point), all the other belonging to the same general case.

6.1 Tangent cycles in the representative space

Proposition 6.1. *Two cycles are tangent when their pencil line is tangent to the fundamental quadric. Therefore, the locus of the representatives of all cycles tangent to a given (real) cycle Ω represented by V (not inside $\mathcal{Q}$) is the cone whose vertex is V and that goes through $\mathcal{Q} \cap polar\,(V)$.*

Proof. Two tangent cycles are defining a tangent pencil ! □

Remark 6.2. When Ω is a point circle, its representative U belongs to the fundamental quadric, and the cone of the tangent cycles degenerates into a doubly coated plane.

Definition 6.3. The Gram matrix $\mathrm{G}_{p,q,\cdots,r}$ of X_p, $X_q, \cdots$, $X_r \in \mathbb{R}^4$ is the matrix of all the products ${}^tX_p \boxed{\mathcal{Q}} X_q$. In this context, notation $W_{pq} = {}^tX_p \boxed{\mathcal{Q}} X_q$ and $w_p^2 = {}^tX_p \boxed{\mathcal{Q}} X_p$ will be used, leading to

$$\mathrm{G}_{pq} = \left(\begin{array}{cc} w_p^2 & W_{pq} \\ W_{pq} & w_q^2 \end{array}\right) \tag{20}$$

Proposition 6.4. *Two cycles Ω_1, Ω_2 are secant, tangent or external when* $\operatorname{signum}\det \mathrm{G}_{12}$ *is (respectively)* $+1$, 0 *or* -1.

Theorem 6.5. *Special cases of the Apollonius problem are (1) cycles from the same pencil and (2) cycles through the same point (tangent bundle). Otherwise, representatives V_j of the three given cycles and their common orthogonal cycle Ω_4 form a basis that splits the problem into four pairs of solutions. One of the solutions is given by $V_0 = \sum k_j\, V_j$ where :*

$$\begin{aligned}
k_1 &= (w_2\, w_3 - W_{23})\,(-w_1\, w_2\, w_3 - w_1\, W_{23} + w_2\, W_{13} + w_3\, W_{12}) \\
k_2 &= (w_1\, w_3 - W_{13})\,(-w_1\, w_2\, w_3 + w_1\, W_{23} - w_2\, W_{13} + w_3\, W_{12}) \\
k_3 &= (w_1\, w_2 - W_{12})\,(-w_1\, w_2\, w_3 + w_1\, W_{23} + w_2\, W_{13} - w_3\, W_{12}) \\
k_4 &= \sqrt{-2\,(w_2\, w_3 - W_{23})\,(w_1\, w_3 - W_{13})\,(w_1\, w_2 - W_{12})\;\mathrm{G}_{123}}\,/w_4
\end{aligned} \tag{21}$$

and the others are obtained by changing k_4 into $-k_4$ (inversion through Ω_4) or changing the signs of w_1, w_2, w_3. A solution is real/imaginary or "unimaginable" (object that would have a non real center) according to the sign of k_4^2. Globally, the number of "imaginable" solutions changes when the tangency condition $\prod \mathrm{G}_{jk}$ vanishes.

Proof. When Ω_j, $j = 1, 2, 3$ is a basis of a non tangent bundle, then Ω_j, $j = 1, 2, 3, 4$ is a basis of the whole representative space. The fundamental quadratic form is described, in this basis, by matrix G_{1234} where $W_{j4} = 0$ for $j = 1, 2, 3$. Computing, in this basis, the tangency condition of Ω_0 and any of the Ω_j leads to 0. Since the w_j are defined as $\sqrt{W_{jj}}$ we have 4 choices of signs leading, due to the possibility of a global proportionality factor, to eight different values. □

6.2 Working out an example : the three excircles

Taking the three excircles as $\Omega_1, \Omega_2, \Omega_3$ leads to a well known situation (Stevanovic, 2003).

1. Representative of point X_1 is :

$$U_0 = \begin{pmatrix} bc\,(b + c - a) \\ ca\,(c + a - b) \\ ab\,(a + b - c) \\ a + b + c \end{pmatrix}$$

   while radius of the incircle is :

$$r = \sqrt{\frac{(a + b - c)\,(c + a - b)\,(b + c - a)}{4\,(a + b + c)}}$$

2. Representative of the incircle, given by (10), is :

$$V_0 = \begin{pmatrix} (a - b - c)^2 \\ (a - b + c)^2 \\ (a + b - c)^2 \\ 4 \end{pmatrix}$$

3. Centers, radiuses and representatives V_a, V_b, V_c of the excircles are obtained by changing one of the sidelengths into its opposite in the respective formulae for the incircle.

4. The representative of the common orthogonal circle, as computed from (19), is :

$$V_4 = \begin{pmatrix} (c+a-b)(a+b-c) \\ (a+b-c)(b+c-a) \\ (b+c-a)(c+a-b) \\ -4 \end{pmatrix}$$

5. The radius of this circle, as computed from (15), is :

$$\omega_4 = \sqrt{\frac{b^2c+ab^2+bc^2+a^2b+ac^2+a^2c+acb}{4(a+b+c)}}$$

while the representative of the center is :

$$U_4 = \begin{pmatrix} 2\,a\left(b^2+c^2\right)-acb+b^3+c^3-a^3 \\ 2\,b\left(c^2+a^2\right)-acb+c^3+a^3-b^3 \\ 2\,c\left(a^2+b^2\right)-acb+a^3+b^3-c^3 \\ 4\ (a+b+c) \end{pmatrix}$$

and the center itself is :

$$b+c : a+c : a+b = X_{10}$$

6. The pairs of solutions of the Apollonius problem, as given by (21), are :

$$S_1 = \begin{pmatrix} b^2+c^2-a^2 \\ c^2+a^2-b^2 \\ a^2+b^2-c^2 \\ 4 \end{pmatrix},\ S_5 = \begin{pmatrix} bc\,(a+b+c)\,(2\,bc+a\,(a+b+c)) \\ ca\,(a+b+c)\,(2\,ca+b\,(a+b+c)) \\ ab\,(a+b+c)\,(2\,ab+c\,(a+b+c)) \\ -4\,abc \end{pmatrix}$$

$$S_2 = \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix},\ S_6 = \begin{pmatrix} (a+b+c)\left(b^2+ab+ac+c^2\right) \\ (b+c)(a-b-c)(a+b-c) \\ (b+c)(a-b-c)(a-b+c) \\ 4\ (b+c) \end{pmatrix}$$

Point S_1 is the representative of the nine-points circle, centered at X_5 while S_5 is related to the Apollonius circle, centered ad X_{970}. Points S_2, S_3, S_4 are the representatives of lines BC, CA, AB while S_6 and S_7, S_8 (obtained cyclically) are the representatives of the last three solutions.

6.3 The special case

Proposition 6.6. *Let Ω_1, Ω_2,Ω_3 be three cycles generating a bundle whose common orthogonal cycle is a point-cycle (ω_5), and ω_4 be any other point. The representative of one of the cycles tangent to Ω_1, Ω_2, Ω_3 is given by $V_0 = \sum_1^3 k_j\, V_j + 4U_4$ where :*

$$k_1 = \left(\frac{w_2 w_3 - W_{2,3}}{(w_1 w_3 - W_{1,3})\,(w_1 w_2 - W_{1,2})}\ \mathrm{G}_{1,2,3,4} - 2\ \mathrm{G}_{2,3,4} \right) \div \Delta_{2,3,4}^{1,2,3} \tag{22}$$

$\Delta_{2,3,4}^{1,2,3}$ is the minor obtained by deleting row 1 and column 4 in $\mathrm{G}_{1,2,3,4}$ and k_2, k_3 are obtained cyclically. Three other cycles are obtained by changing one of the w_1, w_2, w_3 into its opposite. The other solutions are four times the point cycle ω_5.

Proof. In this special case, $\mathrm{G}_{1,2,3} = 0$ and Ω_4 is chosen so that $w_4 = 0$. When assuming that Ω_1, Ω_2,Ω_3 aren't pairwise tangent, a direct substitution shows that Ω_0 is tangent to any of the given cycles. □

Example 6.7. Using $\Omega_1 = 1 : 0 : 0 : 0$ (representative of line BC) etc, leads to $\omega_5 = Sirius$. An efficient choice for ω_4 is any vertex. Using, for example, $U_4 = 0 : c^2 : b^2 : 1$, one re-obtains easily the in/excircles.

Example 6.8. Let (XYZ) be the circumcircle of triangle XYZ. The Apollonius circles relative to (ABH), (BCH), (CAH), i.e. the circles tangents to all of the three given circles, are $(H, 0)$ four times, $(H, 2R)$ once and three other circles, Ta, Tb, Tc.

Circles Ta, Tb, Tc are ever external to each other, and their common orthogonal circle To is real. Condition of (external) tangency is :

$$\left(a^2 - b^2\right)^2 - \left(a^2 + b^2\right) c^2 = 0$$

(etc) or ABC rectangular. The Apollonius circles of Ta, Tb, Tc are (ABH) etc, their inverses in To and two others. Center $X = x : y : z$ and radius ω of the first one are :

$$x = \left(b^2 + c^2 - a^2\right) \times \\ \left(a^8 - 2\left(b^2 + c^2\right) a^6 + 2\left(b^4 - b^2 c^2 + c^4\right) a^4 - 2\left(b^2 + c^2\right)\left(b^2 - c^2\right)^2 a^2 + \left(b^2 - c^2\right)^4\right)$$

$$\omega = 2\,\frac{a^2 b^2 c^2 R}{a^6 + b^6 + c^6 - a^2 b^4 - a^4 b^2 - c^2 b^4 - a^4 c^2 - b^2 c^4 - c^4 a^2 + 4\, a^2 b^2 c^2}$$

while the second is less simple.

6.4 Elementary properties of the Triangle Lie Sphere

Definition 6.9. The Triangle Lie Sphere is the locus of points $X = x : y : z : t : \tau \in \mathbb{P}_{\mathbb{R}}\left(\mathbb{R}^5\right)$ such that ${}^tX \,.\, \boxed{\mathcal{Q}_5} \,.\, X = 0$ where fundamental matrix is defined by :

$$\boxed{\mathcal{Q}_5} = \left[\begin{array}{cc} \boxed{\mathcal{Q}} & 0 \\ 0 & -4S^2 \end{array}\right] = \left[\begin{array}{ccccc} a^2 & -S_c & -S_b & -a^2 S_a & 0 \\ -S_c & b^2 & -S_a & -b^2 S_b & 0 \\ -S_b & -S_a & c^2 & -c^2 S_c & 0 \\ -a^2 S_a & -b^2 S_b & -c^2 S_c & a^2 b^2 c^2 & 0 \\ 0 & 0 & 0 & 0 & -4\,S^2 \end{array}\right]$$

Proposition 6.10. *Using notations of Theorem 4.3, an element of the Lie sphere represents, when $t \neq 0$ and $\tau \neq 0$, an oriented circle :*

$$Y_1 = \left[\begin{array}{c} b^2 r_1^2 + c^2 q_1^2 + 2\, S_a q_1 r_1 - \omega_1^2 \left(p_1 + q_1 + r_1\right)^2 \\ c^2 p_1^2 + a^2 r_1^2 + 2\, S_b r_1 p_1 - \omega_1^2 \left(p_1 + q_1 + r_1\right)^2 \\ a^2 q_1^2 + b^2 p_1^2 + 2\, S_c p_1 q_1 - \omega_1^2 \left(p_1 + q_1 + r_1\right)^2 \\ \left(p_1 + q_1 + r_1\right)^2 \\ 2\, \omega_1 \left(p_1 + q_1 + r_1\right)^2 \end{array} \right]$$

(where signum of ω_1 defines the orientation) or, when $t = 0$ and $\tau \neq 0$, an oriented line :

$$Y_3 = \left[\begin{array}{c} u_3 \\ v_3 \\ w_3 \\ 0 \\ \pm \dfrac{1}{2S} \sqrt{\Delta \,.\, \mathcal{M} \,.\, {}^t\Delta} \end{array} \right]$$

–cf Fact 4.4 for definition of $\mathcal{M}$– or, when $\tau = 0$, a non-oriented point.

Proof. All these results follow directly from Theorem 4.3. □

Proposition 6.11. *Two oriented cycles are tangent if and only if ${}^t Y_j \boxed{\mathcal{Q}_5} Y_k$ vanishes.*

Proof. This result is the rationale behind the former definitions. Using notations of Theorem 4.3, we obtain for two circles :

$$^t Y_1 \boxed{\mathcal{Q}_5} Y_2 = -\left(\left|P_1 P_2\right|^2 - \left(\omega_1 - \omega_2\right)^2 \right) \times 8 S^2 \, \left(p_1 + q_1 + r_1\right)^2 \left(p_2 + q_2 + r_2\right)^2 \tag{23}$$

By continuity, the Proposition holds also when lines are involved. For the sake of completeness, one can nevertheless compute :

$$^t Y_1 \boxed{\mathcal{Q}_5} Y_3 = -\left(\frac{p_1 u_3 + q_1 v_3 + r_1 w_3}{p_1 + q_1 + r_1} + \omega_1 \frac{1}{2S} \sqrt{\Delta_3 \,.\, \mathcal{M} \,.\, {}^t\Delta_3} \right) \times 8 S^2 \, \left(p_1 + q_1 + r_1\right)^2 \tag{24}$$

for a circle and a line, and (after multiplication by conjugate quantity) :

$$-\left(\left(v_4 - w_4\right) u_3 + \left(w_4 - u_4\right) v_3 + \left(-v_4 + u_4\right) w_3\right)^2 \times 8\, S^2$$

for two lines Y_3, Y_4. In the three cases, this is the condition of tangency times a non vanishing factor. □

Remark 6.12. When using the Lie representation, objects that don't belong to $\mathcal{Q}_5$ are meaningless, while $\mathcal{Q}_5$ itself is obtained by double coating the outside of $\mathcal{Q}$ in $\mathbb{P}_{\mathbb{R}}\left(\mathbb{R}^4\right)$. Therefore, imaginary circles are lost : when the radius decreases to 0 in an *isotomic* pencil, the differentiable continuation is going back to positive radiuses (with the other orientation) and not escaping to imaginary values.

Proposition 6.13. *Let Ω_0 be a cycle, but not a point-circle, σ the inversion wrt cycle Ω_0 as described in Theorem 4.12, $V_1 \in \mathbb{P}_{\mathbb{R}}(\mathbb{R}^4)$ a representative of cycle Ω_1 and $Y_1 = (V_1, \tau_1) \in \mathbb{P}_{\mathbb{R}}(\mathbb{R}^5)$ a representative of one of the corresponding oriented cycles. Then applications :*

$$\begin{array}{lll} \sigma^+ & : & \sigma^+(V_1, \tau_1) = (\sigma(V_1), +\tau_1) \\ \sigma^- & : & \sigma^-(V_1, \tau_1) = (\sigma(V_1), -\tau_1) \end{array}$$

are describing inversions wrt each of the oriented cycles corresponding to Ω_0. The conservation law (18) can be rewritten in order to describe a projective invariant by inversion, namely :

$$-\frac{{}^tY_1 \boxed{\mathcal{Q}_5} Y_2}{4S^2\,\tau_1\,\tau_2} = \frac{|P_1P_2|^2 - (\omega_1 - \omega_2)^2}{2\omega_1\,\omega_2}$$

or, for a circle and a line,

$$-\frac{{}^tY_1 \boxed{\mathcal{Q}_5} Y_3}{4S^2\,\tau_1\,\tau_3} = 1 + \frac{p_1u_3 + q_1v_3 + r_1w_3}{\omega_1\,(p_1 + q_1 + r_1)\,\tau_3}$$

and for two lines :

$$-\frac{{}^tY_3 \boxed{\mathcal{Q}_5} Y_4}{4S^2\,\tau_3\,\tau_4} = 1 \pm \cos(\Delta_3, \Delta_4)$$

Remark 6.14. Searby (2009) illustrates how this invariant (named ϵ_{jk}) can be used to summarize tangencies in various situations.

7 Conclusion

We have shown that an efficient description of cycles and pencil of cycles related to the Triangle Plane is obtained by using representatives $V = x : y : z : t$ that belongs to a four coordinates projective space. In this space, orthogonality of cycles in the Triangle Plane is translated into polarity with respect to a fundamental paraboloid $\mathcal{Q}$. The matrix defining this quadric has been related (12) to the Euclidean structure defined by the sidelengths of the reference triangle ABC.

This leads to a geometric representation of the cycles and pencil of cycles. The quadric itself is the locus of representatives of point-circles (aka ordinary points). The locus of representatives of a pencil of cycles is a projective line in the 4D space. The representative lines of two mutually orthogonal pencils are conjugate wrt the fundamental quadric. Finally, inversion wrt a cycle is described as a $\boxed{\mathcal{Q}}$-symmetry.

Many things are going the same way as in the usual representation using the Riemann sphere. The most striking exception is the representation of the cycle required to complete any concentric pencil. In $\mathbb{P}_{\mathbb{C}}(\mathbb{C}^2)$, this role is played by point-circle $\{\infty\}$. In $\mathbb{P}_{\mathbb{R}}(\mathbb{R}^4)$, a natural alternative appears, namely the *horizon circle*, where each point of the line at infinity is counted twice.

All these geometric properties lead to efficient formulae to compute radiuses (15), common orthogonal cycle (19), inverses (17) and even the Apollonius cycles of three given cycles (21).

It is well known that such 4D representation can be embedded into a 5D representation, known as the Lie sphere $\mathcal{Q}_5$. On this quadric $\mathcal{Q}_5$, a point represents an oriented cycle, while the condition of oriented tangency of two cycles is described as a polarity wrt $\mathcal{Q}_5$. The added value here is to describe this Lie sphere in terms adapted to barycentric coordinates in the Triangle Plane, rather than to complex coordinates in $\mathbb{P}_{\mathbb{C}}(\mathbb{C}^2)$. A geometric interpretation of the circular invariant ϵ_{jk} has been given.

To conclude, the so-called circle function (or its barycentric equivalent) is not describing a Kimberling-numbered point in the Triangle Plane. On the contrary, circle functions have to be considered as a limited description of objects whose real life occurs in a 4D space, or even on a quadric embedded in a 5D space.

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