Morley and Lubin revisited

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Contents

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1 Introduction

The present discussion started with a post of Ayme (2009) on "les-mathematiques.net" http://www.les-mathematiques.net/phorum/read.php?8,562009. This post was about the Fuhrmann circle. By the way, the "Morley parameterization" was recalled, resulting into a post of Rideau (2009). The question was : how to characterize points that have such a parameterization. More precisely :

> *Given a degree n and a parameterization of the reference triangle in the complex plane as :*
>
> $z(A) = \alpha^n$; $z(B) = \beta^n$; $z(C) = \gamma^n$ *where* $|\alpha| = |\beta| = |\gamma| = 1$
>
> *determine all the centers P that can be parameterized as :*
>
> $$z(P) = Q(\alpha,\, \beta,\, \gamma) \in \mathbb{Q}[\alpha,\, \beta,\, \gamma]$$

2 Morley's theorem, Lubin's representation

2.1 Morley's theorem

The result known today as "the Morley's theorem" has been discovered as a byproduct of some deeper geometrical results. The original result (1929) can be stated as :

Theorem 2.1 (Morley)**.** *There is a class of triangles. They are equilateral. They are perspective with ABC. They are eighteen.*

Notation 2.2. In Figures 1 and 2, points on the circle are A (label 1), B (label 4), C (label 7) and other points are the thirds of each arc. Concerning the Morley configuration, the centers of eighteen triangles are labeled with a two digit code xy where y is $3j + k$ (j, k ranging in $0, 1, 2$) and either $x = 0$ (points labeled upright of a red $+$) or $x = 1$(points labeled downleft of a blue $\times$).

Definition 2.3. When named with a definite article, "the Morley triangle" is the central one, labeled 00. Its center is X(356) –Morley center–, its perspector is X(357) –Morley-Taylor-Marr center– while the isogonal conjugate of X(357) is named X(358).

2.2 Lubin's parameterization

A large number of research papers have been devoted to the topic. In 1978, Cletus O. Oakley published a list of 150 references ! Among the many proofs given for this theorem, Lubin published, in 1955, a proof using complex numbers. The idea was to start with $A = \alpha^3$, $B = \beta^3$, $C = \gamma^3$, the origin being the center of the circumcircle and the radius R being normalized to $R = 1$. This parameterization can be used in many other situations, and we prefer to call it "Lubin", especially when using this representation in another context.

Remark 2.4. Using complex numbers requires complex-conjugates. Collinearity and cocyclicity are (respectively) :

$$\det_{j=1}^{3}\left([z_j,\, \overline{z_j},\, 1]\right) = 0, \quad \det_{j=1}^{3}\left([z_j\, \overline{z_j},\, z_j,\, \overline{z_j},\, 1]\right) = 0 \tag{1}$$

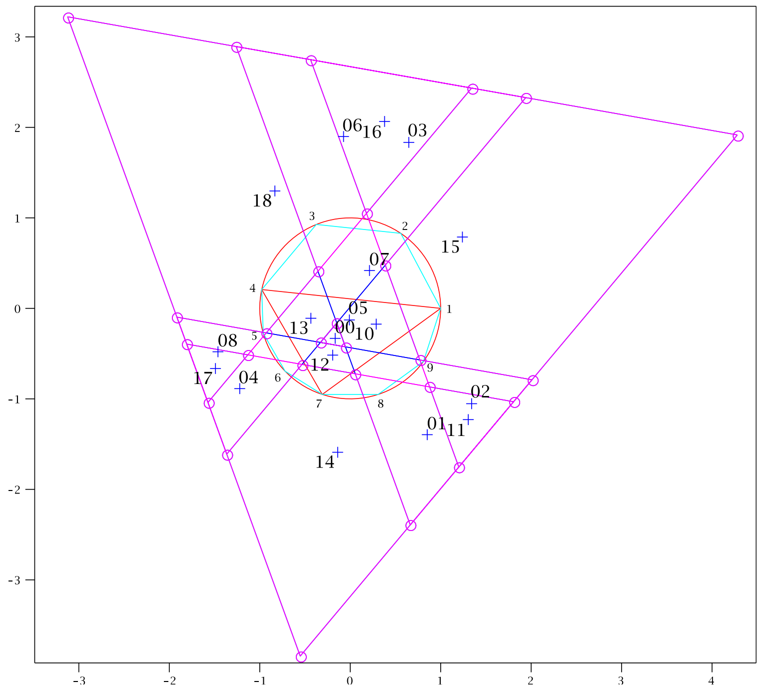

Figure 1: The Morley's configuration (full view)

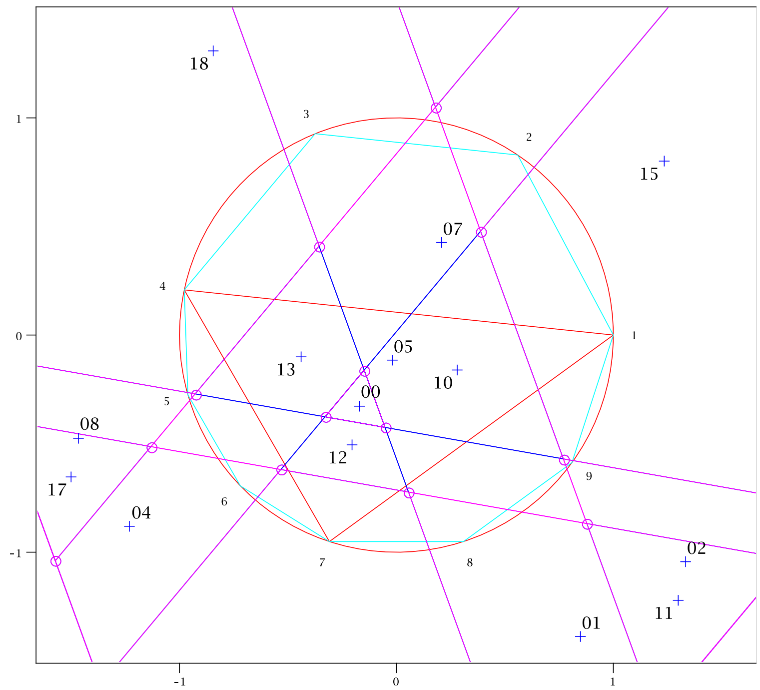

Figure 2: The Morley's configuration (detail)

But complex-conjugacy is not a rational operation. In order to use affixes that remains in a fractional field, we need to use a $\mathbb{R}$-generating family whose points belong to the unit circle, where $\overline{z} = 1/z$ can be used.[1] Another possibility is the Poncelet parameterization using the incircle as unit circle, and the contact points as generating family (cf. Subsection 2.6).

Definition 2.5. We define the Lubin(n) parameterization as the parameterization using n-th powers of the parameters, i.e. :

$$z\,(A) = \alpha^n,\ \text{etc}$$

Remark 2.6. When $n > 1$, this definition is not sufficient for a unique determination of parameters α, etc... and difficulties are expected to occur.

Proposition 2.7. *In any Lubin(n) parameterization, a real number is represented by a fraction of degree* 0*. Therefore, any representable point is represented by a fraction of degree* n*. Conversely, the degree* n *of a Lubin(n) parameterization can be "read" on the affix of any point (except from* O *!).*

Proof. A real number x is characterized by $\overline{x} = x$. But substitution $\alpha \mapsto 1/\alpha$, etc changes the degree of an homogeneous fraction into its opposite. □

Proposition 2.8. *When the Lubin(n) representation* Z *of a point* $P = u : v : w$ *is known, its barycentrics can be expressed as :*

$$\begin{pmatrix} u \\ v \\ w \end{pmatrix} \simeq \begin{pmatrix} z_A\,(z_B - z_C)\left(z_B\,z_C\,\overline{Z} + Z - z_B - z_C\right) \\ z_B\,(z_C - z_A)\left(z_C\,z_A\,\overline{Z} + Z - z_C - z_A\right) \\ z_C\,(z_A - z_B)\left(z_A\,z_B\,\overline{Z} + Z - z_A - z_B\right) \end{pmatrix} \tag{2}$$

Proof. Start from $(u + v + w)\,Z = u\alpha^n + v\beta^n + w\gamma^n$, take the conjugate and eliminate. □

Remark 2.9. Formula (2) don't provide formal barycentric coordinates (i.e. a function of a, b, c). Therefore, identification of a point in the Kimberling inventory has to be done through the "search key".

Proposition 2.10. *If the Lubin(n) affix of* P *is known, the affix of the inverse in circumcircle of* P *is* $1/\overline{Z}$*.*

Proof. Well known property. □

2.3 Standard choice, standard tests

Definition 2.11. ***Standard choice***. In order to obtain a clear definition of every involved things, "the standard choice" will be as follows. Angles $\widehat{A}$, etc are measured in degrees with a protractor (not up to any quotient group). Angles $\widehat{B}$ and $\widehat{C}$ are assumed to be acute, and $\alpha_0 = 1$ is chosen. Orientation is chosen so that points B and C are respectively lying in the upper and the lower half-circle. The number β_0 is the "first n-th root of $B = \beta^n$" i.e. :

$$0 < \left(\overrightarrow{0A}, \overrightarrow{0\beta_0}\right) = 2\widehat{C}/n < 180°/n$$

[1] Answer to Jeffrey Brooks, #18504, 2009/12/25

while γ_0 is defined by :

$$-180°/n < \left(\overrightarrow{0A},\, \overrightarrow{0\gamma_0}\right) = -2\widehat{B}/n < 0$$

Definition 2.12. Standard test. It is convenient to chose angles that are easily recognizable, and also their usual fractions. When using $\widehat{A}, \widehat{B}, \widehat{C} = 42°, 54°, 84°$, we have $a, b, c \approx 1.338, 1.618, 1.989$ together with :

$$z(A) \approx\ 1.0\,;\ z(B) \approx\ -0.9781 + 0.2079\,i\,;\ z(C) \approx\ -0.3090 - 0.9511\,i$$

This leads to the test values :

$$\begin{aligned} z(X(4)) &\approx -0.2872 - 0.7434\,i \\ z(X(1)) &\approx -0.1737 - 0.3145\,i \\ z(X(356)) &\approx -0.1767 - 0.3213\,i \end{aligned}$$

where X(4) i.e. the orthocenter, X(1) i.e. the incenter and X(356) i.e. the Morley center have been chosen in increasing order of difficulty. Any result that don't provides these values is certainly wrong.

2.4 Lubin of first degree

Proposition 2.13. *Using the Lubin(1) representation, i.e.* $z(A) = \alpha^1$, etc *we have* $a = |\gamma - \beta|$, etc. *A rational formulation of this fact is :*

$$Lubin(1)\ :\ \left\{ a^2 = -\frac{(\gamma - \beta)^2}{\beta\,\gamma},\ \text{etc} \right. \tag{3}$$

Proof. Value of a^2 comes from $a^2 = (\gamma - \beta)\,\overline{\gamma - \beta}$. □

Corollary 2.14. *Any point whose barycentrics belong to* $\mathbb{Q}\left(a^2, b^2, c^2, R\right)$ *is representable by a Lubin(1) fraction. For example :*

$$\begin{aligned} z(X(4)) &= \alpha + \beta + \gamma \\ z(X(6)) &= 2\,\frac{\alpha\beta\gamma\,(\alpha + \beta + \gamma) - \sum \alpha^2\beta^2}{6\,\alpha\,\beta\,\gamma - \sum \alpha^2\beta} = \frac{2\,s_2^2 - 6\,s_1 s_3}{s_2 s_1 - 9\,s_3} \end{aligned}$$

Proof. Value of $z(H)$ is the Euler relation $\overrightarrow{OH} = 3\overrightarrow{OG}$. Affix of the Lemoine point is not difficult to compute (in the second part, s_2 is the second symmetric function $\alpha\beta + \beta\gamma + \gamma\alpha$, etc). □

Proposition 2.15. *Let U be a point not on the circumcircle and suppose that U admits Z as Lubin(n) representation. Then isog(P) admits a Lubin(n) representation, given by :*

$$z(isogon(P)) = \frac{z_A z_B z_C\,\overline{Z}^2 - (z_A z_B + z_B z_C + z_C z_A)\,\overline{Z} - Z + z_A + z_B + z_C}{1 - \overline{Z}\,Z} \tag{4}$$

Proof. Suppose that U_1, U_2 is a pair of conjugates. Write $u_1 u_2 = k\, a^2$, $v_1 v_2 = k\, b^2$, etc, use (2) and (3), eliminate k and obtain a pair of equations. By linear combinations, this system con be replaced by a pair of conjugate equations stating that[2] :

$$z_A z_B z_C\, \overline{Z_1 Z_2} + Z_1 + Z_2 - (z_A + z_B + z_C) = 0 \tag{5}$$

where $z_A = z\,(A) = \alpha^n$, etc.

To obtain (4), solve Z_2 as a function of $\overline{Z_2}$, $\overline{Z_1}$, Z_1 from (5), take the complex-conjugate (using $\alpha \mapsto 1/\alpha$, etc) and substitute back into (5). The denominator enforces the fact that isogonal conjugate of a point on Γ is at infinity. □

Remark 2.16. Formula (4) can be used to obtain $z\,(X\,(6))$ from $z\,(X\,(2))$ since K and G form an isogonal pair.

2.5 Lubin of second degree

Definition 2.17. The Lubin(2) representation, i.e. $z\,(A) = \alpha^2$, etc uses points $(1), \cdots, (6)$ of Figure 3 whose standard affixes are (respectively) $\alpha_0^2 = z_A$, $\alpha_0\beta_0$, $\beta_0^2 = z_B$, $\phi\beta_0\gamma_0$, γ_0^2, $\gamma_0\alpha_0$. When using general values of α, β, γ, i.e. $\pm\alpha_0$, $\pm\beta_0$, γ_0, we will have :

$$(2) = \epsilon\,(2)\,\alpha\beta\,;\ (4) = \epsilon\,(4)\,\beta\gamma\,;\ (6) = \epsilon\,(6)\,\gamma\alpha$$

where the ϵ are square roots of unity.

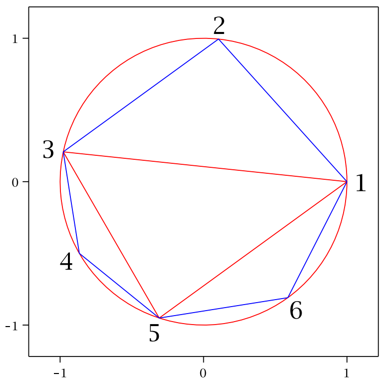

Figure 3: The six reference points of Lubin(2)

Proposition 2.18. *For any choice of α, β, γ, we have $\pi_\epsilon \doteq \epsilon\,(2)\,\epsilon\,(4)\,\epsilon\,(6) = -1$. When using $\alpha = -\alpha_0$, $\beta = \beta_0$, $\gamma = \gamma_0$ then a symmetric representation is obtained, namely*

$$(2) = -\alpha\beta\,;\ (4) = -\beta\gamma\,;\ (6) = -\gamma\alpha \tag{6}$$

Proof. Quantity π_ϵ is invariant by transformation $\alpha \mapsto -\alpha$. We only need to consider the standard choices. When B moves on the upper half-circle, or C moves on the lower half circle, quantity π_ϵ varies continuously (we are not

[2] as recalled by Francois Rideau - #18510 - 2009/12/26

jumping to another branch of the complex square root function. Being discrete, this quantity is constant and can be computed only for the equilateral triangle. If number ω is associated with a $+60°$ rotation, we have $\pi_\epsilon = \omega\,\omega^3\omega^5 = -1$ and conclusion follows. □

Proposition 2.19. *From now on, symmetric representation* (6) *is assumed. Then :*

$$Lubin(2)\ :\ \begin{cases} \cos C = -\dfrac{\alpha^2+\beta^2}{2\,\alpha\,\beta}\ ;\ \sin C = i\,\dfrac{(\beta+\alpha)\,(\beta-\alpha)}{2\alpha\,\beta} \\ \tan\dfrac{A}{2} = i\,\dfrac{\beta-\alpha}{\beta+\alpha} \\ c = i\,R\,\dfrac{(\alpha+\beta)\,(\beta-\alpha)}{\alpha\,\beta} \end{cases}\ , \text{etc} \qquad (7)$$

Proof. Obtain $\widehat{C} = \arg(-\beta/\alpha)$ by the inscribed angle property, then take real and imaginary part. The last formula is $c = 2R\sin C$.[3] To prove the $\tan(A/2)$ formula, use the uniqueness of $t \in \mathbb{C}$ such that $\sin A = 2t/\left(t^2+1\right)$, $\cos A = \left(t^2-1\right)/\left(t^2+1\right)$. Due to the symmetry, these formulae hold for the other two angles. □

Corollary 2.20. *Any point whose barycentrics belong to* $\mathbb{Q}(a,\, b,\, c,\, R)$ *is representable by a Lubin(2) fraction. For example*[4] *:*

$$\begin{array}{lcccc} z\,(X\,(4)) & = & \alpha^2+\beta^2+\gamma^2 & = & s_1^2-2s_2 \\ z\,(X\,(1)) & = & -\alpha\beta-\beta\gamma-\gamma\alpha & = & -s_2 \\ z\,(X\,(8)) & = & -\left(\alpha+\beta+\gamma\right)^2 & = & -s_1^2 \\ z\,(X\,(6)) & = & 2\,\dfrac{\alpha^2\beta^2\gamma^2\left(\alpha^2+\beta^2+\gamma^2\right)-\sum\alpha^4\beta^4}{6\,\alpha^2\,\beta^2\,\gamma^2-\sum\alpha^4\beta^2} & = & ohlala \end{array}$$

Proposition 2.21. *When using Lubin(2), transformation* $R \mapsto -R$, $\alpha \mapsto -\alpha$, $\beta \mapsto \beta$, $\gamma \mapsto \gamma$ *implements the A-continuous transform of Lemoine, leading for example to the A-excircle center when starting from X(1).*

Proof. This transformation results into : $a \mapsto -a$, $b \mapsto b$, $c \mapsto c$. □

2.6 Equivalence of Lubin(2) with Poncelet

In what follows, the Poncelet affix of P will be noted ζ_P, and the Lubin(2) affix will remain noted z_P.

Definition 2.22. The Poncelet's representation starts from the affixes κ, ν, τ of the contacts of the incircle, assuming that $r = 1$. Thereafter, these points are used as a generating family, together with $\overline{\kappa} = \kappa$, etc.

Proposition 2.23. *In this representation, affixes of the vertices are :*

$$\zeta_A,\,\zeta_B,\,\zeta_C = \frac{2\nu\tau}{\nu+\tau},\ \frac{2\tau\kappa}{\tau+\kappa},\ \frac{2\kappa\nu}{\kappa+\nu}$$

[3] As stated by Nikolaos Dergiades - #15817 - 2009/12/26, we can use $c = (\alpha+\beta)\,(\beta-\alpha)\,/\,(\alpha\beta)$ if we are only concerned by substitutions in an homogeneous formula.

[4] $ohlala = \left(2\,s_2^4-8\,s_3s_1s_2^2+2\,s_3^2s_1^2+12\,s_3^2s_2\right)/\left(s_1^2s_2^2-2\,s_3s_1^3-2\,s_2^3+4\,s_3s_1s_2-9\,s_3^2\right)$

while the sides of the triangle are :

$$a=\frac{-2\,i\kappa\,\left(\nu-\tau\right)}{\left(\nu+\kappa\right)\left(\kappa+\tau\right)},\;b=\frac{-2\,i\nu\,\left(\tau-\kappa\right)}{\left(\tau+\nu\right)\left(\nu+\kappa\right)},\;c=\frac{-2\,i\tau\,\left(\kappa-\nu\right)}{\left(\kappa+\tau\right)\left(\tau+\nu\right)}$$

$$R=\frac{-2\kappa\nu\tau}{\left(\nu+\tau\right)\left(\tau+\kappa\right)\left(\kappa+\nu\right)}$$

Proof. The best proof comes from the following proposition. □

Proposition 2.24. *In fact, Poncelet representation is equivalent with Lubin(2), and (for a suitable adjustment of the two basis) :*

$$\zeta_P=\frac{-2\left(z_P+\alpha\,\beta+\alpha\,\gamma+\beta\,\gamma\right)\alpha\,\beta\,\gamma}{\left(\beta+\gamma\right)\left(\alpha+\gamma\right)\left(\alpha+\beta\right)}$$

Proof. The formula given is $\zeta_P=(z_P-z_I)\times R/r$, which is obvious. Applied to A', the A-contact point, this gives $\zeta_{A'}=-\beta\gamma$. The new barycentric basis is nothing but points (2), (4), (6) of the Lubin(2) basis. Geometrically, this comes from the fact that circumcevian triangle of I =X(1) and cevian triangle of X(7) are homothetic, with ratio R/r, wrt point X(56). As it should be, this point is the (ex) similicenter of the circum- and the in- circles. □

Remark 2.25. Knowing κ, ν, τ gives $-\beta\gamma$, etc. Then $(\alpha\beta\gamma)^2$ is known, and $\widetilde{\pi}\doteq\alpha\beta\gamma$ is known apart from a choice of sign. then $\alpha=-\widetilde{\pi}/\kappa$, etc, and the Lubin basis is obtained, apart from a global sign.

2.7 Applications of Lubin(2)

Proposition 2.26. *Euler triangle $\mathcal{T}_E$ has for vertices the midpoints of segments* $[A,H]$, etc. *Its Lubin(2) representation is :*

$$\frac{1}{2}\left(2\alpha^2+\beta^2+\gamma^2\right),\text{ etc}$$

Triangle $\mathcal{T}_E$ is the image wrt homothety $k\,(O,\,2)$ of triangle whose Lubin(2) representation $\left(2\alpha^2+\beta^2+\gamma^2\right)/4$, etc. *The vertices of the later triangle are the midpoints of the medians. The Fuhrmann=X(355) of ABC is the Nagel=X(8) of $\mathcal{T}_E$.*

Proof. Direct computations. □

Proposition 2.27. *Let J_A be the center of the HBC-circumcircle, and circularly. Triangle $J_AJ_BJ_c$ is called the Johnson triangle $\mathcal{T}_J$, and is the image of triangle ABC under a symmetry whose center is X(5), the nine point circle center. The Fuhrmann=X(355) of ABC is the incenter=X(1) of $\mathcal{T}_J$.*

Proof. Direct computations. □

Proposition 2.28. *Lubin(2) of the incircle tangency points are* $I-r\,\beta\gamma$, etc *while radius r of the incircle is :*

$$r=-\frac{\left(\alpha+\beta\right)\left(\beta+\gamma\right)\left(\gamma+\alpha\right)}{2\,\alpha\,\beta\,\gamma}\tag{8}$$

Proof. The A-pedal of X(1) is the A-cevian of X(8) –by definition of the Nagel point. Barycentrics of this A_N are $0 : a-b+c : a+b-c$. Compute $Lubin_2\left(A_N\right)$ using preceding results, and obtain :

$$Lubin_2\left(A_N - I\right) = \frac{(\alpha+\beta)(\beta+\gamma)(\gamma+\alpha)}{2\,\alpha}$$

and conclusion follows since (8) is real. The choice of sign comes from the evidence : $R^2 - OI^2 > 0$, whose Lubin(2) representation is :

$$1 - (\alpha\beta + \beta\gamma + \gamma\alpha)\left(\frac{1}{\alpha\beta} + \frac{1}{\beta\gamma} + \frac{1}{\gamma\alpha}\right) > 0$$

But this quantity is twice (8). As a by product, we obtain another proof of the Euler's result :

$$R^2 - |OI|^2 = 2Rr$$

□

2.8 Fuhrkind theorem

Proposition 2.29 (Fuhrmann theorem)**.** *Define* $A' = (3)+(5)-(4)$, etc. *Point* A' *is the symmetric of the midpoint of internal arc* $\overparen{BC}$ *wrt line* BC. *Then equation of circle through* $A'B'C'$ *(the Fuhrmann circle) is :*$\overline{Z}\,Z - \overline{Z}\,\delta - Z\,\overline{\delta} + \tau = 0$ *where :*

$$\begin{aligned}
\delta &= \alpha^2 + \beta^2 + \gamma^2 + \alpha\,\beta + \beta\,\gamma + \gamma\,\alpha \\
\tau &= (A'\,B'\,C') \div \left(\alpha^2\beta^2\gamma^2\right) \\
\rho &= |\alpha + \gamma + \beta|
\end{aligned}$$

Theorem 2.30. *Given a reference triangle* ABC*, let* P *be a point not on the circumcircle* Γ*. Its isogonal conjugate* P^* *is therefore an ordinary point (not at infinity). Call* A' *the second intersection of line* AP *with circle* Γ*. Construct* A"*, the symmetrical of* A' *wrt line* BC *and act circularly. The Fuhrkind circle of point* P *is defined as the circumcircle of triangle* A"B"C"*. Call the center* $\mathrm{F_K}(P)$ *and the radius* $\rho_F(P)$*. Then, as in Figure 4, segments* $[P^*, \mathrm{F_K}(P)]$ *and* $[O, H]$*=[X(3),X(4)] have the same midpoint X(5) -the nine point center of* ABC*. Point* H *ever belongs to the circle, and its antipode* $\mathrm{N_K}(P)$ *–N like Nagel– is the anticomplement of* P^**. Finally, radius* $\rho_F(P)$ *is given by :*

$$\rho_F(P) = |O\,P^*| = |H\;\mathrm{F_K}(P)| = \left|\frac{Z^2 - \alpha^2\beta^2\gamma^2\overline{Z} - \left(\alpha^2+\beta^2+\gamma^2\right)Z + \left(\alpha^2\beta^2 + \beta^2\gamma^2 + \alpha^2\gamma^2\right)}{1 - Z\,\overline{Z}}\right|$$

Proof. When $|z_3| = 1$ and $Z \neq z_3$, the other intersection of line (Z, z_3) with the unit circle is : $(Z - z_3)/\left(1 - z_3\overline{Z}\right)$. The symmetrical of Z wrt the line (z_1, z_2) is : $\left((z_1 - z_2)\,\overline{Z} + (\overline{z}_1 z_2 - z_1\overline{z}_2)\right) \div \left(\overline{z_1 - z_2}\right)$. When the z_i are on Γ, this simplifies into : $z_1 + z_2 - z_1 z_2\overline{Z}$. Therefore :

$$z\left(A'\right) = \beta^2 + \gamma^2 - \frac{\beta^2\gamma^2\left(1 - \alpha^2\,\overline{Z}\right)}{z - \alpha^2}, \text{ etc}$$

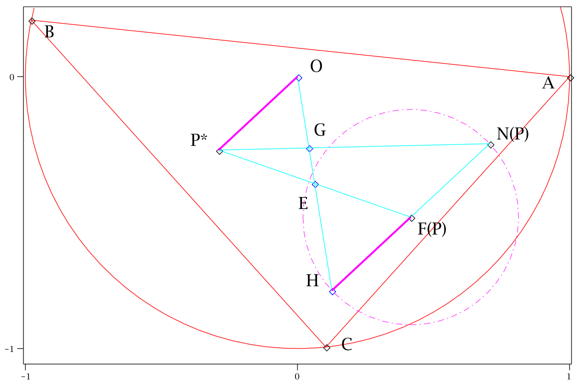

Figure 4: Fuhrkind circle

Equation (1), when written as $\overline{Z}\,Z - \overline{Z}\,\delta - Z\,\overline{\delta} + \tau = 0$ gives directly δ and τ. Coefficient of Z is therefore $\overline{\delta}$, and $\tau \in \mathbb{R}$ (can be verified by taking the complex-conjugate). From the value of δ and (4), we have :

$$\delta + z\,(P^*) = \alpha^2 + \beta^2 + \gamma^2 = 2z\,(X(5))$$

while radius is obtained by $\rho^2 = \delta\,\overline{\delta} - \tau$, that factors in two quantities that are conjugate of each other. A direct substitution proves that H belongs to the circle. Properties relative to $N\,(P)$ are obvious from the Figure. □

Remark 2.31. As it should be, the choice P=X(1) leads to the ordinary Fuhrmann circle, where $z\,(P) = z\,(P^*) = -\alpha\beta - \beta\gamma - \gamma\alpha$, center is X(355) with $z\,(X(355)) = z\,(H) - z\,(I)$, antipode is the Nagel center X(8) with and radius is we go back to δ is the Lubin(2) of X(355), $\delta = (H + N)\,/2$ and $|H - \delta| = \rho$.

Remark 2.32. When taking P on the circumcircle, the Fuhrkind "circle" becomes the Steiner line of the point. Then, formula indicates that "center" of this line is $isogon\,(P)$. It is well known that this point is the direction orthogonal to the Steiner line... and therefore the formula makes sense even in this degenerate case.

3 The Lubin(3) representation

3.1 Proof of the Morley theorem

Definition 3.1. Let j be the complex such that $z \mapsto jz$ is a $+120°$ rotation. The Lubin(3) representation, i.e. $z\,(A) = \alpha^3$, etc uses points $(1)\,, \cdots, (9)$ of Figure 5 whose standard affixes are (respectively)

$$\alpha_0^3 = z_A,\ \alpha_0^2\beta_0,\ \alpha_0\beta_0^2,\ \beta_0^3 = z_B,\ \phi\beta_0^2\gamma_0,\ \phi^2\beta_0\gamma_0^2,\ \gamma_0^3 = z_C,\ \gamma_0^2\alpha_0,\ \gamma_0\alpha_0^2$$

When using general values of α, β, γ, i.e. $j^u\alpha_0,\ j^v\beta_0,\ j^w\gamma_0$, we will have :

$$(2) = \epsilon\,(2)\,\alpha^2\beta\ ;\ (3) = \epsilon\,(3)\,\alpha\beta^2\ ;\ (5) = \epsilon\,(5)\,\beta^2\gamma$$
$$(6) = \epsilon\,(6)\,\beta\gamma^2\ ;\ (8) = \epsilon\,(8)\,\gamma^2\alpha\ ;\ (9) = \epsilon\,(9)\,\gamma\alpha^2$$

where the ϵ are cubic roots of unity.

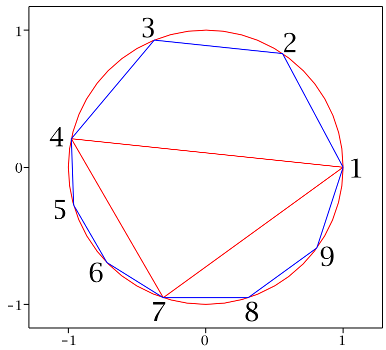

Figure 5: The six reference points of Lubin(2)

Proposition 3.2. *For any choice of α, β, γ, we have :*

$$\pi_\epsilon \doteq \epsilon(2)\,\epsilon(5)\,\epsilon(8) = \frac{1}{\epsilon(3)\,\epsilon(6)\,\epsilon(9)} = j$$

Therefore $\epsilon(2) = \epsilon(5) = \epsilon(8)$ cannot hold, and a symmetrical representation cannot be obtained.

Proof. Relations $\epsilon(2)\,\epsilon(3)$, etc $= 1$ are obvious from $(2)\times(3) = (1)\times(4) = z_A z_B$, etc. Using the same arguments as in proof relative to Lubin(2), we only need to consider the standard choices of an equilateral triangle. If number ω is associated with the $+40°$ rotation, we have $\pi_\epsilon = \omega^1\,\omega^4\omega^7 = \omega^3 = j$.

Since relation $\epsilon(2) = \epsilon(5) = \epsilon(8)$ would imply $\pi_\epsilon = 1$, a symmetrical representation cannot be obtained. In the founding paper Lubin (1955), the standard representation (with $\phi = j$ as it should be) was introduced as an obvious fact. □

Theorem 3.3 (Proof of the Morley Theorem)**.** *Let $A' = (48)\cap(73)$, $B' = (72)\cap(16)$, $C' = (15)\cap(49)$ be the intersection of internal trisectors adjacent to the same sideline. Then (using the standard representation) :*

$$\begin{aligned}
z(A') &= \frac{\beta^3-\gamma^3}{\beta-\gamma}\,\alpha &-& \frac{\beta^2-\gamma^2}{\beta-\gamma}\,\beta\gamma \\
z(B') &= \frac{(\phi\gamma)^3-\alpha^3}{(\phi\gamma)-\alpha}\,\beta &-& \frac{(\phi\gamma)^2-\alpha^2}{(\phi\gamma)-\alpha}\,(\phi\gamma)\,\alpha \\
z(C') &= \frac{\alpha^3-(\phi^2\beta)^3}{\alpha-(\phi^2\beta)}\,\gamma &-& \frac{\alpha^2-(\phi^2\beta)^2}{\alpha-(\phi^2\beta)}\,\alpha\,(\phi^2\beta)
\end{aligned}$$

It can be seen that the following rational fractions :

$$\begin{aligned}
\delta_1 &\doteq |z(A')-z(B')|^2 - |z(A')-z(C')|^2 \\
\delta_2 &\doteq |z(A')-z(B')|^2 - |z(C')-z(B')|^2
\end{aligned}$$

are both divisible by $1+\phi+\phi^2$, but not by $\phi-1$. As a by-product, this is another proof that a symmetrical representation (that would use $\phi = 1$) is impossible.

The most important fact is as follows : all the $2\times3\times3 = 18$ possibilities are leading to equilateral triangles. Multiplying any of the α, β, γ by a power of j

rotates one of the internal trisectors by a multiple of $120°$*, leading to* 9 *triangles (labeled* $0x$ *in Figure 1). Changing* $\phi = j$ *into* $\phi = j^2$ *replaces some internal trisectors by some external ones, leading to* 9 *more triangles (labeled* $1x$ *in the same Figure).*

3.2 The Morley centers

Proposition 3.4. *Point X(356) is defined as the center of equilateral triangle* $A'B'C'$*. From Kimberling (1998-2010), its barycentrics are :*

$$\sin A \left(\cos \frac{A}{3} + 2 \cos \frac{B}{3} \cos \frac{C}{3} \right), \text{ etc}$$

Its standard Lubin(3) affix is :

$$z_{356} = \left(\alpha^2 \beta + \gamma^2 \alpha - \gamma^2 \beta\right) \left(\frac{1}{2} + \frac{i}{6} \sqrt{3} \right) + \left(\alpha^2 \gamma + \beta^2 \alpha - \beta^2 \gamma\right) \left(\frac{1}{2} - \frac{i}{6} \sqrt{3} \right)$$

Proposition 3.5. *Point X(357) is the perspector of triangle* $A'B'C'$ *with* ABC*. From Kimberling (1998-2010), its barycentrics are :*

$$\sin A \div \cos \frac{A}{3}, \text{ etc}$$

Its standard Lubin(3) affix is given in Figure 6.

Proof. Perspectivity is easy to prove using either barycentrics or complex affixes. □

Proposition 3.6. *Point X(358) is the isogonal conjugate of X(357). Barycentrics* $\sin A \cos (A/3)$ *are obvious. Points X(356), X(357), X(358) are aligned.*

Proof. Direct computations. □

Remark 3.7. Two other Morley centers are names in ETC : X(3276) and X(3277), leading to the perspectors X(1134) and X(1136).

3.3 About "symmetrical formulae" for the Morley centers

In the Lubin(3) standard representation, quotients (γ/β, α/γ, β/α) are related to angles ($2A/3 - 120°$, $2B/3$, $2C/3$). If we omit this rotation of $120°$, symmetrical (and erroneous) formulae are obtained. The simplest way to describe what happens, is to use a Lubin(6) representation, with a choice of signs such that $z\,(X(1)) = -\alpha^3\beta^3 - \beta^3\gamma^3 - \gamma^3\alpha^3$.

This leads to substitutions :

$$\begin{aligned} \cos\,(C/3) &= -\left(\alpha^2 + \beta^2\right) / \left(2\,\alpha\beta\right), \text{ etc} \\ \sin\,(C/3) &= i\,\left(\beta^2 - \alpha^2\right) / \left(2\,\alpha\beta\right), \text{ etc} \end{aligned}$$

(B and C are true, A is false). Applied to the barycentrics of X(356), X(357), X(358), and translated back to Lubin(3), one obtains respectively[5] :

[5] as given in #18523 - 2009/12/27

$$z_{357} = \frac{\Phi_1 \phi - \Phi_2}{\Phi_3 \phi + \Phi_4}$$

where

$$\Phi_1 = \begin{pmatrix} \alpha^3\beta^2\gamma^4 + \gamma^4\alpha^4\beta + \alpha^6\beta\,\gamma^2 + 2\,\alpha^5\gamma^2\beta^2 + \alpha^6\beta^2\gamma \\ +\alpha^5\gamma\,\beta^3 + \beta^5\alpha^3\gamma + 2\,\alpha^4\gamma^3\beta^2 + \alpha^3\beta^4\gamma^2 + \gamma^3\alpha^3\beta^3 \\ +\alpha^4\beta^4\gamma + 2\,\alpha^5\gamma^3\beta - \gamma^3\beta^5\alpha + 2\,\gamma^2\alpha^4\beta^3 + \alpha^3\beta^6 \\ +\alpha^4\beta^5 - \beta^6\gamma^3 + \alpha^6\gamma^3 - \beta^5\gamma^4 + \alpha^5\gamma^4 \end{pmatrix}$$

$$\Phi_2 = \begin{pmatrix} \alpha^6\beta\,\gamma^2 + \gamma^5\alpha\,\beta^3 - 2\,\gamma^2\alpha^4\beta^3 - \alpha^4\beta^4\gamma - \alpha^3\gamma^5\beta \\ -2\,\alpha^4\gamma^3\beta^2 - \alpha^5\gamma^3\beta - \gamma^4\alpha^4\beta - \alpha^6\beta^2\gamma - 2\,\alpha^5\gamma\,\beta^3 \\ -\gamma^3\alpha^3\beta^3 - 2\,\alpha^5\gamma^2\beta^2 - \alpha^3\beta^4\gamma^2 - \alpha^3\beta^2\gamma^4 + \gamma^6\beta^3 \\ -\alpha^5\beta^4 - \alpha^6\beta^3 - \alpha^3\gamma^6 + \gamma^5\beta^4 - \gamma^5\alpha^4 \end{pmatrix}$$

$$\Phi_3 = \begin{pmatrix} \alpha^2\beta^3\gamma + \alpha\,\beta^2\gamma^3 + \gamma^4\alpha\,\beta + \beta^4\alpha\,\gamma + 2\,\gamma^2\alpha^2\beta^2 \\ +\alpha^3\beta\,\gamma^2 + \beta^3\alpha^3 + 2\,\gamma^3\alpha^2\beta + 2\,\alpha^3\beta^2\gamma \\ +2\,\beta^3\gamma^2\alpha + \gamma^3\alpha^3 + \beta^2\alpha^4 + \gamma^4\alpha^2 + \beta^4\gamma^2 \end{pmatrix}$$

$$\Phi_4 = \begin{pmatrix} -2\,\alpha^3\beta\,\gamma^2 - 2\,\gamma^2\alpha^2\beta^2 - 2\,\alpha\,\beta^2\gamma^3 - \alpha^3\beta^2\gamma \\ -2\,\alpha^2\beta^3\gamma - \alpha^2\beta^4 - \beta^3\alpha^3 - \beta^4\alpha\,\gamma - \gamma^4\alpha\,\beta \\ -\gamma^3\alpha^2\beta - \beta^3\gamma^2\alpha - \gamma^3\alpha^3 - \gamma^4\beta^2 - \alpha^4\gamma^2 \end{pmatrix}$$

Figure 6: X(357)

$$w_{356} = \frac{2\,s_1 s_3 s_2 + s_1^3 s_3 - s_2^3 - s_1^2 s_2^2}{2\,s_2 s_1}$$

$$w_{357} = \frac{s_1^2 s_3 + s_2 s_3 - s_1 s_2^2}{s_2},\ w_{358} = \frac{s_1 s_3 - s_2^2}{s_1}$$

Wrong affixes are nevertheless aligned, while w_{357} and w_{358} continue to form an isogonal pair.

4 Exploration of ETC : the Kiepert-Brocard list

4.1 The Kiepert RH

Proposition 4.1. *Chose angle ϕ and construct isosceles triangles $BA'C$, $CB'A$, $AC'B$ with basis angle $\angle(BC, BA') = \phi$. Then triangles ABC and $A'B'C'$ are perspective wrt a point P_ϕ :*

$$P_\phi \quad \simeq \quad \frac{1}{4S - (b^2 + c^2 - a^2)\tan\phi},\ \text{etc} \simeq \frac{a}{\sin(A - \phi)},\ \text{etc}$$

and Kiepert RH is the locus of such points. Perspector of this conic is X(523), DeLongchamp point at infinity, and center is X(115). The isogonal of this circumconic is a line, the Brocard axis.

Proof. Using normalized barycentrics, write $A' = B + C + \mu (H - A)$ together with $\tan (\angle (BC,\, BA')) = \tan \phi$ and obtain :

$$A' = a^2 \tan \phi : 2\,S - S_c \tan \phi : 2\,S - S_b \tan \phi$$

Then perspectivity is obvious, and the formula follows. By elimination,

$$\left(b^2 - c^2\right) yz + \left(c^2 - a^2\right) zx + \left(a^2 - b^2\right) xy = 0$$

and this is the Kiepert RH, whose perspector is X(523)= $b^2 - c^2$. The isogonal of this circumconic is a line whose tripolar is X(110)=$isogon\,(X_{523})$: this characterizes the Brocard axis. □

Proposition 4.2. *When using Lubin(1), affix of point P_ϕ on Kiepert RH is :*

$$z_\phi = \frac{4\,\tan \phi \left(s_2^2 - 3\,s_1 s_3\right) + i\,s_1\,\left(1 + \tan^2 \phi\right)\,vdm}{2\,\tan \phi \left(s_1 s_2 - 9\,s_3\right) + i\left(3 + \tan^2 \phi\right)\,vdm}$$

Remark 4.3. Angles $\phi \in [-\pi/2, 0]$ are leading to triangles lying outside of ABC. A transposition acting on B and C don't changes the set of the three isosceles triangles. Nevertheless, orientations are reversed and also the sign of vdm. Due to the special form of the affix, point z_ϕ remains unchanged and P_ϕ is a true center.

4.2 Vierer Gruppe

Example 4.4. Points on Kiepert RH with fixed values of the angle ϕ are given in Table 1, together with points relative to the Brocard angle. More details can be found in Gibert (2004-2010, (Table 38))

Proposition 4.5. *Substitutions* $(\tan \phi \mapsto -\tan \phi)$ *and* $(\tan \phi \mapsto 1/\tan \phi)$ *are generating a "Vierer Gruppe" that corresponds to angles* $\pm\phi \pm \pi/2$. *Abbreviating* $P\,(\phi)\,,\,P\,(-\phi)\,,\,P\,(\pi/2+\phi)\,,\,P\,(\pi/2-\phi)$ into $P_1,\,P_2,\,P_3,\,P_4$, then line P_1P_2 *goes through X(6), the Lemoine center,* P_1P_4 *through X(3), the circumcenter, and* P_1P_3 *through X(5), the Euler (nine points) center.*

Proof. Obvious from the barycentrics. □

Example 4.6. When $\phi = \pi/3$, we have $P_1,\,P_2,\,P_3,\,P_4$=X(14),X(13),X(17),X(18) where values of $\tan\phi$ are, respectively, $\sqrt{3}/3,\,-\sqrt{3}/3,\,-\sqrt{3},\,\sqrt{3}$. When $\phi = \pi/12$, we have $P_1,\,P_2,\,P_3,\,P_4$=X(3392),X(3366),X(3391),X(3367) where values of $\tan\phi$ are, respectively, $2-\sqrt{3},\,-2+\sqrt{3},\,-2-\sqrt{3},\,2+\sqrt{3}$. For example :

$$z_{14} = \frac{2\,\left(s_2^2 - 3\,s_3 s_1\right)\sqrt{3} + 2\,i s_1 vdm}{\left(s_2 s_1 - 9\,s_3\right)\sqrt{3} + 3\,i\,vdm}$$

$$z_{3366} = \frac{4\left(s_2^2 - 3\,s_3 s_1\right) + 2\left(s_2^2 - 3\,s_3 s_1\right)\sqrt{3} - 2 s_1 i\left(2 + \sqrt{3}\right) vdm}{-18\,s_3 + \left(s_2 s_1 - 9\,s_3\right)\sqrt{3} - 2\,s_2 s_1 i\left(11 + 6\sqrt{3}\right) vdm}$$

In the first case, conjugacy wrt $\sqrt{3}$ and vdm are the same, and the 4-group splits in $\sqrt{3}$ pairs. In the second case, the two conjugacies are different (cf Subsection 5.3).

$isog\,(j)$	j	$\phi\,(j)$	$\phi\,(k)$	k	$isog\,(k)$
6	2	0	0	2	6
3390	3392	15	-15	3366	3364
3395	3397	18	-18	3370	3368
???	???	$\arctan(1/3)$	$\arctan(1/3)$	???	???
3386	3388	22.5	-22.5	3373	3371
3312	3317	$\arctan(1/2)$	$-\arctan(1/2)$	3316	3311
62	18	30	-30	17	61
3380	3382	36	-36	???	???
2674	2672	$\arctan\left(3-\sqrt{5}\right)$	$-\arctan\left(3-\sqrt{5}\right)$	2671	2673
372	486	45	-45	485	371
???	???	$\arctan\frac{3+\sqrt{5}}{4}$	$-\arctan\frac{3+\sqrt{5}}{4}$	???	???
???	???	54	-54	3381	3379
16	14	60	-60	13	15
1152	1132	$\arctan(2)$	$-\arctan(2)$	1131	1151
3372	3374	67.5	-67.5	3387	3385
???	1328	$\arctan(3)$	$-\arctan(3)$	1327	???
3369	1140	72	-72	1139	3396
3365	3367	75	-75	3391	3389
3	4	90	-90	4	3
32	76	ω	$-\omega$	83	39
1691	1916	$2\,\omega$	$-2\,\omega$	3407	3094
3095	3406	$\pi/2-2\,\omega$	$\pi/2+2\,\omega$	3399	3398
511	98	$\pi/2-\omega$	$\pi/2+\omega$	262	182

Table 1: Vierer Gruppen of the Kiepert RH

4.3 Cross values

Proposition 4.7. *When intersecting line $P(\phi)$, $P(\psi)$ with line $P(-\phi)$, $P(-\psi)$, one obtains a point on the Euler line. Its affix is :*

$$\frac{\tan\psi\,\tan\phi-1}{\tan\psi\,\tan\phi-3}\,s_1$$

Obviously this affix is invariant by transform $vdm \mapsto -vdm$.

Proof. Obvious from former expressions. This gives another proof of the special result of Proposition 4.5, where $\psi=\pi/2-\phi$ leads to X(3) while $\psi=\pi/2+\phi$ leads to X(5). □

Example 4.8. Using $\phi=\pi/3$, $\psi=\pi/4$ leads to $z=-s_1/\sqrt{3}$ i.e. to X(2043) $=(-1/\sqrt{3})X_4+(1+1/\sqrt{3})X_3$ while $\psi=-\pi/4$ leads to X(2044). Using $\phi=\pi/6$, $\psi=\pi/4$ leads to $z=\left((4-\sqrt{3})/13\right)s_1$ i.e. to X(2045) $=\left((4-\sqrt{3})/13\right)X_4+\left((9+\sqrt{3})/13\right)X_3$ while $\psi=-\pi/4$ leads to X(2046).

Proposition 4.9. *The crossmul of $P(\phi)$ and $P(\phi+\pi/2)$ is the isogonal of $P(-2\phi)$.*

Proof. Direct from barycentrics... when relation is known. Guessed from values listed in ETC. □

5 Exploration of the ETC : the sqrt(3) list

In the Kimberling's Encyclopedia, there are 160 named points whose barycentrics contains $\sqrt{3}$. For example, we have :

$$\begin{aligned} X(559) &= a\left(R(a+b-c)(a+c-b)+abc\sqrt{3}\right), \text{ etc} \\ &= \frac{\left(s_1\,s_2-5\,s_3+i\,\sqrt{3}\,vdm\right)s_2}{3\,s_3+s_1\,s_2-i\,\sqrt{3}\,vdm} \end{aligned}$$

The degree of this fraction is $n=2$, so that we are using a Lubin(2) representation (where $A=\alpha^2$, etc). The length of the Maple string used to write this affix is $L=57$. Writing the affix this way doesn't ensure uniqueness of the expression, but using a $f_1+vdm\,f_2$ shape, where $f_1,\ f_2$ are symmetrical leads to (very) longer expressions. Moreover, the efficiency of the simplification routine ensures an experimental uniqueness...

All the obtained formal expressions have been tested (they give the same numerical values than a direct computation using the barycentrics). A comparison of the results leads to the following classification :

points	*pairs*	*description*	
136	68	$i\sqrt{3}\,vdm$	*paired*
6	6	$i\sqrt{3}\,vdm$	*not paired*
8	4	*Gibert_special*	
6	3	*without_vdm*	
4	2	*at infinity*	
160	83	*total*	

where a "pair" is a set of two points $X(j)$ and $X(k)$ that are exchanged when substituting $\sqrt{3}\mapsto-\sqrt{3}$ in their barycentrics. The emerging classes are as follows.

5.1 The i*vdm*sqrt(3) list

The most numerous group consist of 142 points, forming $68+6=74$ pairs (six points are isolated, i.e. paired with a not named point). All the corresponding affixes contain belong to $\mathbb{Q}\left(s_1,\ s_2,\ s_3,\ i\sqrt{3}\,vdm\right)$ and are therefore dependent on the orientation of the triangle. We obtain Table 2.

Sign of $i$cannot be changed, since it depends on the once for ever chosen orientation of the complex plane. When transposing B and C, the vdm changes, but also the orientations of 60° angles related to the triangle (inducing a change of sign relative to $\sqrt{3}$). Therefore, quantity $vdm\,\sqrt{3}$ is invariant and the point defined by this affix is a center (invariant by all permutations).

5.2 Isolated points in the i*vdm*sqrt(3) list

- X(1250) isogonal conjugate of X(1081)
- X(1251) isogonal conjugate of X(1082).

j	k	n	L	j	k	n	L	j	k	n	L
13	14	1	112	1094	1095	2	517	2927	2928	1	411
15	16	1	79	1250	????	2	215	2945	2946	2	686
17	18	1	109	1251	????	2	252	2952	2953	2	684
61	62	1	80	1276	1277	2	123	2981	????	1	220
202	203	2	557	1337	1338	1	227	2992	2993	1	325
298	299	1	127	1522	1523	1	108	3104	3105	1	139
300	301	1	171	1524	1525	1	90	3106	3107	1	140
302	303	1	118	1526	1527	1	211	3129	3130	1	63
383	1080	1	109	1545	1546	1	89	3131	3132	1	62
395	396	1	113	1605	1606	1	214	3165	3166	1	269
397	398	1	106	1607	1608	1	214	3170	3171	1	249
462	463	1	87	1652	1653	2	263	3179	????	1	231
465	466	1	80	1832	1833	2	708	3180	3181	1	232
470	471	1	100	2004	2005	1	247	3200	3201	1	196
472	473	1	100	2058	2059	1	559	3205	3206	1	196
554	1081	2	143	2151	2152	2	517	3375	3384	2	567
559	1082	2	104	2153	2154	2	550	3376	3383	2	571
616	617	1	127	2306	????	2	478	3411	3412	1	110
618	619	1	126	2307	????	2	219	3438	3439	1	214
621	622	1	118	2378	2379	1	301	3440	3441	1	269
623	624	1	126	2380	2381	1	154	3442	3443	1	213
627	628	1	120	2902	2903	1	268	3457	3458	1	245
629	630	1	127	2912	2913	1	264	3479	3480	1	1038
633	634	1	120	2923	2924	1	411	3489	3490	1	271
635	636	1	127	2925	2926	1	186				

Table 2: The i*vdm*sqrt(3) list

- X(2306)=$X_{1081} *_b X_1 = \left(a^3 : b^3 : c^3\right) \div_b X_{1250}$
- X(2307)=$X_{1082} *_b X_1 = \left(a^3 : b^3 : c^3\right) \div_b X_{1251}$
- X(2981) isogonal conjugate of X(396)
- X(3179) $cevadiv\,(X_{13},\, X_1)$.

5.3 Other points

The other points in the sqrt(3) list are :

1. Points at infinity. X(530), X(531), X(532, X(533) are the points at infinity of lines through the centroid X(2) and, respectively, X(13), X(14), X(17), X(18). For this reason, they are paired –but, obviously, no affix can be computed at all.

2. A group of 4 points 3364, 3365, 3389, 3390 on the Brocard line, and a group of four 3366, 3367, 3391, 3392 on the Kiepert hyperbola, each group being the isogonal conjugate of the other (cf Example 4.6).

3. Three pairs of "cross values" located on the Euler line. Two of them are described in Example 4.8. The remaining pair is the cross of the isodynamic points X(15)-X(16) and the Vecten ($\pm\pi/4$), i.e. X(485)-X(486). These points have symmetrical affixes (containing only s_1, s_2, s_3).

6 Dealing with points at infinity

The ***line*** at infinity of the Triangle Plane cannot be discarded without destroying the efficiency of the barycentric coordinates. But the natural way to introduce infinity in the complex plane of the Lubin's representation is to introduce one infinity ***point*** from $\mathbb{P}_{\mathbb{C}}\left(\mathbb{C}^2\right)$. Therefore another way must be found to describe the direction of lines.

Definition 6.1. When $P \neq Q$, direction of the line PQ will be represented by :

$$dir\,(PQ) = \frac{z_Q - z_P}{\overline{z_Q - z_P}}$$

Theorem 6.2. *Point $dir\,(PQ)$ belongs to the unit circle. In the Lubin(n) representation, the corresponding fraction is homogenous with degree $2n$. And we have the key relation :*

$$\angle\,(UV, UW) = \vartheta \Longleftrightarrow \frac{dir\,(UW)}{dir\,(UV)} = \exp\,(2i\vartheta)$$

Proof. Classification as theorem comes from the fruitfulness, not from the difficulty of the proof. Degree is obvious, and the second part is the inscribed angle property. □

Proposition 6.3. *When P is at infinity and U is finite, then :*

$$\zeta_P \doteq dir\,(UP) = \frac{z_{P+U}}{\overline{z_{P+U}}}$$

belongs to the unit circle and does not depends on point U (and of any proportionality factor acting on the barycentrics). When P is at infinity and U is its isogonal conjugate, ζ_P can be computed formally *as $isog\,(z_U) \div conj\,(isog\,(z_U))$ –using (4)– since $1 - z_U\overline{z_U}$ is real and therefore cancels. Finally, orthogonal directions have opposite ζ.*

Proof. Straightforward computation. □

Fact 6.4. *Among the 231 points at infinity named in ETC, there are :*

- 60 points have a Lubin(1) representation (of degree 2). X(1154) would admit a Lubin(1/2) one.

j	k	L	j	k	L	j	k	L	j	k	L
30	523	8	541		42	782		421	2781		64
511	512	26	542	690	43	804	2782	71	2790	2797	71
520		23	543	2793	127	808		207	2794	2799	123
524	1499	61	688		135	826		58	2847		119
525	1503	61	698		238	888		119	2848		119
526		10	702		320	924		24	2869		143
530		225	706		421	1154	1510	27	2871		141
531		229	710		702	2386		173	2872		172
532		231	732		185	2387		110	2881		150
533		229	736		350	2393		81	2882		247
538		120	754		147	2777		38	3221		161
539		61	778		245	2780	2854	62	3564	3566	61

Among them, 4 points contain vdm in their representation. They are directions from G to the given $X(p)$.

j	k	L	p
530	531	179	$13, 14$
532	533	316	$17, 18$

- 165 points have a Lubin(2) representation (of degree four). Here are the shortest.

513	517	8	834		69	2772	2774	55
514	516	37	891		95	2773	2779	55
515	522	32	900	952	14	2776	2842	52
518	3309	60	912		56	2783	2787	96
519		32	916		74	2800		38
521		57	926	2808	55	2801		42
527		96	928	2807	55	2802	2827	37
528	2826	75	971		60	2803	2828	100
529		92	1938		107	2804	2829	71
535		74	2390		63	2810	2821	56
674		74	2392		55	2815	2841	29
758		56	2771		42	2818		52

- 6 points that don't have a Lubin representation, due to the presence of radicals. They are orthopoints in pairs.

j	k	$Wurzel$
2574	2575	W_3
3307	3307	$W_4/\sqrt{abc}$
3413	3314	W_2

7 Some concluding remarks

1. We have proven that Morley centers cannot have a symmetrical Lubin(3) representation. Expressions obtained are huge and require a great amount

of formal computing tools. Nevertheless, rational fractions have a unique normal form : a great advantage compared to trigonometric expressions. Moreover, this kind of non symmetrical fraction allows to deal with a class of objects as a whole.

2. In fact, Lubin(6) is the most useful representation, since incenter and related points cannot be excluded. There are no additional difficulties when going from Lubin(3) to Lubin(6).

3. The connection with the Poncelet representation, using the incircle, has been elucidated.

4. A method to go back from an exact Lubin representation and obtain formal barycentrics would be great. For the moment, numerical values are required, and this is only useful for points listed in ETC.

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